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Stability Study of Exponential Smooth Transition Double Autoregressive Model with Application

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Abstract In this research, we propose a novel non-linear time series model referred to as the Exponential Smooth Transition Double Autoregressive Model of order p denoted as EXPSTDAR (p). This model is predicated on the double autoregressive framework and incorporates an exponential function. Through the application of a local linearization technique, we derive a stationarity condition for a non-zero singular point of the model. And applying these conditions to the acquired models by modeling real data that represents a weekly average closing price of Iron ore powder in dollars for the period from October 2010 A.D. to May 2025 A.D for several suggested orders of the model.

Keywords Autoregressive conditional Heteroscedastic (ARCH), Double Autoregressive (DAR) model, Exponential Smooth Transition Double Autoregressive (EXPSTDAR), Local linearization approach, non-zero singular point, stability of non-zero singular point, conditional variance

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1. Introduction

we say that $\{x_t\}$is a linear Autoregressive model of order (p) denoted by AR(P) if it satisfies the difference equation

$$x_t + a_1 x_{t-1} + a_2 x_{t-2} + ... + a_p x_{t-p} = z_t \tag{1}$$

where $a_1, a_2, ...a_p$are constants and $z_t \sim i.i.d\ N(0, \sigma_t^2)$the equation (1) can be written: $\alpha(B)x_t = z_t$
where $\alpha(B)$is a polynomial of feed back shift operatar B

$$\alpha(B) = 1 + \alpha_1 B + \alpha_2 B^2 + ... + \alpha_p B^p = \sum_{u=0}^{p} \alpha_u B^u \, , \alpha_0 = 1$$

The general solution of AR(P) given by:

$$x_t = f_{(t)} + a^{-1}(B)z_t$$

where $f_{(t)} = A_1 M_1^t + A_2 M_2^t + ... + A_P M_P^t$ and $\mathrm{M}_1,\ \mathrm{M}_2,\ ...,\ \mathrm{M}_\mathrm{p}$ are the roots of a polynomial $g(\nu) = \nu^P + a_1 \nu^{P-1} + ... + ap_1$, the process is asymptotically stationary if and only if:
$|M_i| < 1 for 1 \leq i \leq p$. Moreover the particular solution$X_t = a^{-1}(B)z_t$can be written as:

$$x_t = [\sum_{i=1}^{p} \frac{k_i}{1 - BM_i}]z_t$$

where k_i,$1 \leq i \leq p$ are arbitrary constants. Then the general solution of AR(p) can be written as:

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$\therefore x_t = A_1 M_1^t + A_2 M_2^t + ... + A_P M_P^t + [\sum_{i=1}^{k} \frac{k_i}{1-BM_i}] z_t$

Assuming thatM_1, M_2, ..., M_p are distinct roots of a characteristic polynomial $g(\nu) = \nu^p + a_1 \nu^{p-1} + a_2 v^{p-2} ... + a_p$ and $A_1, A_2 ..., A_p$ be an arbitrary constants[32] . In 1982, Engle introduced a family of non-linear time series model known as the Autoregressive Conditional Heteroscedastic (ARCH) model, which is based on a martingale difference series [2].

In 1985 ,Ozaki, T. proposed an exponential autoregressive model ;EXPAR model and find a stationary conditions of the this model by using a dynamical approach Known as local linearization method which is an approximated method used to approximate the nonlinear autoregressive method to a linear difference equation in terms of the model parameters and the stationarity condition determined by insure that all the roots of a characteristic equation of a difference equation lies inside the unite circle [31] . In addition many proposed nonlinear autoregressive model used this dynamical approach due to Ozaki for example [22, 23, 24, 25, 26, 27, 28, 29, 30].

In 1986 Bollersiev expanded ARCH model this by presenting the Generalized Autoregressive Conditional Heteroscedastic (GARCH) model [1]. Later, in 2007, Ling proposed a newly developed autoregressive model called the Double Autoregressive (DAR) model. The Double AR(p) model combines the features of the Autoregressive AR(p) model with the ARCH framework [8].

Since then, several variations of DAR model have been proposed and extensively studied. For instance, in 2016, Li and Zhang introduced the threshold DAR model [9, 15, 16, 17]. In 2017 Li, Zhu, Liu and Li,w,k presented the mixture DAR model [10, 11, 12, 13, 14]. Subsequently, in 2018, Zhu and Li,G analyzed linear double autoregression [38]. More recently in 2020, Jiang and Zhu proposed the augmented DAR model [7]. In 2022 Hiba and Azher introduced the Log DAR [22] and the Exp DAR [23].In 2023, Tan and Zhu presented the DA-LDAR [34].

Finally, in2025 Saber and Azher used a local linearization method to study stationarity of a proposed double autoregressive model known as Sec DAR(p) model that based on secant and secant hyperbolic functions [30].

The Double Autoregressive of order P (DAR(P)) model proposed by Ling in 2007 is represented mathematically as:

$$y_t = \sum_{i=1}^{p} \varphi y_{t-i} + z_t \sqrt{w + \sum_{i=1}^{p} \alpha_i y_{t-i}^2} \tag{2}$$

where The parameters $\omega > 0 and \alpha_i$ are constants and $\{\mathrm{z_t}\}$is a sequence of independent and identically random variables following a standard normal distribution, with mean 0 and unit variance. denoted by $z_t \sim i.i.d\ N(0, 1)$. Additionally, $\mathrm{M} = \{-\mathrm{P}, ..., 0, 1, 2, ...\}$ and $\mathrm{y}_s$is assumed to be independent of $\{z_t : t \geq 1\}$for$s \leq 0$. Let $F_t$represent the $\sigma - field$generated by $\{\mathrm{n}_t, ..., \mathrm{n}_1, \mathrm{y}_0, ..., \ \mathrm{y}_{-\mathrm{p}}\}$ for $\mathrm{t} \in \mathrm{M}$,under these assumptions, the conditional variance of y_t.
is given by

$$Var(y_t \,| F_{t-1}) = w + \sum_{i=1}^{p} \alpha_i y_{t-i}^2$$

The DAR(P) model described in equation (1) is a specific case of mixed ARMA-ARCH models. For the weak stationarity of the ARMA model, it is necessary to ensure geometric ergodicity of the Markov chain representation derived from the model [8].

Many reserchers deriving specific stability conditions in terms of model parameters using approximation methods. These often involve linearization techniques such as the local linearization method [35, 18, 19, 20, 21].

The local linearization method has been employed in several studies to analyze stability conditions for various models. For example, Mohammad and Salim (2007) applied this method to study the stability of the Logistic Autoregressive (LSTAR(p)) model [28]. In 2010 Mohammad and Ghannam proposed a Cauchy Autoregressive model and derived its stability conditions [25]. Similarly, Salim and Younis (2012) analyzed the stability of nonlinear autoregressive models [33]. Mohammad and Ghaffar (2016) utilized the method to investigate the stationarity of GARCH models [24]. while Mohammad and Mudhir (2020) studied stationarity in EGARC(Q, P) models [26]. Noori and Mohammad (2021) focused on the stationarity of GJR-GARCH(Q, P)models using this techniqu [29]. Lastly omar and Azher in 2025 used this method in study the stationarity condition of sec DAR model [30].

