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Exploring Sigma Coloring within Lexicographic Products of Innovative Graph Families

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Abstract Sigma coloring is a vertex coloring strategy that distinguishes adjacent vertices by the sum of colors in their open neighborhood. The minimum number of colors needed in a sigma coloring of a graph G is called its sigma chromatic number $\sigma(G)$. The bounds of $\sigma(G)$ are sharp. For every graph G, $\sigma(G) \leq \chi(G)$. In this article, a clear investigation is made under sigma coloring on the lexicographic product of a path graph with various graph families including path graph, cycle graph, star graph, double star graph, and comb graph. Additionally, the sigma chromatic number for each case is found and verified with chromatic number for bounds.

Keywords Sigma coloring, sigma chromatic number, lexicographic product, path, cycle, star, double star, comb graph.

AMS 2010 subject classifications 05C15, 05C76

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1. Introduction

In graph theory, graph coloring plays essential role and becomes inevitable solution in scheduling, networking, resource allocation and assigning frequencies in wireless communication [10, 11]. Many types of coloring concepts was defined and discussed by researchers. Sigma coloring is one of the type of neighborhood$-$ distinguishing coloring and introduced by Gary chartrand and Zhang in 2008 [3]. Gary chartrand and Zhang presented his first paper in 2010 [4], obtaining sigma chromatic number for complete graphs, complete $r-$ partite graphs with $r \geq 2$ and cycles. He also stated that $\sigma(G) \leq \chi(G)$ where $\chi(G)$ is the minimum number of colors used in the proper vertex coloring G. Preethi K Pillai and J suresh kumar recently determined sigma coloring for Barbell graph, Twig graph, shell graph, tadpole graph, lollipop graph, fusing all the vertices of cycle and duplication of every edge by a vertex in C_n in 2023 [6]. They also obtained sigma chromatic number of some operations on graphs in 2021 [8]. C. Yogalakshmi and B.J. Balamurugan investigated sigma coloring under mycielski transformation and conclude that $\sigma(G) \leq \chi(G)$ for path, complete bipartite graph, complete graph, cycle,wheel graph and helm graph in 2024[1] . Wenhui Ma, Guanghua Dong, Yingyu Lu and Ning Wang consider lexicographic product of path with path for antimagicness in 2018[9] , which inspires to use this product in sigma coloring. In this paper, our primary goal is to find sigma chromatic number of lexicographic product of some new graph families.

Let G be simple connected graph and $\tau : V(G) \rightarrow N$, where N is set of positive integers, be a coloring of the vertices in G. We call $\tau(v)$ as the color of the vertex,v. For any $v \in V(G)$, let $\sigma(v)$ denotes the sum of colors of the

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vertices adjacent to v then τ is called a sigma coloring of G if for any two adjacent vertices $u, v \in V(G)$,$\sigma(u) \neq \sigma(v)$. Another definition and terminologies related to sigma coloring can be seen in [12, 13, 14, 15, 16]. The least number of colors needed in a sigma coloring of a graph G is called its sigma chromatic number $\sigma(G)$. The lexicographic product $G[H]$ of graphs[9] G and H is a graph such that the vertex set of G[H] is the cartesian product $V(G) \times V(H)$, and any two vertices (u, v) and (x, y) are adjacent in $G[H]$ if and only if either u is adjacent with x in G or $u = x$ and v is adjacent with y in H. This is equivalent to replacing each vertex of G by a copy of H, and linking these copies by complete bipartite graphs according to the edges of G. In general, the lexicographic product is non-commutative: $G[H] \neq H[G]$.

The path graph P_n, [2] is a graph with n vertices that can be enumerated such that two vertices are adjacent if and only if they are consecutive in the enumeration. The cycle graph C_n, [2] is a graph with n vertices that can be enumerated such that two vertices are adjacent if and only if they are consecutive in the enumeration or are the first and last vertex in the enumeration. A tree containing exactly one vertex that is not a pendant vertex is called a star graph $K_{1,n}$ [2]. The double star graph $K_{1,n,n}$ is obtained from a star graph $K_{1,n}$ by joining a new pendant edge to each existing pendant vertex. The comb graph [5] is represented by $P_n \odot K_1$. The P_n is a path graph with $(n-1)$ edges and n vertices. The graph is constructed by connecting each vertex in the path with a pendant edge.

