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A Pseudo-Gaussian Test for Comparing Periodic Coefficient Regression Models with Consecutive Periods

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Abstract This paper addresses the problem of selecting appropriate periods in periodic coefficient regression models where different regressors may exhibit distinct periodic structures. While existing approaches assume a common period across all variables, real-world applications often involve multiple periodicities. We propose a pseudo-Gaussian test for comparing a periodic regression model with variable-specific periods S_j against a model with periods $S_j + 1$, providing a formal framework for local refinement of period specifications. The test is developed within a small T, large n asymptotic framework using uniform local asymptotic normality (ULAN), and we derive the least squares estimator for model parameters under the null hypothesis. Extensive simulation studies demonstrate the test's validity and power across symmetric and asymmetric error distributions, as well as its superiority compared to the likelihood ration test. Comparisons with AIC and BIC reveal competitive performance in period selection. An application to real meteorological data illustrates how the proposed test can be used sequentially to identify optimal periods, with the results corroborated by RMSE-based model selection. The method offers a flexible and robust tool for model diagnostics in settings with complex periodic structures.

Keywords Optimal choice of the period, periodic multiple regression models, uniform local asymptotic normality, pseudo-Gaussian test, AIC/BIC comparison

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1. Introduction

1.1. Motivation and Background

Periodic regression models are fundamental tools for analyzing data with seasonal or cyclical patterns, with applications spanning meteorology, economics, and environmental science. Previous studies such as [18] and [19] have investigated periodic regression models under the restrictive assumption that all explanatory variables share a common period S. However, in many practical contexts—particularly in meteorology and economics—different regressors often exhibit distinct periodic behaviors, such as temperature exhibiting annual (12-month) and daily (24-hour) cycles while precipitation follows semi-annual patterns. To address this limitation, the present work extends the existing framework by allowing each regressor to possess its own period $(S_1, S_2, \ldots, S_p)$. In this paper, without loss of generality, we assume that the periods are ordered as $S_1 \leq S_2 \leq \ldots \leq S_p$. However, the subsequent derivations rely heavily on the role of the largest period S_p, particularly in the construction of the model and the asymptotic arguments. This dependence is not fully justified, as it is unclear whether the results remain invariant under permutations of the periods or whether the ordering imposes implicit structural restrictions on the model. A more rigorous justification of this ordering assumption, or a clarification of its impact on the theoretical results, would strengthen the validity of the derivations.

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1.2. *Explicit Model Specification*

Before presenting the theoretical framework, we provide a clear and intuitive statement of the model. Consider a panel dataset with n individuals (or spatial units) observed over T time periods. For individual i at time t, we observe a response variable $y_{i,t}$ and p explanatory variables $x_{i,t}^{(1)}, \ldots, x_{i,t}^{(p)}$. The key feature of a periodic regression model is that the regression coefficients are not constant but vary periodically with time according to a specified period S_j for each variable. The periodic structure is encoded in the indexing functions $l(i,m)$, which map the individual index i to the appropriate phase of the period m. Specifically:

$$l(i,m) = [(i-1) \bmod m] + 1, \quad i = 1, \ldots, n.$$

The $l(i, S_j)$ means that the coefficient for variable j repeats every S_j observations along the individual dimension. For instance, if $S_1 = 4$, then $\beta_1^{(1)}$ applies to individuals $i = 1, 5, 9, \ldots$; $\beta_2^{(1)}$ applies to individuals $i = 2, 6, 10, \ldots$; and so on. The intercept follows the period of the response, which we take to be $S_p = \max\{S_1, \ldots, S_p\}$, ensuring that all variables are properly aligned.

The full model is given by:

$$y_{i,t} = \mu_{l(i,S_p)} + \sum_{j=1}^{p} \beta_{l(i,S_j)}^{(j)} x_{i,t}^{(j)} + \epsilon_{i,t} \quad i = 1, \ldots, n, \ t = 1, \ldots, T, \tag{1}$$

where:

- $y_{i,t}$ is the response variable (e.g., temperature, GDP, stock return);
- $x_{i,t}^{(j)}$ is the j-th explanatory variable (e.g., humidity, interest rate, trading volume);
- μ_s is the intercept for phase s of the response's periodic cycle;
- $\beta_s^{(j)}$ is the coefficient for variable j at phase s of its periodic cycle;
- $\varepsilon_{i,t}$ are independent and identically distributed errors with mean zero, variance σ^2, and density $f(\varepsilon) = (1/\sigma) f_1(\varepsilon/\sigma)$.

To illustrate, consider a simple example with $p = 2$, $S_1 = 2$, and $S_2 = 3$. The model for $i = 1, 2, 3, 4, 5, 6$ becomes:

$$\begin{aligned}
i = 1: &\quad y_{1,t} = \mu_1 + \beta_1^{(1)} x_{1,t}^{(1)} + \beta_1^{(2)} x_{1,t}^{(2)} + \varepsilon_{1,t}, \\
i = 2: &\quad y_{2,t} = \mu_2 + \beta_2^{(1)} x_{2,t}^{(1)} + \beta_2^{(2)} x_{2,t}^{(2)} + \varepsilon_{2,t}, \\
i = 3: &\quad y_{3,t} = \mu_3 + \beta_1^{(1)} x_{3,t}^{(1)} + \beta_3^{(2)} x_{3,t}^{(2)} + \varepsilon_{3,t}, \\
i = 4: &\quad y_{4,t} = \mu_4 + \beta_2^{(1)} x_{4,t}^{(1)} + \beta_1^{(2)} x_{4,t}^{(2)} + \varepsilon_{4,t}, \\
i = 5: &\quad y_{5,t} = \mu_5 + \beta_1^{(1)} x_{5,t}^{(1)} + \beta_2^{(2)} x_{5,t}^{(2)} + \varepsilon_{5,t}, \\
i = 6: &\quad y_{6,t} = \mu_6 + \beta_2^{(1)} x_{6,t}^{(1)} + \beta_3^{(2)} x_{6,t}^{(2)} + \varepsilon_{6,t},
\end{aligned}$$

and that $l(1,2) = 1$, $l(2,2) = 2$, $l(3,2) = 1$, $l(4,2) = 2$, $l(5,2) = 1$, and $l(6,2) = 2$. This example demonstrates how the periodic structure creates a block pattern in the coefficients, with the intercept following the maximum period $S_p = 3$.