2. Preliminaries

2.1. Properties of ARCH and DAR models

The Autoregressive conditional Heteroscedastic (ARCH) model is predicated upon a martingale difference sequence$\{x_t\}$, defined in terms of the conditional expectation of its first and second moments, given a non-decreasing collection of $\sigma - fields$structured as follows:

$$...... \subset F_{t-1} \subset F_t \subset F_{t+1} \subset ......$$

Here, $F_t = \sigma(x_t, x_{t-1}, ..., x_0)$ denotes a $\sigma - field$or filter. A martingale series associated with this filtering is a stochastic process$\{x_t\}$, such that $x_t$is F_t-measurable and satisfies$E(x_t \mid F_{t-1}) = 0$. This condition implies that $x_t$is orthogonal to all random variables within the filter F_{t-1}, consequently, for $s < t$, where x_s belongs to F_s , which is hierarchically nested within $F_{t-1}$and F_t, the following holds $E(x_t\ x_s) = 0$ for

$$s \neq t$$

A key characteristic of heteroscedasticity in this context is that the Variance is not constant and varies over time. This phenomenon, referred to as conditional variance, is expressed as $E(x_t^2|F_{t-1}) = \sigma_t^2$. The ARCH model imposes specific constraints on the relationship between the innovations process z_t and the conditional standard deviation process $\{\sigma_t\}$, such that:

$$x_t = \sigma_t z_t \tag{3}$$

Here, $\{\sigma_t\}$and $\{z_t\}$represent independent stochastic processes adhering to the following specifications:
.1 The process $\sigma_t$is measurable with respect to F_{t-1} or equivalently, it is F_{t-1} measurable. This implies thatσ_t depends solely on the information available up to time$t - 1$.
.2 The process $\{z_t\}$ consists of independent and identically distributed random variables that follow a standard normal distribution, *i.e.*,$Z_t \sim i.i.d\ N(0, 1)$.
.3 It is assumed that$\sigma_t > 0$ ensuring positivity.

Under these conditions, the standard deviation σ_t is recognized as the volatility ofx_t. The joint behavior of these processes leads to the following statistical properties:
1. The expected value of $x_t$can be computed as $E(x_t) = E(\sigma_t z_t) = E(\sigma_t)E(z_t)$, since $E(z_t) = 0$for a standard normal random variable, it follows that $E(x_t) = 0$.
2. The covariance between $x_t$and $x_{t\pm k}$, denoted as $cov(x_t, x_{t\pm k})$, is given by$E(z_t)E(\sigma_t z_{t\pm k})$. Given the independence of $z_t$and the fact that $E(z_t) = 0$, the result simplifies to $cov(x_t, x_{t\pm k}) = 0$ for all $k > 0$.
Consequently, the process $\{x_t\}$ satisfies the properties of a weak white noise process, characterized by zero mean and orthogonality between observations at distinct time points separated by any lag $k > 0$.

To address the problem of volatility the ARCH model developed by R. Engle is commonly utilized. In its standard form, the ARCH model is defined using the $\sigma - field, F_{t-1} = \sigma(x_s; s < t)$,which encapsulates the information set generated by the past values of the stochastic process $\{x_t\}$.The ARCH model expresses the squared volatility or conditional variance as a linear function of past squared terms from the series, denoted as $\{x_t^2\}$ [1, 3].
The general formulation of the ARCH(q) model can be expressed as follows:

$$\begin{aligned} x_t &= \sigma_t z_t \\ \sigma_t^2 &= \alpha_0 + \textstyle\sum_{i=1}^{q} \alpha_i x_{t-i}^2 \,, for\, i = 1 to q. \end{aligned} \tag{4}$$

Here, $\{z_t\}$represents an independent and identically distributed *(i.i.d)* standard normal random variable, $Z_t \sim i.i.d\ N(0, 1)$. The parameters satisfy the conditions $\alpha_0 >\ 0$ and$\alpha_i \geq 0 \forall\ i = 1, 2, ..., q$. The variance conditional on the information set F_{t-1} is given by:

$$Var(x_t \left| F_{t-1} \right) = E(x_t \left| F_{t-1} \right) = \alpha_0 + \sum_{i=1}^{q} \alpha_i x_{t-i}^2 \, for \, i = 1 to q.$$

Furthermore, the unconditional Variance of the process, denoted as σ_X^2, is given by:

$$Var(x_t) = \sigma_x^2 = \alpha_0/(1 - \sum_{i=1}^{q} \alpha_i) \, for \, i = 1 to q.$$

where the existence of unconditional Variance requires $0 < \sum_{i=1}^{q} \alpha_i < 1$. In this framework, the conditional variance σ_t^2 depends on time (t), while the unconditional variance σ_X^2 remains constant over time [4, 5, 36].
Ling (2007) extended the ARCH model by proposing the Double Autoregressive (DAR) model. This model incorporates two components:
(i) a linear autoregressive component that accounts for mean dynamics.
(ii) a conditional heteroscedastic autoregressive component representing the variance dynamics.
The DAR(P) model is expressed as:

$$\begin{aligned} & y_t = \sum_{i=1}^{p} \varphi_i y_{t-i} + n_t \sqrt{w + \sum_{i=1}^{p} \alpha_i y_{t-i}^2} \\ & for \, i = 1 to p. \end{aligned} \quad (5)$$

In this formulation, $\omega > 0$, $\alpha_0 > 0$ and $\alpha_i, \varphi_i, i = 1, \ldots.., p$are constants, and$\{n_t\}$ is an i.i.d random sequence such that $n_t \sim N(0, 1)$.
In the kind DAR (1) model , the necessary and sufficient condition for stationarity is$\phi_1^2 + \alpha_1 < 1$. Figure (1) illustrates the regions in the (ϕ_1, α_1)-plane that fulfill this stationarity condition (denoted as Region B). which is a region lies under the parabola $\alpha_1 = 1 - \phi_1^2$. Additionally, another condition, $y_0 = E[\ln \left| \phi_1 + \sqrt{\alpha_1} y \right|] < 0$ (Region A), ensure further stationarity properties. Through the application of Jensen's inequality and an analysis of Figure (1), it becomes evident that the condition $y < 0$addresses cases where some roots of $\phi(z) = 1 - \sum_{i=1}^{p} \phi_i z^i = 0 (for \, i = 1 \, to p)$ lie on or outside the unit circle, and scenarios where $E(y_t^2)$ diverges as t approaches to infinity.

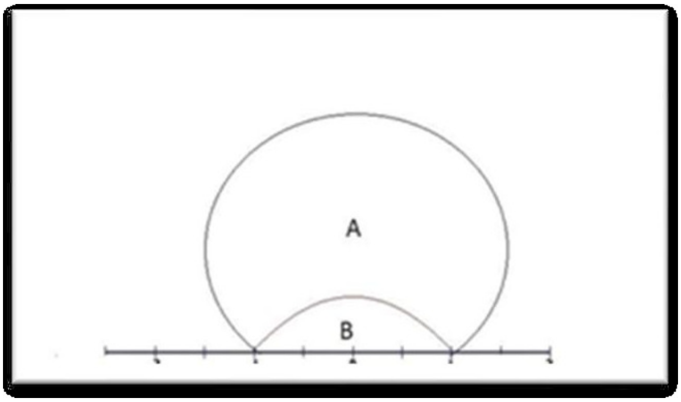


Figure 1. $\phi_1^2 + \alpha_1 < 1 \, as (\phi_1, \alpha_1) \in B$ [8]

The weakest sufficient conditions for $E(y_t^2) < \infty$are given by$(\sum_{i=1}^{P} |\phi_i|)^2 + \sum_{i=1}^{P} \alpha_i < 1$, where the summations run over i from 1 to P. Except in the case of$P = 1$, this condition is stricter than others. For instance, if $\alpha_1 = \alpha_2 = \ldots = \alpha_P = 0$, the necessary and sufficient condition for the stationarity of usual AR(P) model is considerably less stringent than$\sum_{i=1}^{P} |\phi_i| < 1$ [8].