1.1. Conceptual Foundation

This section presents foundational concepts and previously established results concerning sigma coloring.

The neighborhood of v, $N(v)$, is the set of all vertices adjacent to v. The closed neighborhood of v, $N[v]$, is the set $N(v) \cup v$. If two vertices u and v have $N[u] = N[v]$, they are called **strong twins**.[4]

Lemma 1.1.1 [4]

Let G be a nontrivial connected graph. Then $\sigma(G) = 1$, if and only if, every two adjacent vertices of G have different degrees.

Lemma 1.1.2 [4]

If u and v are two adjacent vertices in a graph G such that $N[u] = N[v]$, then $c(u) \neq c(v)$ for every sigma coloring c of G.

Lemma 1.1.3 [4]

If H is a complete subgraph of order k in a graph G such that $N[u] = N[v]$ for every two vertices u and v of H, then $\sigma(G) \geq k$

Lemma 1.1.4 [4]

Let G be a nontrivial connected graph of order n. Then $\sigma(G) = n$ if and only if $G = K_n$.

2. Main results

The following theorems presents results on the sigma chromatic number of lexicographic products for new classes of graphs.

Theorem 1.2.1

For all integers $(m, n \geq 2)$, the sigma chromatic number of lexicographic product of path graph with path graph $P_m[P_n]$ given by $\sigma(P_m[P_n]) = 4$

Proof:

Vertex Structure:

For clarity and consistency, throughtout this theorem, a $m-$ path P_m is represented by vertices of sequence $u_1, u_2, ..., u_m$ and a $n-$ path P_n is represented by vertices of sequence $v_1, v_2, ..., v_n$. In the graph $P_m[P_n]$, the vertex set is given by $V(P_m[P_n]) = \{(u_i, v_j); 1 \leq i \leq m, 1 \leq j \leq n\}$.

Let categorize the vertices as follows:

- **Horizontal external vertices:** All vertices of the form $(u_1, v_j)(1 \leq j \leq n)$
- **Vertical external vertices:** All vertices of the form $(u_i, v_1)(1 \leq i \leq m)$

- **Internal vertices:** All remaining vertices not included in the above two categories

The cardinality of $|V(P_m[P_n])|$ is $m \times n$ vertices

Coloring Construction:
Let the vertex color $\tau : V(P_m[P_n]) \longrightarrow \{1, 2, 3, 4\}$ as follows
For horizontal external vertices:-

$$\tau(u_1, v_j) = \begin{cases} 1, & \text{if } j \equiv 1 \pmod 4, \\ 2, & \text{if } j \not\equiv 1 \pmod 4, \end{cases} \qquad \text{for} 1 \le j \le n.$$

For vertical external vertices:-

$$\tau(u_i, v_1) = \begin{cases} 1, & \text{if } i \equiv 1 \pmod 4, \\ 3, & \text{if } i \not\equiv 1 \pmod 4, \end{cases} \qquad \text{for } 1 \le i \le m.$$

For internal vertices:-

$$\tau(u_i, v_j) = \begin{cases} 1, & \text{if } i \equiv 1 \pmod 4 \text{ and } j \equiv 1 \pmod 4, \\ 2, & \text{if } i \equiv 1 \pmod 4 \text{ and } j \not\equiv 1 \pmod 4, \end{cases} \qquad \text{for } 5 \le i \le m,\ 2 \le j \le n.$$

$$\tau(u_i, v_j) = \begin{cases} 4, & \text{if } j \text{ is even}, \\ 3, & \text{if } j \text{ is odd}, \end{cases} \qquad \text{for } 2 \le j \le m,\ i \not\equiv 1 \pmod 4,\ 2 \le i \le n.$$