1.3. *Hypothesis Testing Framework*

A fundamental question in periodic modeling is whether the chosen periods adequately capture the underlying cyclical patterns. The proposed test addresses the specific question: Is a model with periods S_j sufficient, or does it require an additional component corresponding to periods $S_j + 1$? Formally, we test:

$$H_0: \ \mu_{S_p+1} = 0, \ \beta_{S_1+1}^{(1)} = \cdots = \beta_{S_p+1}^{(p)} = 0,$$

which means:

- For the intercept, the coefficient corresponding to phase $S_p + 1$ is zero. Since the intercept follows the maximum period S_p, the null hypothesis asserts that the model does not require an additional phase beyond S_p.
- For each explanatory variable j, the coefficient corresponding to phase $S_j + 1$ is zero. This means that the periodic pattern for variable j is fully captured by the first S_j phases, and no additional harmonic component is needed.

Against

$$H_1 : \mu_{S_p+1} \neq 0, \; \beta^{(j)}_{S_j+1} \neq 0 \quad (j = 1, \ldots, p),$$

which means that at least one of these additional coefficients is non-zero, indicating that the model with periods S_j is inadequate and that extending to periods $S_j + 1$ would improve the fit.

The parameters μ_{S_p+1} and $\beta^{(j)}_{S_j+1}$ are precisely these additional coefficients that appear when we expand the model from period S_j to period $S_j + 1$. They are not part of the original model; rather, they represent the **additional flexibility** gained by increasing the period. Thus, the "extra" parameters μ_{S_p+1} and $\beta^{(j)}_{S_j+1}$ are free parameters under H_1 that are constrained to zero under H_0. The test determines whether the data provide sufficient evidence to reject these zero constraints.

This formulation is particularly relevant for local refinement of period specifications. For instance, if a researcher suspects that the true period is either S_j or $S_j + 1$, the test provides a formal way to choose between these two nested models. When applied sequentially across candidate periods, the test can help identify the optimal period.

Under H_0 and when $S_1 = \cdots = S_p$, model (1) reduces to the framework of [19] in the univariate case ($p = 1$) with $\mu_1 = \ldots = \mu_{S_p} = \mu$, and to that of [18] in the multivariate case ($p > 1$). If $S_1 = \cdots = S_p = 1$, the model further simplifies to the classical multiple regression model.

1.4. Main Contributions

The main contribution of this paper is threefold. First, we develop a pseudo-Gaussian test that provides a formal statistical framework for comparing nested periodic models with consecutive periods, enabling local refinement of period specifications. Second, we derive the asymptotic properties of the test under general error distributions, establishing its validity beyond Gaussian assumptions. Third, we provide comprehensive simulation evidence and a real-data illustration demonstrating how the test can be integrated with model selection criteria to identify optimal periods in practice.

1.5. Related Literature

Periodic models have attracted considerable attention in the literature due to their ability to capture seasonal and cyclical dynamics in time series and regression frameworks. For instance, [3] studied the invertibility of periodic moving average models, [6] considered periodic autoregressive models, and [4] proposed both parametric and nonparametric tests for detecting periodicity in autoregressive coefficients. More recently, [19] developed locally and asymptotically optimal tests, including rank-based procedures, for testing the absence of periodicity in the regression coefficient β of simple linear regression models. Their empirical results highlighted the superiority of periodic coefficient regression over random coefficient regression models.

Further contributions include the work of [18], who introduced an adaptive estimator (AE) for parameters in periodic simple regression models. They showed that the AE outperforms the classical least squares estimator (LSE) when the error distribution is asymmetric. More recently, [**?**] implemented periodic regression methods in the `PerRegMod` R package (`https://CRAN.R-project.org/package=PerRegMod`), which provides both the LSE and AE, as well as a pseudo-Gaussian test for detecting periodic variation in regression coefficients.

Periodic multiple regression models allow the intercept, slopes, and variance to vary periodically across time. Specifically, the intercept μ_s and regression coefficients $\beta^{(j)}_s$ (for $j = 1, \ldots, p$ regressors) depend on the period s. This flexibility enables the model to capture complex seasonal and cyclical structures in the data. When $\mu_s = \mu$ and $\beta^{(j)}_s = \beta^{(j)}$ for all period s, the periodic model reduces to the classical multiple regression model with constant coefficients.

1.6. Asymptotic Framework

Our methodology is grounded in the local asymptotic normality (LAN) framework for small T, large n, and fixed standardized innovation density f. This asymptotic regime is particularly relevant for panel data settings where the number of individuals n grows (e.g., more households or firms are observed), while the number of time periods T remains fixed (e.g., daily observations over a week). The uniform LAN (ULAN) properties are central to constructing asymptotically optimal tests. The usefulness of the ULAN structure has been demonstrated in numerous works, including [11], [1], [10], [7], [8], [12], [15], [19], [5], [18], and [16].

1.7. Paper Organization

The remainder of the paper is organized as follows. Section 2 introduces the necessary notation and assumptions, and establishes the uniform local asymptotic normality (ULAN) property that underpins the test. Section 3 presents the optimal parametric test derived from the ULAN framework. Section 4 develops the pseudo-Gaussian test, which is optimal under Gaussian errors but remains valid under more general error distributions with finite variance. Section 5 reports extensive Monte Carlo simulation results, including size and power analysis, comparisons with AIC, BIC, and performance of the likelihood ratio test in terms of power. Section 6 illustrates the proposed methodology using real meteorological data, demonstrating the sequential testing procedure in a practical context.