2.2. Exponential Smooth Transition Double Autoregressive model EXPSTDAR(P)

The exponential smooth transition function is defined as:

$$G(x,c,\gamma) = 1 - e^{-\gamma(x-c)^2}, \gamma > 0 \tag{6}$$

where $G : R \to [0,1]$, ca position parameter and γbe smooth (shape) parameter.
Figures (2) and (3) of an exponential smooth transition function shows the effect of the parameterγ and an effect of the parameter c respectively, and G satisfies the following properties:

1. $\lim\limits_{x\to c} G(x,\gamma,c) = 0$
2. $\lim\limits_{x\to\mp\infty} G(x,\gamma,c) = 1$
3. $\lim\limits_{\gamma\to\infty} G(x,\gamma,c) = 1$
4. $\lim\limits_{\gamma\to\infty} G(x,\gamma,c) = 0$

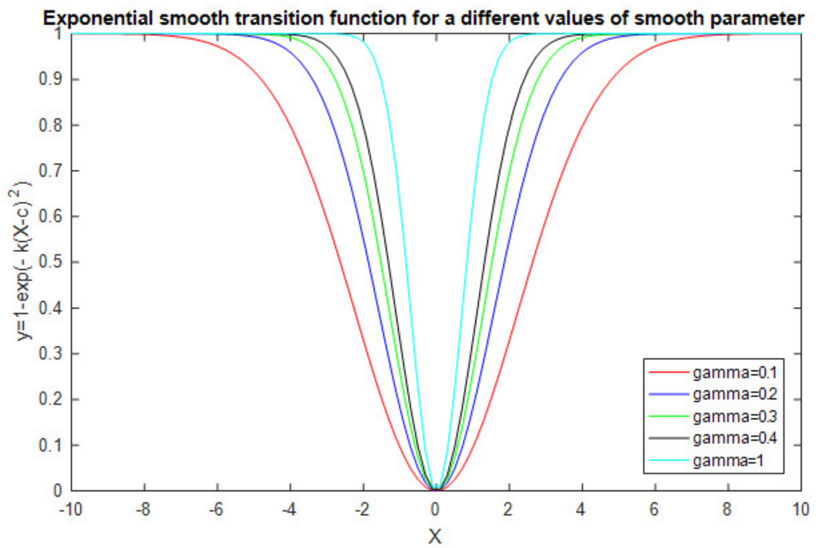


Figure 2. Graph of exponential smooth transition function for a different values of .

In nonlinear time series modes especially nonlinear autoregressive models the nonlinearity characterized by the jump phenomenon and existing the limit cycle. some models such as threshold autoregressive model proposed by Tong ,H. the jump which is a transition from one regime to another happened in suddenly manner depending on threshold parameter [24]. In exponential autoregressive model due to Ozaki, T. the transition happened through exponential function . Our suggested model that based on the dynamical properties that locate the position of transition via the parameter c and the smoothness of transition via the smooth parameter γ , it is clear that the value of exponential smooth transition function ranged from 0 to 1 as shown in graphs and the previous four dynamical properties.
This behavior of the function requires the suggest of new type of double autoregressive model. Known as exponential smooth transition double autoregressive model of order P, denoted (EXPSTDAR)

$$y_t = \sum_{i=1}^{p} \alpha_i y_{t-i} + n_t \sqrt{A + \sum_{i=1}^{p} \beta_i (1 - e^{-\gamma(y_{t-i}-c)^2})} \tag{7}$$

where $\{\alpha_i\}_{i=1}^p, \{\beta_i\}_{i=1}^p, A > 0$are constants and $\mathrm{n_t} \sim \mathrm{iidN}(0,1)$with zero mean and unit variance.

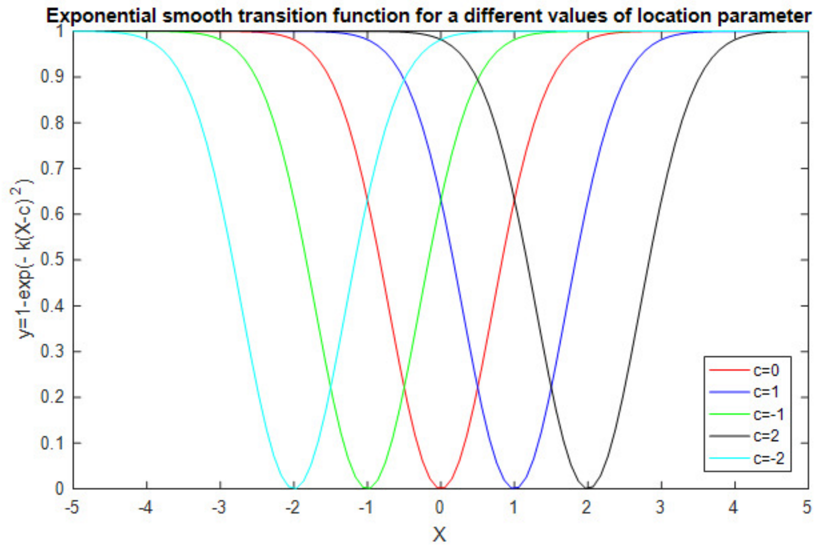


Figure 3. Graph of exponential smooth transition function for a different values of c.

Let $F_t$be a -field generated by $\{\mathrm{n_t}, \mathrm{n_{t-1}}, ..., \mathrm{n_1}, \mathrm{y_0}, ..., \mathrm{y_{-p}}\}$ term where $m = \{-p, ..., 0, 1, 2, ...\}$

To study stationary conditions of EXPSTDAR(P) model must be the condition variance σ_t^2 converges to the unconditional variance σ_y^2see [27, 37].

We discuss the stationarity of a non zero singular point and find the condition that makes the non zero singular point is stable.

In order to find conditional expectation of (EXPSTDAR(P)) model, we taking conditional expectation with respect to filter F_t, after squaring both sides (7) we get:

$$\begin{aligned} &y_t^2 = \textstyle\sum_{i=1}^p \alpha_i^2 y_{t-i}^2 + 2\sum_{i\neq j}^p \alpha_i\alpha_j y_{t-i}y_{t-j} + \\ &2n_t \textstyle\sum_{i=1}^p \alpha_i y_{t-i}\sqrt{A + \sum_{i=1}^p \beta_i(1 - e^{-\gamma(y_{(t-i)}-c)^2})} \\ &+n_t^2(A + \textstyle\sum_{i=1}^p \beta_i(1 - e^{-\gamma(y_{(t-i)}-c)^2}) \end{aligned} \tag{8}$$

By taking the conditional expectation with respect to a filter F_{t-1} then.

$$\begin{aligned} &E(y_t^2/F_{t-1}) = \textstyle\sum_{i=1}^p \alpha_i^2 E(y_{t-1}^2 \,|F_{t-i-1}\,) + \\ &2E(n_t)\textstyle\sum_{i=1}^p \alpha_i E(y_{t-i}\,|F_{t-i-1}\,)\sqrt{(A + \sum_{i=1}^p \beta_i E(1 - e^{-\gamma(y_{(t-i)}-c)^2}\,|F_{t-i-1}\,)} \\ &+2\textstyle\sum_{i\neq j}^p \alpha_i\alpha_j E((y_{t-i}y_{t-j})\,|F_{t-i-1}\,) + \\ &E(n_t^2)(A + \textstyle\sum_{i=1}^p \beta_i E(1 - e^{-\gamma(y_{(t-i)}-c)^2}\,|F_{t-i-1}\,) \end{aligned} \tag{9}$$

Since $\mathrm{n_t} \sim \mathrm{iidN}(0, 1)$ and $E(y_i y_j) = 0 \forall i \neq j$, then:

$$E(y_t^2\,|F_{t-1}\,) = \sum_{i=1}^p \alpha_i^2 E(y_{t-i}^2\,|F_{t-i-1}\,) + A + \sum_{i=1}^p \beta_i E\left[(1 - e^{-\gamma(y_{t-i}-c)^2})\,|F_{t-i-1}\right]$$

$$\sigma_t^2 = \sum_{i=1}^p \alpha_i^2 \sigma_{t-i}^2 + A + \sum_{i=1}^p \beta_i E\left[(1 - e^{-\gamma(y_{t-i}-c)^2})\,|F_{t-i-1}\right] \tag{10}$$