By using modular concept in coloring construction of τ, it is evident that no two adjacent vertices have twin neighborhood sum. Here, take τ is a $\sigma-$ coloring with $\sigma(P_m[P_n]) \le 4$. Consider $\sigma(P_m[P_n]) = 1$, then the horizontal external vertices (u_1, v_j) and $(u_1, v_{j+1})(2 \le j \le n-2)$ are of uniformity in degree and are adjacent resulting in identical neighborhood sum which violates the condition of $\sigma-$ coloring. According to lemma 1.1.1, $\sigma(P_m[P_n]) \ne 1$. Next, assume $\sigma(P_m[P_n]) = 2$ or 3, then for instance, if the graph $P_m[P_n]$ is colored using less than 4 colors. Then, for any $(u, v) \,\epsilon\, V(P_m[P_n])$, then $\sigma(u) = \sigma(v)$ such that u and v are adjacent and may or may not equal in degree ends in identical neighborhood sum where the rule of $\sigma-$ coloring breaks. From the coloring of τ, it can be concluded that $\sigma(P_m[P_n]) = 4$

Theorem 1.2.2
For all integers $(m \ge 2, n \ge 3)$, the sigma chromatic number of lexicographic product of path graph with cycle graph $P_m[C_n]$ given by $\sigma(P_m[C_n]) = \begin{cases} 6 \text{ if n=3} \\ 5 \text{ if n is odd} \\ 4 \text{ if n is even} \end{cases}$

Proof:
Vertex Structure:
Let the vertices of $m-$ path P_m be represented as $u_1, u_2, ..., u_m$ and the cycle C_n of length n has the vertices $v_1, v_2, ..., v_n$. In the graph $P_m[C_n]$, the vertex set is given by $V(P_m[C_n]) = \{(u_i, v_j); 1 \le i \le m, 1 \le j \le n\}$.
Let categorize the vertices as follows:

- **Horizontal external vertices:** All vertices of the form $(u_1, v_j)(1 \le j \le n)$
- **Vertical external vertices:** All vertices of the form $(u_i, v_1)(1 \le i \le m)$
- **Internal vertices:** All remaining vertices not included in the above two categories

The cardinality of $|V(P_m[C_n])|$ is $m \times n$ vertices.
Case(i): If $n = 3$, then it have two sub cases for 'value of m'
Sub case(i): $m = 2$
Coloring Construction:
Let the vertex color $\tau : V(P_m[C_n]) \longrightarrow \{1, 2, 3, 4, 5, 6\}$ as follows

$$\tau(u_1, v_1) = 3, \qquad \tau(u_1, v_2) = 1, \qquad \tau(u_1, v_3) = 2, \qquad \tau(u_2, v_1) = 6, \qquad \tau(u_2, v_2) = 4, \qquad \tau(u_2, v_3) = 5.$$

By using modular concept in coloring construction of τ, it is true that no two adjacent vertices receives equal neighborhood sum. Assume τ is a $\sigma-$ coloring with $\sigma(P_m[C_n]) \leq 6$. Let us examine the possibility of using fewer colors, Consider $\sigma(P_m[C_n]) = 1$, then the horizontal external vertices (u_1, v_j) and $(u_1, v_j + 1)(1 \leq j \leq n - 1)$ are of uniformity in degree and are adjacent receives twin neighborhood sum which violates the condition of $\sigma-$ coloring. By lemma 1.1.1, it is evident that $\sigma(P_m[C_n]) \neq 1$. It is to be noted that all the vertices in $P_m[C_n]$ are uniformity in degree and adjacency structure implies that the graph is a complete graph. Then, for any $(u, v)\ \epsilon\ V(P_m[C_n])$, then $\sigma(u) = \sigma(v)$ such that u and v are adjacent and equal in degree results in identical neighborhood sum. Since, each vertex contributes one another during neighborhood sum. This results in that at least six colors are needed to color the vertices to satisfy the condition of $\sigma-$ coloring. Hence $\sigma(P_m[C_n]) = 6$.
Sub case(ii): For $m \geq 3$
Coloring Construction:
Let the vertex color $\tau : V(P_m[C_n]) \longrightarrow \{1, 2, 3, 4, 5\}$ as follows

$$\tau(u_i, v_1) = \begin{cases} 2, & \text{if } i \text{ is odd,} \\ 1, & \text{if } i \text{ is even,} \end{cases} \qquad \text{for } 1 \leq i \leq m.$$

$$\tau(u_i, v_2) = \begin{cases} 1, & \text{if } i \text{ is odd,} \\ 4, & \text{if } i \text{ is even,} \end{cases} \qquad \text{for } 1 \leq i \leq m.$$

$$\tau(u_i, v_3) = \begin{cases} 3, & \text{if } i \text{ is odd,} \\ 5, & \text{if } i \text{ is even,} \end{cases} \qquad \text{for } 1 \leq i \leq m.$$