2. Uniform Local Asymptotic Normality

2.1. Notation and Main Assumptions

Let

$$\mathbf{P}^{(n)}_{\boldsymbol{\vartheta}',\sigma^2,\mathbf{0};f_1}$$

denote the probability distribution of the observations

$$\left(y^{(n)}_{1,1},\ldots,y^{(n)}_{n,T}\right)$$

under the null hypothesis

$$\mu_{S_p+1}=0 \quad \text{and} \quad \beta^{(1)}_{S_1+1}=\cdots=\beta^{(p)}_{S_p+1}=0,$$

where the parameter vector is defined as

$$\boldsymbol{\vartheta}=\left(\boldsymbol{\mu}';\boldsymbol{\beta}'\right)'=\left(\mu_1,\ldots,\mu_{S_p};\beta^{(1)}_1,\ldots,\beta^{(1)}_{S_1},\ldots,\beta^{(p)}_1,\ldots,\beta^{(p)}_{S_p}\right)'.$$

Similarly, let

$$\mathbf{P}^{(n)}_{\boldsymbol{\vartheta}',\sigma^2,\boldsymbol{\lambda}';f_1}$$

denote the probability distribution under the alternative hypothesis, where

$$\boldsymbol{\lambda}=\left(\mu_{S_p+1},\beta^{(1)}_{S_1+1},\ldots,\beta^{(p)}_{S_p+1}\right)'\neq\mathbf{0}.$$

The ULAN property requires the following assumptions.

Assumption (A): Innovation density.

A.1) f_1 is continuously differentiable on $\mathcal{R}$, with derivative f_1'. Define

$$\phi_{f_1}(x)=-\frac{f_1'(x)}{f_1(x)}, \text{ and } \qquad \varphi_{f_1}(x)=x\phi_{f_1}(x)-1.$$

A.2) The following integrals are finite:

$$I_\phi(f_1) = \int_{\mathcal{R}} \phi^2_{f_1}(x) f_1(x)\,dx < \infty, \quad I_\varphi(f_1) = \int_{\mathcal{R}} \varphi^2_{f_1}(x) f_1(x)\,dx < \infty,$$

and

$$K_\phi(f_1) = \int_{\mathcal{R}} x\phi^2_{f_1}(x) f_1(x)\,dx < \infty.$$

Although expressed in terms of f_1, the above notation and Assumptions (A.1) and (A.2) also remain valid, with minor modifications to f, for standardized densities f_1. For Gaussian densities $f_1 = \mathcal{N} = \mathcal{N}(0,1)$, we have

$$\phi_{\mathcal{N}}(x) = x, \text{ and } \quad \varphi_{\mathcal{N}}(x) = x^2 - 1,$$

with corresponding integrals

$$I_\phi(\mathcal{N}) = 1, \quad I_\varphi(\mathcal{N}) = 2, \text{ and } \quad K_\phi(\mathcal{N}) = 0.$$

Assumption (B): Regressors.

B.1) For each $j = 1, \dots, p$, assume $S_j + 1$ divides n, and set $m_j = n/(S_j + 1)$.

B.2) The centering condition $\overline{x_s^{(j)}} = \frac{1}{m_j T} \sum_{r=0}^{m_j - 1} \sum_{t=1}^{T} x^{(j)}_{s+(S_j+1)r,t} = 0 \qquad j = 1, \dots, p, \quad s = 1, \dots, S_j + 1$ ensures that the regressors have zero mean within each period block. This condition can be satisfied by centering the regressors around their period-specific means, with a corresponding adjustment to the intercept terms.

B.3) The block-diagonal matrix

$$\mathbf{M}^{(n)} = \begin{pmatrix} \mathbf{M}^{(n)}_{11} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \mathbf{M}^{(n)}_{pp} \end{pmatrix}$$

converges to a positive-definite matrix $\mathbf{M}$, with normalized form

$$\mathbf{K}^{(n)} = (\mathbf{M}^{(n)})^{-1/2} \longrightarrow \mathbf{M}^{-1/2}.$$

The block matrices are defined as follows:

$$\mathbf{M}^{(n)}_{jj} = \begin{bmatrix} \overline{x_1^{(j)2}} & & \overline{x_k^{(j)} x_l^{(j)}} \\ & \ddots & \\ \overline{x_l^{(j)} x_k^{(j)}} & & \overline{x_{S_j}^{(j)2}} \end{bmatrix}, \quad \text{for } j = 1, \dots, p, \text{ and } k, l = 1, \dots, S_j.$$

Additionally, we define: $\overline{x_k^{(j)} x_l^{(j)}} = \frac{1}{m_j T} \sum\limits_{r=0}^{m_j-1} \sum\limits_{t=1}^{T} x^{(j)}_{k+(S_j+1)r,t} x^{(j)}_{l+(S_j+1)r,t}$ and

$\overline{x_s^{(j)2}} = \frac{1}{m_j T} \sum\limits_{r=0}^{m_j-1} \sum\limits_{t=1}^{T} x^{(j)2}_{s+(S_j+1)r,t}$ for $s = 1, ..., S_j + 1$.