By Taylor expansion of the exponential function $(1-e^{-\gamma(y_{t-i}-c)^2})$ around c and by property Eq. (1), then the exponent $(y_{t-i}-c)^2$ is very small and then we can truncate the terms of a higher power

$$\begin{array}{l} 1-e^{-\gamma(y_{t-i}-c)^2}=1-(1-\gamma(y_{t-i}-c)^2-\frac{\gamma^2}{2}(y_{t-i}-c)^4+......\\ \simeq\gamma(y_{t-i}^2-2cy_{t-i}+c^2)\\ E\left[(1-e^{-\gamma(y_{t-i}-c)^2})\,|F_{t-i-1}\right]\simeq\gamma E(y_{t-i}^2\,|F_{t-i-1})-2cE(y_{t-i}\,|F_{t-i-1})+c^2 \end{array}$$

Since $E(y_{t-i}\,|F_{t-i-1})=0$ by martingale difference property and $E(y_{t-i}^2\,|F_{t-i-1})=\sigma_{t-i}^2$, then:

$$E\left[(1-e^{-\gamma(y_{t-i}-c)^2})\,|F_{t-i-1}\right]\simeq\gamma\left(\sigma_{t-i}^2+c^2\right) for \quad i=1,2,...,p \tag{11}$$

To find the unconditional variance we use definition of fixed point since the unconditional variance is constant(does not depend on t) and then we can put.

$$\begin{array}{l} \sigma_t=\sigma_{t-1}=........=\sigma_{t-p}=\sigma_y\\ \sigma_y^2=\sum_{i=1}^p\alpha_i^2\sigma_y^2+A+\gamma\sum_{i=1}^p\beta_i\sigma_y^2-\gamma c^2\sum_{i=1}^p\beta_i\\ \sigma_y^2-\sum_{i=1}^p\alpha_i^2\sigma_y^2-\gamma\sum_{i=1}^p\beta_i\sigma_y^2=A+\gamma c^2\sum_{i=1}^p\beta_i\\ \sigma_y^2=\frac{A+\gamma c^2\sum_{i=1}^p\beta_i}{1-\sum_{i=1}^p(\alpha_i^2+\gamma\beta_i)},\gamma>0,A>0 \end{array}$$

σ_y^2 represents the unconditional variance of the EXPSTDAR(p) model, and its exist if $0<\sum_{i=1}^p(\alpha_i^2+\gamma\beta_i)<1$, and $c^2\sum_{i=1}^p\beta_i>0$

In order to find a strictly locally stationary condition of a non-zero singular part of the model by using local linearization technique . Let s be a non-zero singular point of the model also a singular point is an equilibrium point such that there is no other equilibrium point in its neighborhood then we can put $y_t=y_{t-1}=......=y_{t-p}=s$ then substitute it in EXPSTDAR(P) we have:

$$\begin{array}{l} s=\sum_{i=1}^p\alpha_i s+n_t\sqrt{A+\sum_{i=1}^p\beta_i(1-e^{-\gamma(s-c)^2})}\\ s^2=\sum_{i=1}^p\alpha_i^2s^2+2\sum_{i=1}^p\alpha_i sn_t\sqrt{A+\sum_{i=1}^p\beta_i(1-e^{-\gamma(s-c)^2})}\\ +n_t^2(A+\sum_{i=1}^p\beta_i(1-e^{-\gamma(s-c)^2})) \end{array}$$

By taking the mathematical expectation we get:

$$\begin{array}{l} E(s^2)=\sum_{i=1}^p\alpha_i^2s^2+2\sum_{i=1}^p\alpha_iE(n_t)\sqrt{A+\sum_{i=1}^p\beta_iE(1-e^{-\gamma(s-c)^2})}\\ +E(n_t^2)(A+\sum_{i=1}^p\alpha_iE(1-e^{-\gamma(s-c)^2})) \end{array}$$

Since $E(n_t)=0$ and $E(n_t^2)=1$

$$s^2(1-\sum_{i=1}^p\alpha_i^2)=A+\sum_{i=1}^p\alpha_iE(1-e^{-\gamma(s-c)^2})$$

And from (11):

$$E(1-e^{-\gamma(s-c)^2})=\gamma(s^2+c^2) \tag{12}$$

$$\begin{array}{l} s^2(1-\sum_{i=1}^p\alpha_i^2)=A+\sum_{i=1}^p\beta_i(\gamma s^2+\gamma c^2)\\ =A+\gamma s^2\sum_{i=1}^p\beta_i+\gamma c^2\sum_{i=1}^p\beta_i\\ s^2(1-\sum_{i=1}^p\alpha_i^2)-\gamma s^2\sum_{i=1}^p\beta_i=A+\gamma c^2\sum_{i=1}^p\beta_i\\ s^2=\frac{A+\gamma c^2\sum_{i=1}^p\beta_i}{(1-\sum_{i=1}^p\alpha_i^2)-\gamma\sum_{i=1}^p\beta_i} \end{array}$$

$$s^2=\frac{A+\gamma c^2\sum_{i=1}^p\beta_i}{1-\sum_{i=1}^p(\alpha_i^2+\gamma\beta_i)} \tag{13}$$

$$s = \mp\sqrt{\frac{A + \gamma c^2 \sum_{i=1}^{p} \beta_i}{1 - \sum_{i=1}^{p}(\alpha_i^2 + \gamma\beta_i)}}, A > 0$$

Then s exist if $\frac{A+\gamma c^2 \sum_{i=1}^{p} \beta_i}{1-\sum_{i=1}^{p}(\alpha_i^2+\gamma\beta_i)} > 0$
Proposition (1): The EXPSTDAR(p) is asymptotically stationary if all the roots of the characteristic equation : $Z^p - \sum_{i=1}^{p} k_i Z^{p-i} = 0$
Where $k_i = \frac{\alpha_i}{2} + \frac{\gamma}{s}\beta_i(s-c)e^{-\gamma(s-c)^2}, i = 1, 2, ..., p$
lies inside the unit circle or $|\mathrm{Z_i}| < 1$ for $i = 1, 2, ..., p$
Proof: The EXPSTDAR (p) model has the form:

$$y_t = \sum_{i=1}^{p} \alpha_i y_{t-i} + n_t \sqrt{A + \sum_{i=1}^{p} \beta_i (1 - e^{-\gamma(y_{t-i}-c)^2})}$$

By squaring both sides we get

$$y_t^2 = \sum_{i=1}^{p} \alpha_i^2 y_{t-i}^2 + 2\sum_{i\neq j} \alpha_i\alpha_j y_{t-i}y_{t-j} + 2n_t \sum_{i=1}^{p} \alpha_i y_{t-i}\sqrt{A + \sum_{i=1}^{p} \beta_i(1 - e^{-\gamma(y_{(t-i)}-c)^2})}$$
$$+n_t^2(A + \sum_{i=1}^{p} \beta_i(1 - e^{-\gamma(y_{(t-i)}-c)^2})$$

By taking the mathematical expection of both sides we get

$$E(y_t^2) = \sum_{i=1}^{p} \alpha_i^2 E(y_{t-i}^2) + 2\sum_{i=1}^{p} \alpha_i\alpha_j E(y_{t-i}y_{t-j}) +$$
$$2E(n_t)\sum_{i=1}^{p} \alpha_i E(y_{t-i})\sqrt{A + \sum_{i=1}^{p} \beta_i E(1 - e^{-\gamma(y_{(t-i)}-c)^2})}$$
$$+E(n_t^2)(A + \sum_{i=1}^{p} \beta_i E(1 - e^{-\gamma(y_{(t-i)}-c)^2})$$

Since $E(y_{t-i}y_{t-j}) = 0$ for $i \neq j, E(n_t) = 0$ and $E(n_t^2) = 1$ for all t we get

$$E(y_t^2) = \sum_{i=1}^{p} \alpha_i^2 E(y_{t-i}^2) + A + \sum_{i=1}^{p} \beta_i(1 - e^{-\gamma(y_{(t-i)}-c)^2}) \tag{14}$$

Note that equation (14) is noise free or the white noise input is suppressed and this is corresponds to a local linearization approximation method for more details see [31] pp.37.