From the above, under the vertex coloring of τ, it is clear that no two adjacent vertices receives same neighborhood sum. Take τ is a $\sigma-$ coloring with $\sigma(P_m[C_n]) \leq 5$. Let us analyze the possibility of using fewer colors, consider $\sigma(P_m[C_n]) = 1$, then the vertical external vertices (u_i, v_1) and $(u_{i+1}, v_1)(1 \leq i \leq m - 1)$ are of uniformity in degree and are adjacent receives identical neighborhood sum which contradicts the rule of $\sigma-$ coloring. By lemma 1.1.1, it is evident that $\sigma(P_m[C_n]) \neq 1$. In graph $P_m[C_n]$, if $2 \leq \sigma(P_m[C_n]) \leq 4$, then for instance, if the graph $P_m[C_n]$ is colored using less than 5 colors. Then, for any $(u, v)\ \epsilon\ V(P_m[C_n])$, then $\sigma(u) = \sigma(v)$ such that u and v are adjacent and may or may not equal in degree ends in identical neighborhood sum violates the rule of $\sigma-$ coloring. From the coloring of τ, it can be concluded that $\sigma(P_m[C_n]) = 5$
Case(ii): For n is odd, then it leads to two sub cases for 'value of m'.
Coloring Construction:
Define the vertex color $\tau : V(P_m[C_n]) \longrightarrow \{1, 2, 3, 4, 5\}$ as follows
Sub case(i): For $m = 2, 3, 4$.
For horizontal external vertices:-
$\tau(u_1, v_1) = 2$;

$$\tau(u_1, v_j) = \begin{cases} 1, & \text{if } j \text{ is even,} \\ 3, & \text{if } j \text{ is odd,} \end{cases} \qquad \text{for } 2 \leq j \leq n.$$

For vertical external vertices:-

$$\tau(u_i, v_1) = \begin{cases} 2, & \text{if } i \equiv 0 \pmod 4, \\ 1, & \text{if } i \not\equiv 0 \pmod 4, \end{cases} \qquad \text{for } 2 \le i \le m.$$

For internal vertices:-

$$\tau(u_i, v_j) = \begin{cases} 4, & \text{if } j \text{ is even}, \\ 5, & \text{if } j \text{ is odd}, \end{cases} \qquad \text{for } 2 \le i \le m,\ 2 \le j \le n.$$

Sub case(ii): For $m \ge 5$.

$$\tau(u_i, v_j) = \begin{cases} 3, & \text{if } i \equiv 1 \pmod 4 \text{ and } j = 1, 4, \\ 1, & \text{if } i \equiv 1 \pmod 4 \text{ and } j = 2, \\ 2, & \text{if } i \equiv 1 \pmod 4 \text{ and } j = 3, \end{cases} \qquad \text{for } 1 \le i \le m$$

$$\tau(u_i, v_j) = \begin{cases} 1, & \text{if } i \equiv 1 \pmod 4 \text{ and } j \text{ is odd}, \\ 2, & \text{if } i \equiv 1 \pmod 4 \text{ and } j \text{ is even}, \end{cases} \qquad \text{for } 1 \le i \le m, 5 \le j \le n.$$

$$\tau(u_i, v_j) = \begin{cases} 5, & \text{if } i \not\equiv 1 \pmod 4 \text{ and } j = 1, 4, \\ 3, & \text{if } i \not\equiv 1 \pmod 4 \text{ and } j = 2, \\ 4, & \text{if } i \not\equiv 1 \pmod 4 \text{ and } j = 3, \end{cases} \qquad \text{for } 1 \le i \le m.$$

$$\tau(u_i, v_j) = \begin{cases} 3, & \text{if } i \not\equiv 1 \pmod 4 \text{ and } j \text{ is even}, \\ 4, & \text{if } i \not\equiv 1 \pmod 4 \text{ and } j \text{ is odd}, \end{cases} \qquad \text{for } 1 \le i \le m,\ 5 \le j \le n.$$