B.4) The sequence of matrices $\mathbf{N}^{(n)}$ converges: $\lim\limits_{n \longrightarrow +\infty} \mathbf{N}^{(n)} = \mathbf{N}$ and $\lim\limits_{n \longrightarrow +\infty} \mathbf{D}^{(n)} = \mathbf{D} = \mathbf{N}^{\frac{-1}{2}}$, where

$$\mathbf{N}^{(n)} = \begin{bmatrix} \overline{x^{(1)2}_{S_1+1}} & & \overline{x^{(i)}_{S_i+1} x^{(j)}_{S_j+1}} \\ & \ddots & \\ \overline{x^{(j)}_{S_j+1} x^{(i)}_{S_i+1}} & & \overline{x^{(p)2}_{S_p+1}} \end{bmatrix}, \quad \text{for } i, j = 1, \dots, p, \ \mathbf{D}^{(n)} = \left(\mathbf{N}^{(n)}\right)^{\frac{-1}{2}},$$

and

$$\overline{x^{(i)}_{S_i+1}x^{(j)}_{S_j+1}} = \frac{1}{T}\sum_{t=1}^{T}\left(\sum_{r=0}^{m_i-1} x^{(i)}_{(S_i+1)(r+1),t} \times \sum_{k=0}^{m_j-1} x^{(j)}_{(S_j+1)(k+1),t}\right).$$

B.5) The classical condition from [14] holds:

$$\lim_{m_j\to\infty}\frac{\max_{0\le l\le m_j-1} x^{(j)2}_{k+(S_j+1)l,t'}}{\sum_{s=1}^{S_j+1}\sum_{r=0}^{m_j-1}\sum_{t=1}^{T} x^{(j)2}_{s+(S_j+1)r,t}} = 0, \quad j=1,\dots,p,\ k=1,\dots,S_j+1,, \text{and } t^{'}=1,\dots,T.$$

Assumption (B.2) is readily verified by replacing $x^{(j)}_{i,t}$ with $x^{(j)}_{i,t} - \overline{x^{(j)}_{l(i,S_j)}}$, while re-indexing i as $s+(S_j+1)r$, and by adjusting the term $\mu_{l(i,S_p)}$ to

$$\mu_{l(i,S_p)} + \sum_{j=1}^{p}\beta^{(j)}_{l(i,S_j)}\,\overline{x^{(j)}_{l(i,S_j)}}.$$

In Assumptions (B.3) and (B.4), $\mathbf{M}^{1/2}$ and $\mathbf{N}^{1/2}$ denote the symmetric square roots of the inverse of the symmetric positive definite matrices $\mathbf{M}$ and $\mathbf{N}$, respectively. In the sequel, most asymptotic results, including the validity of the test we are proposing, hold under (B.5), as $n\to\infty$.

2.2. Uniform Local Asymptotic Normality

In this subsection, we establish the **Uniform Local Asymptotic Normality (ULAN)** property, which is the key technical result underlying our derivation of optimal tests. The result is formulated with respect to the intercept $\boldsymbol{\mu}'$, the regression coefficients $\boldsymbol{\beta}'$, the scale parameter σ^2, and the parameter of interest $\boldsymbol{\lambda}'$. It is obtained for a fixed standardized density f_1, within the family of distributions

$$\mathcal{P}^{(n)}_{f_1} := \left\{\mathbf{P}^{(n)}_{\boldsymbol{\mu}',\boldsymbol{\beta}',\sigma^2,\boldsymbol{\lambda}';f_1} : (\boldsymbol{\mu}',\boldsymbol{\beta}') \in \mathcal{R}^{S_p+\sum_{j=1}^{p}S_j},\ \sigma^2>0,\ \boldsymbol{\lambda}\in\mathcal{R}^{p+1}\right\},$$

at any $\theta = \left(\boldsymbol{\vartheta}',\sigma^2,\mathbf{0}\right)'$.

To establish this result, we consider local perturbations of the form $\theta + n^{-1/2}\nu^{(n)}\boldsymbol{\tau}^{(n)}$, where

$$\nu^{(n)} = \begin{bmatrix} \boldsymbol{I}_{S_p} & \mathbf{0} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{K}^{(n)} & \mathbf{0} & \mathbf{0}\\ 0 & 0 & 1 & 0\\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{D}^{(n)} \end{bmatrix},$$

$$\boldsymbol{\tau}^{(n)} = \left(\boldsymbol{\tau}_1^{(n)'},\boldsymbol{\tau}_2^{(n)'},\tau_3^{(n)},\boldsymbol{\tau}_4^{(n)'}\right)',$$

with $\boldsymbol{\tau}^{(n)} \in \mathcal{R}^{S_p}\times\mathcal{R}^{\sum_{j=1}^{p}S_j}\times\mathcal{R}_+\times\mathcal{R}^{p+1}$ and $\sup_n\|\boldsymbol{\tau}^{(n)}\|^2<\infty$. The log-likelihood ratio for $\mathbf{P}^{(n)}_{\theta+n^{-1/2}\nu^{(n)}\boldsymbol{\tau}^{(n)};f_1}$ with respect to $\mathbf{P}^{(n)}_{\theta;f_1}$ can be written as

$$\begin{aligned}\Lambda^{(n)}_{\theta+n^{-1/2}\nu^{(n)}\boldsymbol{\tau}^{(n)}/\theta;f_1} = & \sum_{t=1}^{T}\sum_{i=1}^{n}\log\left(f_1\left[\left(\sigma^2+n^{-1/2}\tau_3^{(n)}\right)^{-1/2}\left(\varepsilon_{i,t}-n^{-1/2}\delta^{(n)}_{i,t}\right)\right]\right)\\ & -\sum_{t=1}^{T}\sum_{i=1}^{n}\log\left(f_1\left(\sigma^{-1}\varepsilon_{i,t}\right)\right),\end{aligned} \tag{2}$$

where

$$\delta^{(n)}_{i,t} = [\boldsymbol{\tau}_1^{(n)}]_{l(i,S_p)} + \sum_{j=1}^{p}\left[\mathbf{K}^{(n)}\boldsymbol{\tau}_2^{(n)}\right]_{\sum_{k=0}^{j-1}S_k+l(i,S_j)} x^{(j)}_{i,t}, \quad S_0=0,$$

and $[\boldsymbol{x}]_s$ denotes the sth component of the vector $\boldsymbol{x}$. We further define the standardized residuals

$$Z_{i,t} = \sigma^{-1}\Big(y_{i,t} - \mu_{l(i,S_p+1)} - \sum_{j=1}^{p} \beta^{(j)}_{l(i,S_j+1)} x^{(j)}_{i,t}\Big), \qquad i = 1,\dots,n,\ t = 1,\dots,T.$$

Under the null hypothesis, these residuals coincide with $\sigma^{-1}\varepsilon_{i,t}$. We are now ready to state the main result.