The local linearization method consists of lineariztion the model near the non-zero singular point or in sufficiently small neighborhood around s and by using a vibrational equation:
By using a variational equation:

$$y_{t-i} = s + s_{t-i} \tag{15}$$

For $i = 1, 2, ..., p$ such that the radius s_t sufficiently small such that $|s_t|^n \to 0$for $n \geq 2$
By substuting equation (15) in a equation (16) we get:

$$E((s+s_t)^2) = \sum_{i=1}^{p} \alpha_i^2 E(s+s_{t-i})^2 + A + \sum_{i=1}^{p} \beta_i E(1 - e^{-\gamma(s+s_{t-i}-c)^2}) \tag{16}$$

By using Taylor approximation we get:

$$1 - e^{-\gamma(s+s_{t-i}-c)^2} = 1 - e^{-\gamma((s-c)+s_{t-i})^2}$$
$$= 1 - e^{-\gamma((s-c)^2+2s_{t-i}(s-c)+s_{t-i}^2)}$$
$$= 1 - e^{-\gamma(s-c)^2}e^{-2\gamma(s-c)s_{t-i}}$$
$$= 1 - e^{-\gamma(s-c)^2}(1 - 2\gamma(s-c)s_{t-i})$$
$$= 1 - e^{-\gamma(s-c)^2} + 2\gamma(s-c)e^{-\gamma(s-c)^2}s_{t-i}$$

But from Eq. (12):

$$E(1-e^{-\gamma(s-c)^2})=\gamma(s^2+c^2)$$

Then:

$$E(1-e^{-\gamma(s+s_{t-i}-c)^2})=\gamma(s^2+c^2)+2\gamma(s-c)e^{-\gamma(s-c)^2}s_{t-i} \quad (17)$$

Substitute equation (17) in (17) we get:

$$\begin{aligned}
&E(s^2+2ss_t+s_t{}^2)=\textstyle\sum_{i=1}^{p}\alpha_i^2(s^2+2ss_{t-i}+s_{t-i}{}^2)+A\\
&+\textstyle\sum_{i=1}^{p}\beta_i E(1-e^{-\gamma(s-c)^2}+2\gamma(s-c)e^{-\gamma(s-c)^2}s_{t-i})\\
&s^2+2ss_t=\textstyle\sum_{i=1}^{p}\alpha_i^2(s^2+2ss_{t-i})+A\\
&+\textstyle\sum_{i=1}^{p}\beta_i\gamma(s^2+c^2)+\sum_{i=1}^{p}\beta_i 2\gamma(s-c)e^{-\gamma(s-c)^2}s_{t-i}\\
&s^2(1-\textstyle\sum_{i=1}^{p}\alpha_i^2-\gamma\sum_{i=1}^{p}\beta_i)+2ss_t=\sum_{i=1}^{p}\alpha_i^2 ss_{t-i}+A\\
&+\textstyle\sum_{i=1}^{p}\beta_i\gamma c^2+\sum_{i=1}^{p}\beta_i 2\gamma(s-c)e^{-\gamma(s-c)^2}s_{t-i}
\end{aligned}$$

But from equation (13) $s^2(1-\sum_{i=1}^{p}\alpha_i^2-\gamma\sum_{i=1}^{p}\beta_i)=A+\gamma c^2\sum_{i=1}^{p}\beta_i$ then we get:

$$\begin{aligned}
&A+\gamma c^2\textstyle\sum_{i=1}^{p}\beta_i+2ss_t=\sum_{i=1}^{p}\alpha_i^2 ss_{t-i}+A+\gamma c^2\sum_{i=1}^{p}\beta_i+\sum_{i=1}^{p}\beta_i 2\gamma(s-c)e^{-\gamma(s-c)^2}s_{t-i}\\
&2ss_t=\textstyle\sum_{i=1}^{p}\alpha_i^2 ss_{t-i}+\sum_{i=1}^{p}\beta_i 2\gamma(s-c)e^{-\gamma(s-c)^2}s_{t-i}
\end{aligned}$$

Dividing both sides by $(2s)$ since $s\neq 0$ we get.

$$s_t=\frac{1}{2}\sum_{i=1}^{p}\alpha_i^2 s_{t-i}+\sum_{i=1}^{p}\beta_i\frac{\gamma}{s}(s-c)e^{-\gamma(s-c)^2}s_{t-i}$$

$$s_t=\sum_{i=1}^{p}(\frac{\alpha_i^2}{2}+\frac{\gamma}{s}\beta_i(s-c)e^{-\gamma(s-c)^2})s_{t-i} \quad (18)$$

The equation (18) represent a linear difference equation of order p AR(p) without noise term and it's converge to zero (asymptotically stationary) if all roots of the characteristic equation

$$z^p-\sum_{i=1}^{p}k_i z^{p-i}=0$$

Lies inside the unit circle where:$k_i=\frac{\alpha_i^2}{2}+\frac{\gamma}{s}\beta_i(s-c)e^{-\gamma(s-c)^2}, i=1,2,.......,p$
And $s=\sqrt{\frac{A+\gamma c^2\sum_{i=1}^{p}\beta_i}{1-\sum_{i=1}^{p}(\alpha_i^2+\gamma\beta_i)}}$

3. Application

3.1. Data Descriptions

The stationarity condition of the EXPSTDAR(p) model is applied to a dataset Comprising 760 observations, which represent the weekly average closing price of iron ore powder in us dollars. This dataset covers the period from October 2010 to May 2025 and was sourced from the website (https://sa.investing.com/commodities/iron-ore-62-cfr-futures-historical-data)

3.2. DATA Analysis

Initially, a real data is used to construct EXPSDAR time series models, which is subsequently visualized through a plotted graph. Figure 3.1 illustrates the time series representation of the weekly average of Historical real data for Iron ore powder from October 2010 to May 2025.

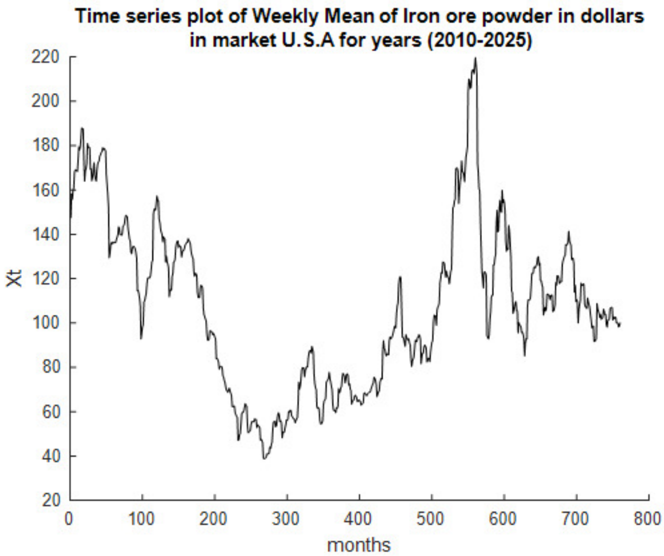


Figure 4. Time series plot of the Weekly average of closing price in U.S Dollar of Historical real data for Iron ore powder from October 2010 to May 2025.

The original series is transformed into a Returns series using a specific formula defined as follows:

$$r_t = \log(p_t/p_{t-1})$$

Here $r_t$represents the Returns series, while $p_t$and p_{t-1}denote the observed data at time t and$t-1$, respectively. This transformation was implemented in MATLAB using the$function\, r_t = \mathrm{price2ret}(P_t)$. Figure 3.2 illustrates the resulting Returns series, which demonstrates noticeable volatility and fluctuations at certain points. Due to these characteristics, EXPDAR(p) proves to be a useful model for analyzing the fluctuations associated with such phenomena. Figure (3) depicts the autocorrelation and partial.

autocorrelation functions of the studied dataset. Observations indicate that volatility at specific points falls outside the confidence interval boundaries, calculated as $\pm 1.96/\sqrt{N}$,where N refers to the sample size [26].