From the above, under the coloring construction of τ, it is evident that no two adjacent vertices receives identical neighborhood sum. Consider τ is a $\sigma-$ coloring with $\sigma(P_m[C_n]) \le 5$. Assume $\sigma(P_m[C_n]) = 1$, then the vertical external vertices (u_i, v_1) and $(u_{i+1}, v_1)(1 \le i \le m-1)$ are of uniformity in degree and are adjacent receives same neighborhood sum. By lemma 1.1.1, it is evident that $\sigma(P_m[C_n]) \ne 1$. Next, let us take $2 \le \sigma(P_m[C_n]) \le 4$, then for instance, if the graph $P_m[C_n]$ is colored using less than 5 colors. Then, for any $(u, v) \,\epsilon\, V(P_m[C_n])$, then $\sigma(u) = \sigma(v)$ such that u and v are adjacent and may or may not equal in degree leads to identical neighborhood sum, which again contradicts the rule of $\sigma-$ coloring. By the coloring of τ given above, it can be concluded that $\sigma(P_m[C_n]) = 5$

Case(iii): For n is even.

Coloring Construction:

Define the vertex color $\tau : V(P_m[C_n]) \longrightarrow \{1, 2, 3, 4\}$ as follows

$$\tau(u_i, v_j) = \begin{cases} 1, & \text{if } j \text{ is odd}, \\ 2, & \text{if } j \text{ is even}, \end{cases} \qquad i \equiv 1 \pmod 4,\ \text{for } 1 \le i \le m,\ 1 \le j \le n.$$

$$\tau(u_i, v_j) = \begin{cases} 3, & \text{if } j \text{ is odd}, \\ 4, & \text{if } j \text{ is even}, \end{cases} \qquad i \not\equiv 1 \pmod 4,\ \text{for } 1 \le i \le m,\ 1 \le j \le n.$$

From the above vertex color construction, it is evident that no two adjacent vertices receives identical neighborhood sum. Take τ is a $\sigma-$ coloring with $\sigma(P_m[C_n]) \leq 4$. Let us scrutinize the potential for minimizing the number of colors, for it consider $\sigma(P_m[C_n]) = 1$, then the horizontal vertical vertices (u_1, v_j) and $(u_1, v_{j+1})(1 \leq j \leq n-1)$ are of uniformity in degree and are adjacent receives identical neighborhood sum . By lemma 1.1.1, it is true $\sigma(P_m[C_n]) \neq 1$. Let us take the assumption that $2 \leq \sigma(P_m[C_n]) \leq 3$, then for instance, if the graph $P_m[C_n]$ is colored using less than 4 colors. Then, for any $(u, v) \epsilon V(P_m[C_n])$, then $\sigma(u) = \sigma(v)$ such that u and v are adjacent and may or may not equal in degree ends in equal neighborhood sum which breaks the rule of $\sigma-$ coloring. Considering the coloring of τ from above, it can be concluded that $\sigma(P_m[C_n]) = 4$

Theorem 1.2.3
For all integers $(m, n \geq 2)$, the sigma chromatic number of lexicographic product of path graph with star graph $P_m[K_{1,n}]$ is given by $\sigma(P_m[K_{1,n}]) = 4$.