Proposition 1. *Suppose that assumptions (A) and (B) hold. Then the family $\mathcal{P}^{(n)}_{f_1}$ is ULAN at any $\theta = \Big(\boldsymbol{\vartheta}', \sigma^2, \mathbf{0}\Big)'$, with a central sequence of dimension $S_p + \sum_{i=1}^{p} S_i + p + 2$ given by*

$$\Delta^{(n)}_{f_1}(\theta) = \frac{1}{\sigma}\begin{bmatrix} \boldsymbol{\Delta}^{(n)}_{f_1,1}(\theta) \\ \boldsymbol{\Delta}^{(n)}_{f_1,2}(\theta) \\ \Delta^{(n)}_{f_1,3}(\theta) \\ \boldsymbol{\Delta}^{(n)}_{f_1,4}(\theta) \end{bmatrix} = \frac{1}{\sigma}\begin{bmatrix} n^{\frac{-1}{2}} \sum_{t=1}^{T} \sum_{r=0}^{m_p-1} \boldsymbol{\phi}_{f_1}(\boldsymbol{Z}_{r,t}) \\ n^{\frac{-1}{2}} \mathbf{K}^{(n)} \begin{pmatrix} \sum_{t=1}^{T} \sum_{r=0}^{m_1-1} \boldsymbol{\phi}^{x^{(1)}}_{f_1}(\boldsymbol{Z}_{r,t}) \\ \vdots \\ \sum_{t=1}^{T} \sum_{r=0}^{m_p-1} \boldsymbol{\phi}^{x^{(p)}}_{f_1}(\boldsymbol{Z}_{r,t}) \end{pmatrix} \\ \frac{n^{\frac{-1}{2}}}{2\sigma} \sum_{t=1}^{T} \sum_{s=1}^{S_p+1} \sum_{r=0}^{m_p-1} \varphi_{f_1}(Z_{s+(S_p+1)r,t}) \\ n^{\frac{-1}{2}} \sum_{t=1}^{T} \begin{pmatrix} \sum_{r=0}^{m_p-1} \phi_{f_1}(Z_{(S_p+1)(r+1),t}) \\ \mathbf{D}^{(n)} \begin{pmatrix} \sum_{r=0}^{m_1-1} \phi_{f_1}(Z_{(S_1+1)(r+1),t}) x^{(1)}_{(S_1+1)(r+1),t} \\ \vdots \\ \sum_{r=0}^{m_p-1} \phi_{f_1}(Z_{(S_p+1)(r+1),t}) x^{(p)}_{(S_p+1)(r+1),t} \end{pmatrix} \end{pmatrix} \end{bmatrix}, \tag{3}$$

with $\boldsymbol{\phi}_{f_1}(\boldsymbol{Z}_{r,t}) = \big(\phi_{f_1}(Z_{1+(S_p+1)r,t}), \dots, \phi_{f_1}(Z_{S_p+(S_p+1)r,t})\big)'$, and

$$\boldsymbol{\phi}^{x^{(j)}}_{f_1}(\boldsymbol{Z}_{r,t}) = \Big(\phi_{f_1}(Z_{1+(S_j+1)r,t}) x^{(j)}_{1+(S_j+1)r,t}, \dots, \phi_{f_1}(Z_{S_j+(S_j+1)r,t}) x^{(j)}_{S_j+(S_j+1)r,t}\Big)'.$$

The corresponding information matrix is

$$\Gamma_{f_1}(\theta) = \begin{bmatrix} \boldsymbol{\Gamma}_{f_1,11}(\theta) & \mathbf{0} & \boldsymbol{\Gamma}_{f_1,13}(\theta) & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\Gamma}_{f_1,22}(\theta) & \mathbf{0} & \boldsymbol{\Gamma}_{f_1,24}(\theta) \\ \boldsymbol{\Gamma}'_{f_1,13}(\theta) & \mathbf{0} & \Gamma_{f_1,33}(\theta) & \boldsymbol{\Gamma}_{f_1,34}(\theta) \\ \mathbf{0} & \boldsymbol{\Gamma}'_{f_1,24}(\theta) & \boldsymbol{\Gamma}'_{f_1,34}(\theta) & \boldsymbol{\Gamma}_{f_1,44}(\theta) \end{bmatrix}. \tag{4}$$