The second step involves detecting the presence of heteroscedastic variance by analyzing the series of squared errors for the returns series. This is determined through the relationship:

$e_t = (y_t - \bar{y}_t)^2$,where $\bar{y}_t$represents the mean of returns. Following this, the autocorrelation and partial autocorrelation functions are plotted in Figure 3.4. shows that there are more than 0.05% correlations lies outside the confidence interval .

The autocorrelation and partial autocorrelation function indicate that certain correlation values fall outside confidence interval.

The third phase involves conducting various tests to detect the presence of heteroscedasticity. One of the most popular and widely used methods for this purpose is the Ljung-Box test. This test evaluates the null hypothesis(H_0) against the alternative hypothesis(H_1) ,where:

$$H_0 : \rho_1 = \rho_2 = \ldots = \rho_k = \ldots = \rho_m = 0$$
$$H_1 : \rho_k \neq 0\, for\, some\, values\, of\; k$$

The test statistic is calculated as:

$$Q_{(m)} = n(n+2)\sum \frac{\widehat{\rho}_k^{\,2}}{n-k} \sim \chi^2(m-p)$$

Here, $Q_{(m)}$follows the chi-square distribution with$(\mathrm{m}-\mathrm{p})$ degrees of freedom, where:

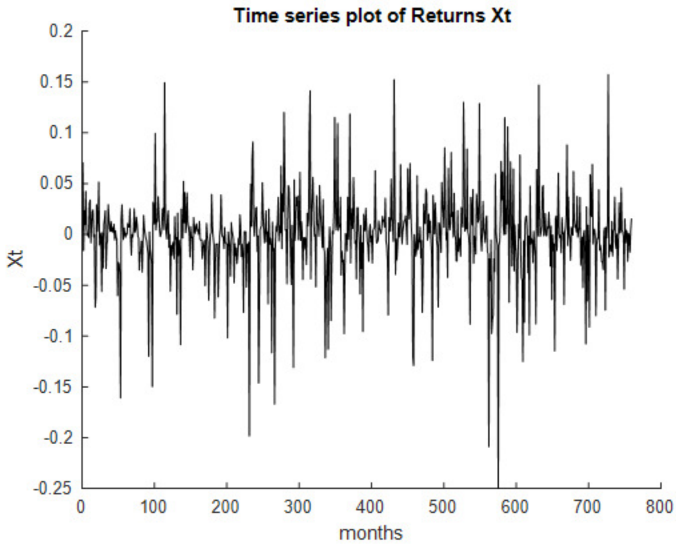


Figure 5. represents the weekly mean returns series of historical contract data for Iron ore powder.

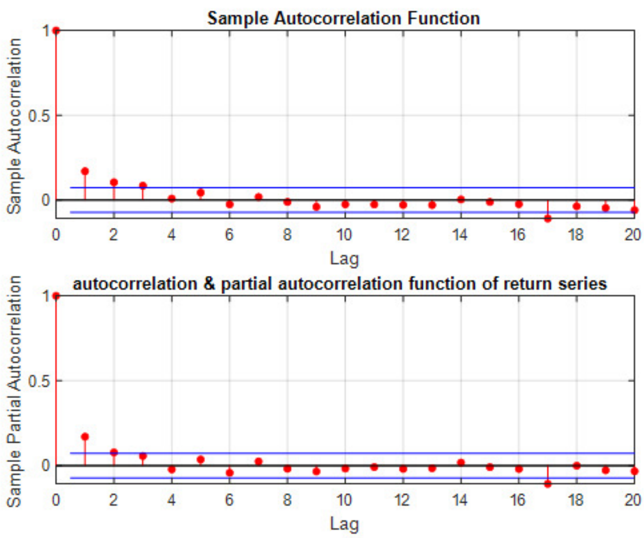


Figure 6. The autocorrelation and partial autocorrelation function.

- $n$represents the sample size.
- m is the maximum lag, typically approximated as $m = \ln(n)$.
- p is the number of parameters in the model.
- $\hat{\rho}_k^2$represents the estimated squared correlation coefficients of the residual series.

If $Q(m) < \chi^2_\alpha(m-p)$,then the null hypothesis H_0 is accepted of a significance level a (commonly set at α= 0.05). This indicates that all correlation coefficients fall within the acceptable range for the given significance level. Conversely, if H_0 is rejected, the alternative hypothesis $\mathrm{H}_1$is accepted, signifying the presence of heteroscedasticity. This outcome is represented by $\mathrm{h} = 1$ in Table3.1.

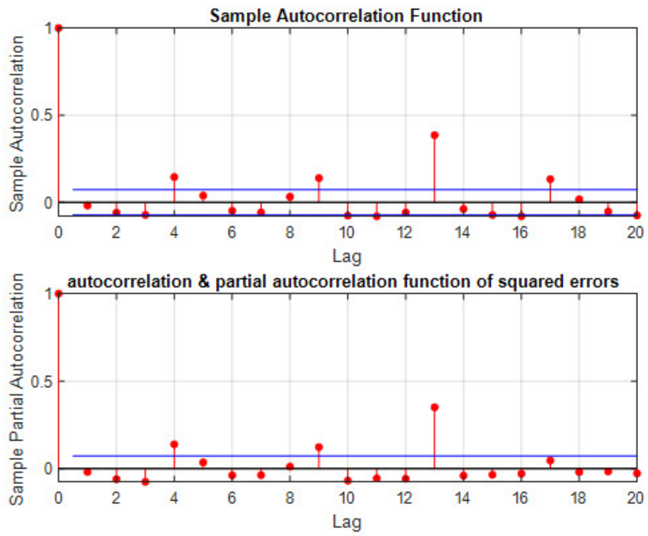


Figure 7. (a) autocorrelation and partial autocorrelation function.

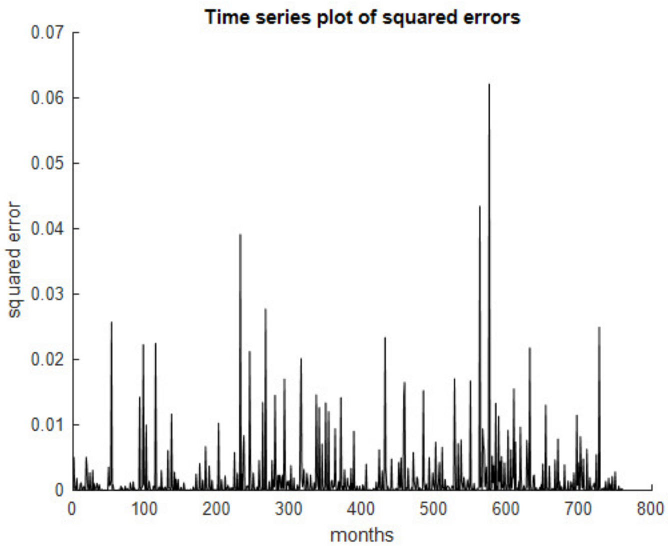


Figure 8. (b) time series plot of square Errors Return.

Table 1. presents the results of a Ljung-Box test conducted on the returns series.