Proof:
Vertex Structure:
Let the vertices of $m-$ path represented as $u_1, u_2, ..., u_m$ and the star graph $K_{1,n}$ has v_1 as apex vertex and $v_2, v_3, ..., v_n$ are connected to it. Then the vertex set of $V(P_m[K_{1,n}]) = \{(u_i, v_j); 1 \leq i \leq m, 1 \leq j \leq n\}$.
Let categorize the vertices as follows:

- **Horizontal external vertices:** All vertices of the form $(u_1, v_j)(1 \leq j \leq n)$
- **Vertical external vertices:** All vertices of the form $(u_i, v_1)(1 \leq i \leq m)$
- **Internal vertices:** All remaining vertices not included in the above two categories

The cardinality of $|V(P_m[K_{1,n}])|$ is $m(n+1)$ vertices.
Coloring Construction:

$$\tau(u_i, v_j) = \begin{cases} 1, & \text{if } i \text{ is odd and } j = 1, \\ 2, & \text{if } i \text{ is odd and } 2 \leq j \leq n+1, \end{cases} \qquad \text{for } 1 \leq i \leq m.$$

$$\tau(u_i, v_j) = \begin{cases} 3, & \text{if } i \text{ is even and } j \text{ is odd}, \\ 4, & \text{if } i \text{ is even and } j \text{ is even}, \end{cases} \qquad \text{for } 1 \leq i \leq m, \ 1 \leq j \leq n+1.$$

From the above procedure, it is evident that no pairwise adjacent vertices receives same neighborhood sum. Take τ is a $\sigma-$ coloring with $\sigma(P_m[K_{1,n}]) \leq 4$. Suppose $\sigma(P_m[K_{1,n}]) = 1$, then the vertical external vertices (u_i, v_1) and $(u_{i+1}, v_1)(2 \leq i \leq m-2)$ being adjacent and are of same degree receives same neighborhood sum, which violates the condition of $\sigma-$ coloring. Therefore by lemma 1.1.1, $\sigma(P_m[K_{1,n}]) \neq 1$. In the structure of $P_m[K_{1,n}]$, when $2 \leq \sigma(P_m[K_{1,n}]) \leq 3$, then for instance, if the graph $P_m[K_{1,n}]$ is colored using less than 4 colors. Then, for any $(u, v) \epsilon V(P_m[K_{1,n}])$, then $\sigma(u) = \sigma(v)$ such that u and v are adjacent and may or may not equal in degree ends in identical neighborhood sum, where the rule breaks. By the coloring of τ assigned above, each vertex yields distinct neighborhood sum and it conclude that $\sigma(P_m[K_{1,n}]) = 4$.

Theorem 1.2.4

For all integers $(m, n \geq 2)$, the sigma chromatic number of lexicographic product of path graph with double star graph $P_m[K_{1,n,n}]$ is given by $\sigma(P_m[K_{1,n,n}]) = 4$.

Proof:
Vertex Structure:
For convenience, throughout this theorem, a $m-$ path denoted by $u_1, u_2, ..., u_m$. Then, let $K_{1,n,n}$ be the double star graph whose apex vertex is v and $v_1, v_2, ..., v_n$ are connected to it. Let $w_1, w_2, ..., w_n$ are the

vertices connected to $v_1, v_2, ..., v_n$ correspondingly in $K_{1,n,n}$. Let $P_m[K_{1,n,n}]$ has vertex set $V(P_m[K_{1,n,n}]) = \{(u_i, v)(u_i, v_j), (u_i, w_k); 1 \leq i \leq m, 1 \leq j \leq n, 1 \leq k \leq n\}$.
Let categorize the vertices as follows:

- **Horizontal external vertices:** All vertices of the form $(u_1, v), (u_1, v_j)(1 \leq j \leq n)$ and $(u_1, w_k)(1 \leq k \leq n)$.
- **Vertical external vertices:** All vertices of the form $(u_i, v)(1 \leq i \leq m)$
- **Internal vertices:** All remaining vertices not included in the above two categories

The cardinality of $|V(P_m[K_{1,n,n}])|$ is $m(2n+1)$ vertices.
Coloring Construction:
Let the vertex color is defined as $\tau : V(P_m[K_{1,n,n}]) \longrightarrow \{1, 2, 3, 4\}$ as follows
For horizontal external vertices:-
$\tau(u_1, v_j) = 2$ for $1 \leq j \leq n$;
$\tau(u_1, w_k) = 1$ for $1 \leq k \leq n$.
For vertical external vertices:-

$$\tau(u_i, v) = \begin{cases} 4, & \text{if } i \equiv 1 \pmod 4, \\ 3, & \text{if } i \not\equiv 1 \pmod 4, \end{cases} \qquad \text{for } 2 \leq i \leq m.$$