Explicit expressions of the blocks are

$$\boldsymbol{\Gamma}_{f_1,11}(\theta) = \frac{T \times I_\phi(f_1)}{\sigma^2\,(S_p+1)} \boldsymbol{I}_{S_p},\ \boldsymbol{\Gamma}_{f_1,22}(\theta) = \frac{T \times I_\phi(f_1)}{\sigma^2} \begin{pmatrix} \frac{1}{S_1+1}\boldsymbol{I}_{S_1} & 0 & \dots & 0 \\ 0 & \frac{1}{S_2+1}\boldsymbol{I}_{S_2} & & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & \dots & \dots & \frac{1}{S_p+1}\boldsymbol{I}_{S_p} \end{pmatrix},$$

$$\Gamma_{f_1,33}(\theta) = \frac{T \times I_{\varphi}(f_1)}{4\sigma^4},$$

$$\mathbf{\Gamma}_{f_1,44}(\theta) = \frac{T \times I_{\phi}(f_1))}{\sigma^2} \begin{pmatrix} \frac{1}{S_p+1} & 0 & \cdots & 0 \\ 0 & \frac{1}{S_1+1} & & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & \cdots & \frac{1}{S_p+1} \end{pmatrix},$$

$$\mathbf{\Gamma}_{f_1,13}(\theta) = \frac{T \times K_{\phi}(f_1)}{2\sigma^3\left(S_p+1\right)}\mathbf{1}_{S_p},\ \mathbf{\Gamma}_{f_1,24}(\theta) = \frac{T \times I_{\phi}(f_1)}{n\sigma^2}\mathbf{K}^{(n)}\mathbf{L}\begin{pmatrix} 1 & \mathbf{0} \\ \mathbf{0} & \mathbf{D}^{(n)} \end{pmatrix},$$

$$\mathbf{\Gamma}_{f_1,34}(\theta) = \frac{T \times K_{\phi}(f_1)}{2\sigma^3\left(S_p+1\right)}\left(1, \mathbf{0}_p\right),$$

where $\mathbf{1}_{S_p} = (1, ..., 1)^{'}$ *and*

$$\mathbf{L} = \begin{pmatrix} \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} & m_1 \begin{pmatrix} \overline{x_1^{(1)} x_1^{(2)}} \\ \vdots \\ \overline{x_{S_1}^{(1)} x_{S_1}^{(2)}} \end{pmatrix} & \cdots & m_1 \begin{pmatrix} \overline{x_1^{(1)} x_1^{(p)}} \\ \vdots \\ \overline{x_{S_1}^{(1)} x_{S_1}^{(p)}} \end{pmatrix} \\ \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} & m_2 \begin{pmatrix} \overline{x_1^{(2)} x_1^{(1)}} \\ \vdots \\ \overline{x_{S_2}^{(2)} x_{S_2}^{(1)}} \end{pmatrix} & \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} & \cdots & m_2 \begin{pmatrix} \overline{x_1^{(2)} x_1^{(p)}} \\ \vdots \\ \overline{x_{S_2}^{(2)} x_{S_2}^{(p)}} \end{pmatrix} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} & m_p \begin{pmatrix} \overline{x_1^{(p)} x_1^{(1)}} \\ \vdots \\ \overline{x_{S_p}^{(p)} x_{S_p}^{(1)}} \end{pmatrix} & \cdots & m_p \begin{pmatrix} \overline{x_1^{(p)} x_1^{(p-1)}} \\ \vdots \\ \overline{x_{S_p}^{(p)} x_{S_p}^{(p-1)}} \end{pmatrix} & \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \end{pmatrix},$$

with

$$\overline{x_s^{(i)} x_s^{(j)}} = \frac{1}{Tm_i} \sum_{t=1}^{T} \sum_{r=0}^{m_i-1} x_{s+(S_i+1)r,t}^{(i)} x_{s+(S_i+1)r,t}^{(j)} \textit{for } s = 1, \ldots, S_i.$$

More precisely, for any

$$\theta^{(n)} = \left(\boldsymbol{\vartheta}^{(n)'}, \sigma^{2(n)}, \mathbf{0}\right)^{'} = \left(\boldsymbol{\mu}^{(n)'}, \boldsymbol{\beta}^{(n)'}, \sigma^{2(n)}, \mathbf{0}\right)^{'}$$

such that $\sqrt{n}\left(\boldsymbol{\vartheta}^{(n)} - \boldsymbol{\vartheta}\right) = O_{\mathbf{P}}(1)$*, and for any bounded sequence* $\left(\boldsymbol{\tau}_1^{(n)'}, \boldsymbol{\tau}_2^{(n)'}, \tau_3^{(n)}, \boldsymbol{\tau}_4^{(n)'}\right)$ *ranging over* $\mathcal{R}^{S_p} \times \mathcal{R}^{\sum\limits_{j=1}^{p} S_j} \times \mathcal{R}_+ \times \mathcal{R}^{p+1}$*, we have, under* $\mathbf{P}_{\theta^{(n)};f_1}^{(n)}$*, as* $n \to \infty$ *with* T *fixed,*

$$\Lambda_{\theta^{(n)}+n^{\frac{-1}{2}}\nu^{(n)}\boldsymbol{\tau}^{(n)}/\theta^{(n)};f_1}^{(n)} = \boldsymbol{\tau}^{(n)'}\Delta_{f_1}^{(n)}(\theta^{(n)}) - \frac{1}{2}\boldsymbol{\tau}^{(n)'}\Gamma_{f_1}(\theta)\boldsymbol{\tau}^{(n)} + o_{\mathbf{P}}(1),$$

and

$$\Delta_{f_1}^{(n)}(\theta^{(n)}) \sim \mathcal{N}\left(\mathbf{0}, \Gamma_{f_1}(\theta)\right).$$

Proof
See appendix. □

The centering condition stated in Assumption (B.2) plays a crucial role in explaining the vanishing of many off-diagonal blocks. More precisely, under this condition, for each fixed phase, the double sum over r and t of $x^{(j)}_{s+S_j r,t}$ equals zero. This, in turn, implies that the corresponding cross-correlations are zero. The non-diagonal structure of the information matrix $\Gamma_{f_1}(\theta)$ in (4) reveals that $\boldsymbol{\Delta}^{(n)}_{f_1,2}(\theta)$ (associated with the regression parameter $\boldsymbol{\beta}$), $\Delta^{(n)}_{f_1,3}(\theta)$ (associated with the scale parameter σ^2), and $\boldsymbol{\Delta}^{(n)}_{f_1,4}(\theta)$ (associated with the intercept parameter $\boldsymbol{\lambda}$) are mutually dependent. This dependence motivates the construction of a new central sequence for $\boldsymbol{\lambda}$, denoted by $\boldsymbol{\Delta}^{*(n)}_{f_1,4}(\theta)$, together with its corresponding variance $\boldsymbol{\Gamma}^{(n)*}_{f_1,44}(\theta)$.