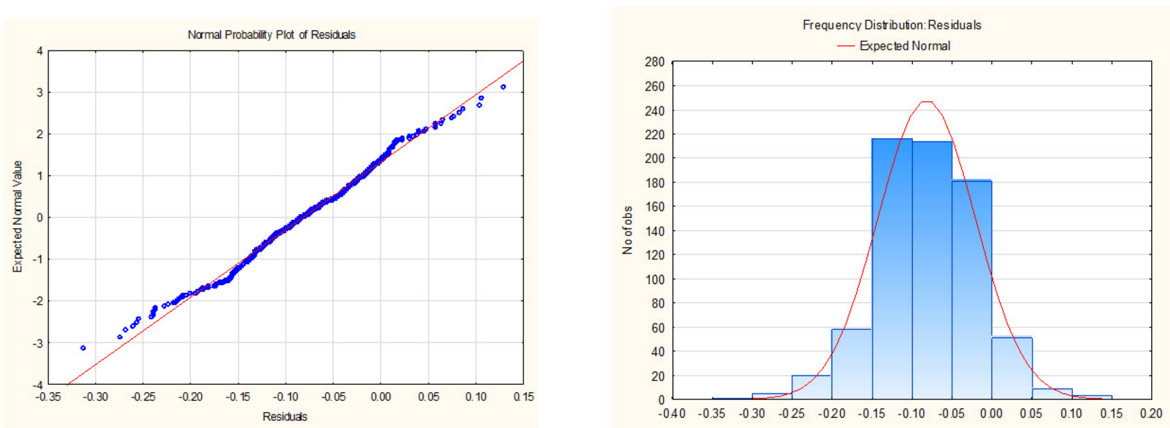
Lags	h-value	p-value	Q-test	Critical value
Lag 1	1	0.0273	21.9990	3.8415
Lag 2	1	0.0026	30.3405	5.9915
Lag 3	1	0.0008	35.7819	7.8147
Lag 4	1	0.0031	35.8367	9.4877
Lag 5	1	0.0052	37.3211	11.0705
Lag 6	1	0.0122	37.8193	12.5916
Lag 7	1	0.0294	38.0718	14.0671
Lag 8	1	0.0702	38.1651	15.5073
Lag 9	1	0.0977	39.3973	16.9190
Lag 10	1	0.1765	39.8997	18.3070

3.3. Modeling and constructing the EXPSTDAR(p) model

We will employ the EXPSTDAR(p) model on a series of data and carefully monitor and validate the convergence of the conditional variance to the unconditional variance. We will also estimate parameters, make necessary adjustments as needed, and proceed to analyze the stationarity of the model according to the condition that mentioned in proposition (1). The software utilized for parameter estimation is statistica. V.12 the part of nonlinear estimation quazi -Newton method, and the time series simulation plot was carried out by using MATLAB R2021a. In examining the stability scenario for a sixth order EXPSTDAR(p) model, we will first consider an EXPSTDAR(1) model as an example of asymptotically stationary model, followed by an EXPSTDAR(6) model to demonstrate non-stationarity as an application of the result obtained in proposition (1).
Example (1)
EXPSTDAR (1):

$$y_t = 0.109989668 y_{t-1} + \eta_t \sqrt{0.0250525666 + 0.100055489(1 - e^{-0.100074968(y_{t-1}-0.130029637)^2})} \quad \ldots(20)$$

$\eta_t \sim$ iid $\mathrm{N}(0,1)$. The unconditional variance is $\sigma_y^2 = 0.0614$which is more than 0. Figure 3.5 shows the normality distributed and frequency distribution of standard residuals a the EXPSTDAR (1) model .

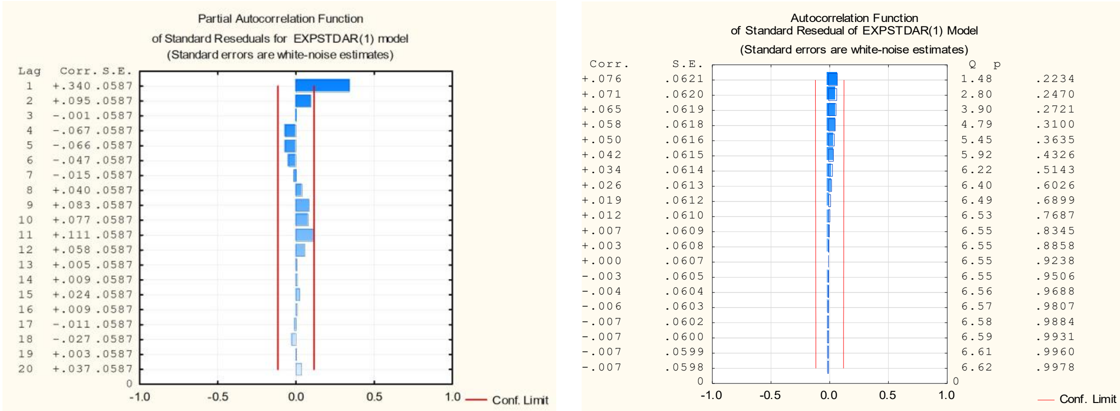


(a) (b)

Figure 9. displays two plots: (a) a normal probability plot of the residuals and (b) a plot showing the frequency distribution of the standard residuals. EXPSTDAR (1) model.

To insure the independency of standard residuals figure 3.6 shows the autocorrelation function and partial autocorrelation function of the standard residuals and of the values of correlations in both functions lies inside the confidence interval for 20 lags with confidence level 0.05.
Equation (2.10) yields a nonzero singular point with a real value of$\sigma_x = \xi = 0.1590$.The corresponding values for the AIC , CAIC and BIC are-2101.3, -2107.23 and -2078.15, respectively.
By applying proposition (1), we may express the characteristic equation as
$v - 0.0079 = 0$, then $v = 0.0079 < 1$. This shows that the model possesses a stable non-zero singular point and the model an asymptotically stationary according to proposition (1)
In order to confirm this result, we plot the simulation a time series of this model starting from a different initial value, as shown in Figure 3.6, which illustrates that the time series of this model is stationary.

Solving the equation determines that the solution is $\gamma - 0.0079 = 0$. This shows that the model possesses a stable singular point that is not equal to zero.
In order to confirm this result, we plot the trajectories of this model, as shown in Figure 3.6, which illustrates that the model is stable or approaching a stationary state.



(a) (b)

Figure 10. The graph and correlations of (a) Partial autocorrelation function. (b)Autocorrelation function.

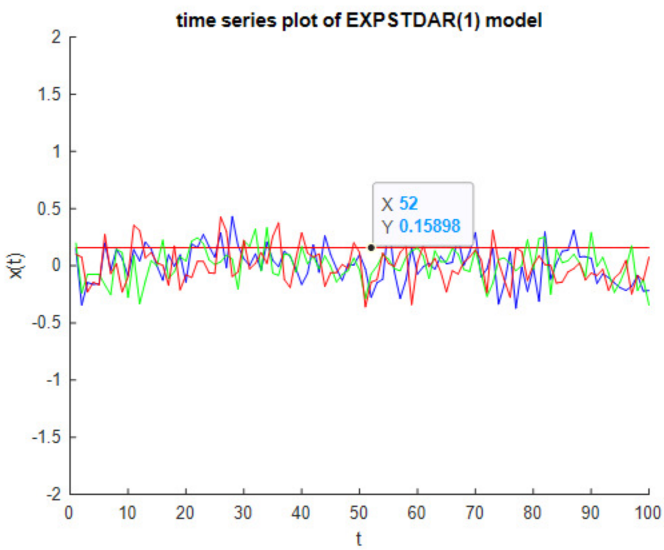


Figure 11. The simulation plot of time series generated by EXPSTDAR (1) model in Example (1) starting with a different initial values.