For internal vertices:-

$$\tau(u_i, v_j) = \begin{cases} 2, & \text{if } i \equiv 1 \pmod 4, \\ 4, & \text{if } i \not\equiv 1 \pmod 4, \end{cases} \qquad \text{for } 2 \leq i \leq m,\ 1 \leq j \leq n.$$

$$\tau(u_i, w_k) = \begin{cases} 1, & \text{if } i \equiv 1 \pmod 4, \\ 3, & \text{if } i \not\equiv 1 \pmod 4, \end{cases} \qquad \text{for } 2 \leq i \leq m,\ 1 \leq k \leq n.$$

From the above, constructed coloring of τ, it is clear that no pairwise adjacent vertices receives same neighborhood sum. Assume τ is a $\sigma-$ coloring with $\sigma(P_m[K_{1,n,n}]) \leq 4$. Let us inspect the potential for minimizing the number of colors, for it consider $\sigma(P_m[K_{1,n,n}]) = 1$, then the horizontal external vertices (u_i, v) and $(u_{i+1}, v)(2 \leq i \leq m-2)$ being adjacent and are of uniformity in degree receives identical neighborhood sum, which contradicts the rule of $\sigma-$ coloring. Therefore by lemma 1.1.1, $\sigma(P_m[K_{1,n,n}]) \neq 1$. Then taking the structure of $P_m[K_{1,n,n}]$ into account and $2 \leq \sigma(P_m[K_{1,n,n}]) \leq 3$, then for instance, if the graph $P_m[K_{1,n,n}]$ is colored using less than 4 colors. Then, for any $(u, v) \ \epsilon \ V(P_m[K_{1,n,n}])$, then $\sigma(u) = \sigma(v)$ such that u and v are adjacent and may or may not equal in degree results in neighborhood sum which again breaks the rule of $\sigma-$ coloring. Consider the coloring of τ assigned above successfully ensures that it can be concluded that $\sigma(P_m[K_{1,n,n}]) = 4$.

Theorem 1.2.5

For all integers $(m, n \geq 2)$, the sigma chromatic number of lexicographic product of path graph with comb graph $P_m[P_n \odot K_1]$ is given by $\sigma(P_m[P_n \odot K_1]) = 4$.

Proof:
Vertex Structure:
Let us consider P_m be the path of vertices $u_1, u_2, ..., u_m$. Then, let $P_n \odot K_1$ be the comb graph of vertices is $v_1, v_2, ..., v_n$, $w_1, w_2, ..., w_n$ where they are connected to each other correspondingly. Let $P_m[P_n \odot K_1]$ has vertex set $V(P_m[P_n \odot K_1]) = \{(u_i, v_j), (u_i, w_k); 1 \leq i \leq m, 1 \leq j \leq n, 1 \leq k \leq n\}$.
Let categorize the vertices as follows:

- **Horizontal external vertices:** All vertices of the form $(u_1, v_j)(1 \leq j \leq n)$ and $(u_1, w_k)(1 \leq k \leq n)$.
- **Vertical external vertices:** All vertices of the form $(u_i, v_1)(1 \leq i \leq m)$.