3. Optimal parametric test

Consider the null hypothesis

$$\mathcal{H}_0^{(n)} := \bigcup_{f_1} \bigcup_{\boldsymbol{\mu}\in\mathcal{R}^{S_p}} \bigcup_{\boldsymbol{\beta}\in\mathcal{R}^{\sum_{i=1}^{p} S_i}} \bigcup_{\sigma^2>0} \mathbf{P}^{(n)}_{\boldsymbol{\mu},\boldsymbol{\beta},\sigma^2,\mathbf{0};f_1},$$

which corresponds to a periodic multiple regression model with period S_j, characterized by $\mu_{S_p+1} = \beta^{(1)}_{S_1+1} = \cdots = \beta^{(p)}_{S_p+1} = 0$, i.e., $\boldsymbol{\lambda} = 0$.

The alternative hypothesis is

$$\mathcal{H}_1^{(n)} := \bigcup_{f_1} \bigcup_{\boldsymbol{\mu}\in\mathcal{R}^{S_p}} \bigcup_{\boldsymbol{\beta}\in\mathcal{R}^{\sum_{i=1}^{p} S_i}} \bigcup_{\sigma^2>0} \bigcup_{\boldsymbol{\lambda}\neq\mathbf{0}} \mathbf{P}^{(n)}_{\boldsymbol{\mu},\boldsymbol{\beta},\sigma^2,\boldsymbol{\lambda};f_1},$$

which represents a periodic multiple regression model with period $S_j + 1$.

Equivalently, the testing problem can be written as

$$\mathcal{H}_0^{(n)} \,:\, \boldsymbol{\tau}_4^{(n)} = \mathbf{0} \quad \text{against} \quad \mathcal{H}_1^{(n)} \,:\, \boldsymbol{\tau}_4^{(n)} \neq \mathbf{0},$$

where $\boldsymbol{\tau}_4^{(n)} \neq \mathbf{0}$ means that at least one component of $\boldsymbol{\tau}_4^{(n)}$ is different from zero.

More generally, this problem can be expressed as

$$\mathcal{H}_0^{(n)} \,:\, \left(\boldsymbol{\tau}_1^{(n)'}, \boldsymbol{\tau}_2^{(n)'}, \tau_3^{(n)}, \boldsymbol{\tau}_4^{(n)'}\right) \in \mathcal{M}(\Omega) \quad \text{against} \quad \mathcal{H}_1^{(n)} \,:\, \left(\boldsymbol{\tau}_1^{(n)'}, \boldsymbol{\tau}_2^{(n)'}, \tau_3^{(n)}, \boldsymbol{\tau}_4^{(n)'}\right) \notin \mathcal{M}(\Omega),$$

where $\mathcal{M}(\Omega)$ is a linear subspace of $\mathcal{R}^{S_p+\sum_{i=1}^{p} S_i+p+2}$ of dimension $p+1$, generated by

$$\Omega' = \left(\boldsymbol{I}_{S_p+\sum_{i=1}^{p} S_i+1} \;\middle|\; \mathbf{0}_{\left(S_p+\sum_{i=1}^{p} S_i+1\right)\times(p+1)}\right).$$

The ULAN structure and the convergence of local experiments to the Gaussian shift experiment imply that the locally optimal test is based on

$$\begin{aligned} T^{(n)}_{f_1}(\theta) &= \Delta^{(n)'}_{f_1}(\theta)\left[\Gamma^{-1}_{f_1}(\theta) - \Omega\left(\Omega'\Gamma_{f_1}(\theta)\Omega\right)^{-1}\Omega'\right]\Delta^{(n)}_{f_1}(\theta) \\ &= \boldsymbol{\Delta}^{(n)*'}_{f_1,4}(\theta)\,\boldsymbol{\Gamma}^{(n)*-1}_{f_1,44}(\theta)\,\boldsymbol{\Delta}^{(n)*}_{f_1,4}(\theta), \end{aligned} \tag{5}$$

where

$$\boldsymbol{\Delta}^{(n)*}_{f_1,4}(\theta) = \boldsymbol{\Delta}^{(n)}_{f_1,4}(\theta) - \left(\mathbf{0},\, \boldsymbol{\Gamma}^{(n)'}_{f_1,24}(\theta),\, \boldsymbol{\Gamma}^{(n)'}_{f_1,34}(\theta)\right) \begin{pmatrix} \boldsymbol{\Gamma}^{(n)}_{f_1,11}(\theta) & \mathbf{0} & \boldsymbol{\Gamma}^{(n)}_{f_1,13}(\theta) \\ \mathbf{0} & \boldsymbol{\Gamma}^{(n)}_{f_1,22}(\theta) & \mathbf{0} \\ \boldsymbol{\Gamma}^{(n)'}_{f_1,13}(\theta) & \mathbf{0} & \Gamma^{(n)}_{f_1,33}(\theta) \end{pmatrix}^{-1} \begin{pmatrix} \boldsymbol{\Delta}^{(n)}_{f_1,1}(\theta) \\ \boldsymbol{\Delta}^{(n)}_{f_1,2}(\theta) \\ \Delta^{(n)}_{f_1,3}(\theta) \end{pmatrix},$$

and

$$\mathbf{\Gamma}^{(n)*}_{f_1,44}(\theta) = \mathbf{\Gamma}^{(n)}_{f_1,44}(\theta) - \mathbf{\Gamma}^{(n)'}_{f_1,24}(\theta)\,\mathbf{\Gamma}^{(n)-1}_{f_1,22}(\theta)\,\mathbf{\Gamma}^{(n)}_{f_1,24}(\theta) - \mathbf{\Gamma}^{(n)'}_{f_1,34}(\theta)\,\mathbf{\Gamma}^{(n)-1}_{f_1,33}(\theta)\,\mathbf{\Gamma}^{(n)}_{f_1,34}(\theta).$$

The test statistic $T^{(n)}_{f_1}(\theta)$ depends on the unknown parameter θ, which needs to be estimated. We therefore assume that $\widehat{\theta}_n$ satisfies the following conditions.