Example (2): the EXPSTDAR (6) In this example we discuss the following obtained model of sixth order

$$y_t = 2.08y_{t-1} + 0.04y_{t-2} + 0.197y_{t-3} + 0.001y_{t-4} + 0.099y_{t-5} + 0.121y_{t-6} + \eta_t \sqrt{\begin{array}{l} 0.025 + 0.04(1 - e^{-0.11(y_{t-i}-0.088)^2}) + \\ 0.407(1 - e^{-0.11(y_{t-i}-0.088)^2}) + 0.09(1 - e^{-0.11(y_{t-i}-0.088)^2}) \\ +0.132(1 - e^{-0.11(y_{t-i}-0.13)^2}) + 0.089(1 - e^{-0.11(y_{t-i}-0.088)^2}) + \\ 0.121(1 - e^{-0.11(y_{t-i}-0.088)^2}) \end{array}} \quad (19)$$

$$y_t = \sum_{i=1}^{6} \alpha_i y_{t-i} + n_t \sqrt{0.000108175663 + \sum_{i=1}^{6} \beta_i (1 - e^{-0.113448713(y_{t-i}-0.0879618415)^2})} \quad (20)$$

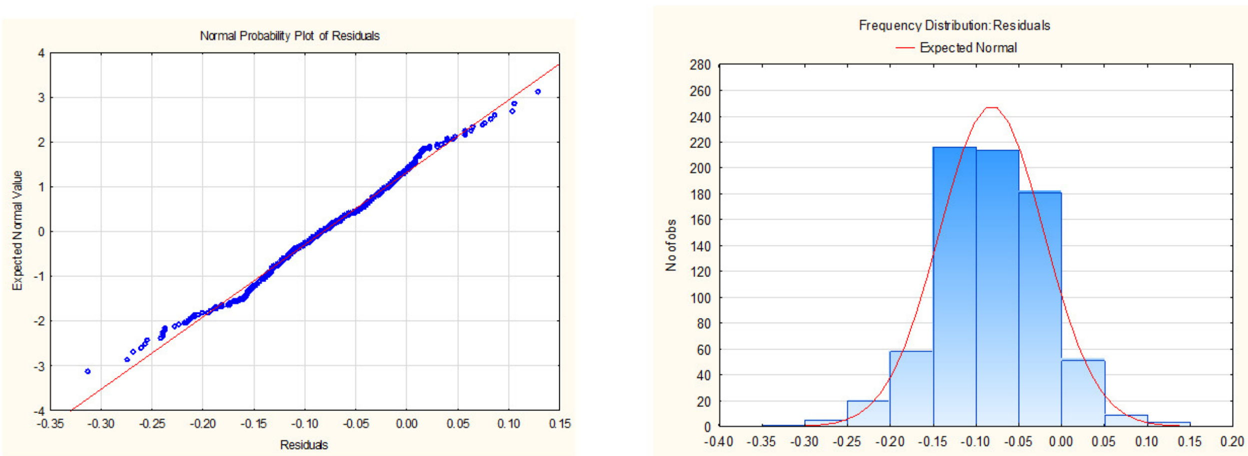
Where the estimated values of model parameters are:

Table 2

$\hat{\alpha}_1$	$\hat{\alpha}_2$	$\hat{\alpha}_3$	$\hat{\alpha}_4$	$\hat{\alpha}_5$	$\hat{\alpha}_6$
1.58044596	0.0420903434	0.197982361	0.00124007535	0.0999909139	0.121992102
$\hat{\beta}_1$	$\hat{\beta}_2$	$\hat{\beta}_3$	$\hat{\beta}_4$	$\hat{\beta}_5$	$\hat{\beta}_6$
1.20955716	0.407279138	-0.098078056	0.132055855	0.0899330746	0.121993805

$$\eta_t \sim iidN(0,1), the\sigma_t^2 = 0.07 > 0, AIC = -1951.85, CAIC = -1967.21, BIC = -1882.49$$

Figure 3.9 represents a normal probability plot and frequency distribution plot of the residuals of the EXPSTDAR (6) model.



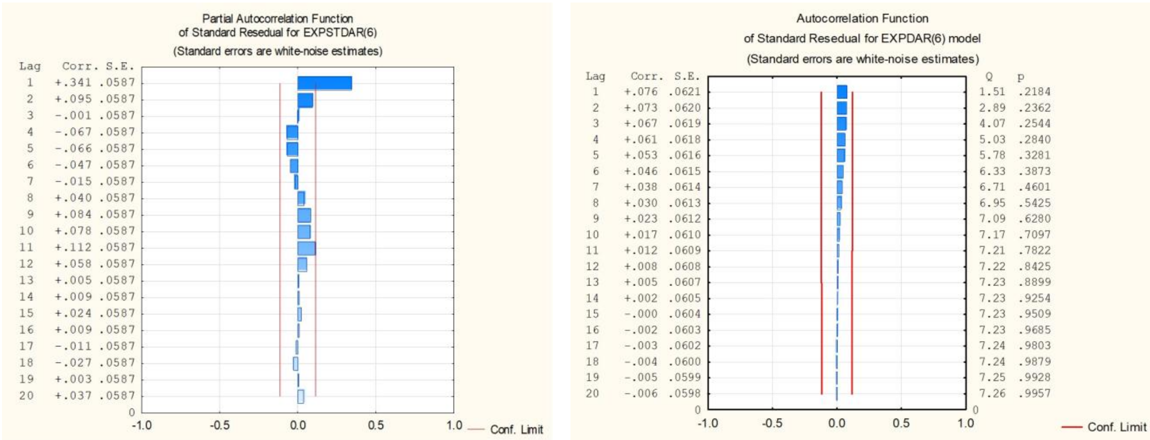
(a) (b)

Figure 12. displays two plots: (a) a normal probability plot of the residuals and (b) a plot showing the frequency distribution of the fitted residuals. EXPSTDAR(6) model.

and figure (13) shows the independency of the standard residuals of EXPSTDAR (6) model via the autocorrelation and partial autocorrelation functions.
The nonzero singular point of the model (20) is calculated using equation (12), resulting in a value of $\sigma_{\text{y}} = 0.0114$. .
The characteristic equation can be expressed by using proposition (1) in the form:

$$v^6 - 2.132v^5 + 0.309v^4 - 0.094v^3 + 0.1v^2 + 0.063v + 0.085 = 0 \quad (21)$$

The roots of the characteristic equation (22) are:
$v_1 = 1.9806$, $v_2 = 0.6809$, $v_3 = 0.0982 + 0.5352\text{i}$, $v_4 = 0.0982 - 0.5352\text{i}$, $v_5 = -0.3625 + 0.2882\text{i}$, and $v_6 = -0.3625 - 0.2882\text{i}$. From this, we may deduce that the time series of this model is non- stationary according to proposition (1) since the first root lies outside the unite circle $|v_1| > 1$,.
To verify this outcome, we plot the simulation of a time series of this model starting from a different initial values shown in Figure (14), which demonstrates that the time series of EXPSTDAR (6) model is non-stationary.



(a) (b)

Figure 13. The graph and correlations of (a) Partial autocorrelation function. (b)Autocorrelation function.

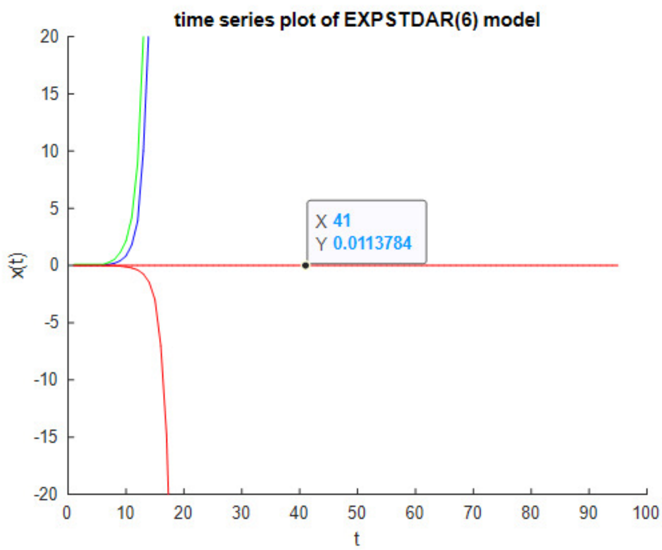


Figure 14. simulation plot of a time series generated by EXPSTDAR (6) model.

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