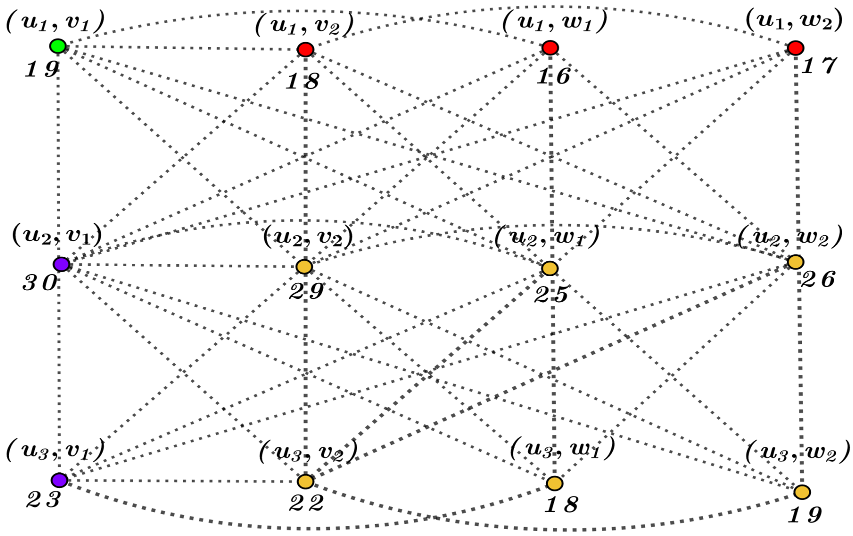

Figure 1. $P_3[P_2 \odot K_1]$

- **Internal vertices:** All remaining vertices not included in the above two categories

Coloring Construction:

Let the vertex color is defined as $\tau : V(P_m[P_n \odot K_1]) \longrightarrow \{1, 2, 3, 4\}$ as follows
For horizontal external vertices:-
$\tau(u_1, v_j) = 2$; for $1 \leq j \leq n$;
$\tau(u_1, w_k) = 2$; for $1 \leq k \leq n$.
For vertical external vertices:-

$$\tau(u_i, v_1) = \begin{cases} 1, & \text{if } i \equiv 1 \pmod 4, \\ 3, & \text{if } i \not\equiv 1 \pmod 4, \end{cases} \qquad \text{for } 2 \leq i \leq m.$$

For internal vertices:-

$$\tau(u_i, v_j) = \begin{cases} 4, & \text{if } i \not\equiv 1 \pmod 4 \text{ and } j \text{ is even}, \\ 3, & \text{if } i \not\equiv 1 \pmod 4 \text{ and } j \text{ is odd}, \end{cases} \qquad \text{for } 2 \leq i \leq m,\ 2 \leq j \leq n.$$

$\tau(u_i, w_k) = 4$ if $i \not\equiv 1 \pmod 4$; for $2 \leq i \leq m, 1 \leq k \leq n$
$\tau(u_i, v_j) = 2$ if $i \equiv 1 \pmod 4$; for $2 \leq i \leq m, 2 \leq j \leq n$
$\tau(u_i, w_k) = 2$ if $i \equiv 1 \pmod 4$; for $2 \leq i \leq m, 1 \leq k \leq n$

From the above, coloring construction of τ, it is obvious that no two adjacent vertices receives identical neighborhood sum. Assume τ is a $\sigma-$ coloring with $\sigma(P_m[P_n \odot K_1]) \leq 4$. Suppose let us take $\sigma(P_m[P_n \odot K_1]) = 1$, then the vertical external vertices (u_i, v_1) and $(u_{i+1}, v_1)(2 \leq i \leq m-2)$ being adjacent and are of uniformity in degree receives identical neighborhood sum, which contradicts the rule of $\sigma-$ coloring. So, by lemma 1.1.1, $\sigma(P_m[P_n \odot K_1]) \neq 1$. Then considering the structure of $P_m[P_n \odot K_1]$ into account and $2 \leq \sigma(P_m[P_n \odot K_1]) \leq 3$, then for instance, if the graph $P_m[P_n \odot K_1]$ is colored using less than 4 colors. Then, for any $(u, v)\ \epsilon\ V(P_m[P_n \odot K_1])$, then $\sigma(u) = \sigma(v)$ such that u and v are adjacent and may or may not equal in degree gets equal neighborhood sum, violates the rule of $\sigma-$ coloring. Taking the coloring of τ into consideration from above, it can be concluded that $\sigma(P_m[P_n \odot K_1]) = 4$. The sigma coloring of $P_3[P_2 \odot K_1]$ is illustrated in figure1, which clearly demonstrates how 4 colors are used to avoid neighborhood sum conflicts.

3. Conclusion

Finally, it was acclaimed that sigma coloring properties are satisfied by lexicographic product of path with path, path with cycle, path with star, path with double star and path with comb graphs and also sigma chromatic number of its kind is obtained. It can be concluded that $\sigma(G) \leq \chi(G)$ under lexicographic of graphs, where G includes graphs as mentioned above.

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