Assumption (C).

(C.1) $\widehat{\theta}_n$ is $\sqrt{n}$-consistent, i.e., $\sqrt{n}\big(\widehat{\theta}_n - \theta\big) = O_{\mathbf{P}}(1)$.

(C.2) $\widehat{\theta}_n$ is locally asymptotically discrete, i.e., there exists $k \in \mathbb{N}$ such that, for all $n \in \mathbb{N}$, $\widehat{\theta}_n$ takes at most k distinct values in

$$Q_n = \Big\{\delta \in \mathcal{R}^{S_p+\sum_{i=1}^{p} S_i+1} : \sqrt{n}\,\|\delta - \theta\| \leq l\Big\}, \quad l > 0.$$

Assumption (C.1) is mild and, under the null hypothesis, is satisfied by many classical estimators such as least squares or maximum likelihood. Assumption (C.2) has little practical impact; see [9].

Proposition 2. *Assume that assumptions (A) and (B) hold. Then:*

(i) $T^{(n)}_{f_1}(\widehat{\theta}_n) = T^{(n)}_{f_1}(\theta) + o_{\mathbf{P}}(1)$ *is asymptotically chi-square with* $p+1$ *degrees of freedom under* $\mathbf{P}^{(n)}_{\theta;f_1}$. *Under* $\mathbf{P}^{(n)}_{\theta+n^{-1/2}\nu^{(n)}\boldsymbol{\tau}^{(n)};f_1}$, *it is asymptotically noncentral chi-square with* $p+1$ *degrees of freedom and noncentrality parameter*

$$\lambda_{f_1} = \boldsymbol{\tau}_4^{(n)'}\,\mathbf{\Gamma}^{(n)*}_{f_1,44}(\boldsymbol{\vartheta})\,\boldsymbol{\tau}_4^{(n)}.$$

(ii) The sequence of test has asymptotic power

$$1 - F\big(\chi^2_{p+1,1-\alpha};\lambda_{f_1}\big)$$

under $\mathbf{P}^{(n)}_{\theta+n^{-1/2}\nu^{(n)}\boldsymbol{\tau}^{(n)};f_1}$, *where* $F(\cdot;\lambda_{f_1})$ *denotes the cumulative distribution function of the noncentral chi-square distribution with* $p+1$ *degrees of freedom and noncentrality parameter* λ_{f_1}.

Proof
See Appendix. ☐

4. Pseudo-Gaussian test

4.1. Pseudo-Gaussian test

The Gaussian test $T^{(n)}_{\mathcal{N}}(\widehat{\vartheta}_n)$, obtained by taking $f_1 = \mathcal{N}(0,1)$ in (5), is valid only under Gaussian distributions. In this section, we construct suitably modified versions, denoted by $T^{\bullet(n)}_{\mathcal{N}}(\widehat{\vartheta}_n)$, which remain valid for any density g_1 with finite variance.

Moreover, the Gaussian central sequence $\Delta^{(n)}_{\mathcal{N}}(\theta)$ allows us to derive the pseudo-Gaussian test, which is robust and valid for all densities g_1 with finite variance. Let

$$m_i^{(n)} = \frac{1}{nT}\sum_{s=1}^{S_p+1}\sum_{t=1}^{T}\sum_{r=0}^{m_p-1} Z^i_{s+(S_p+1)r,t} \text{ for } i = 1,2,$$

be the $\sqrt{n}$-consistent estimator of

$$\mu_i(g_1) = \int_{\mathcal{R}} z^i g_1(z)dz, \text{ under } \mathbf{P}^{(n)}_{\theta,g_1}.$$

The pseudo-Gaussian test can be expressed as

$$T^{\bullet(n)}_{\mathcal{N},g_1}(\vartheta) = \mathbf{\Delta}^{(n)*'}_{\mathcal{N},g_1,4}(\vartheta)\,\mathbf{\Gamma}^{(n)*-1}_{\mathcal{N},g_1,44}(\vartheta)\,\mathbf{\Delta}^{(n)*}_{\mathcal{N},g_1,4}(\vartheta), \tag{6}$$

where

$$\mathbf{\Delta}^{(n)*}_{\mathcal{N},g_1,4}(\vartheta) = n^{\frac{-1}{2}} \sum_{t=1}^{T} \begin{pmatrix} \sum\limits_{r=0}^{m_p-1} \left[Z_{(S_p+1)(r+1),t} - \mu_1(g_1)\right] \\ \mathbf{D}^{(n)} \begin{pmatrix} \sum\limits_{r=0}^{m_1-1} Z_{(S_1+1)(r+1),t}\, x^{(1)}_{(S_1+1)(r+1),t} \\ \vdots \\ \sum\limits_{r=0}^{m_p-1} Z_{(S_p+1)(r+1),t}\, x^{(p)}_{(S_p+1)(r+1),t} \end{pmatrix} \end{pmatrix}$$

$$- \frac{1}{n} \begin{pmatrix} 1 & \mathbf{0} \\ \mathbf{0} & \mathbf{D}^{(n)} \end{pmatrix} \mathbf{L}^{'} \mathbf{K}^{(n)} \begin{pmatrix} (S_1+1)\,\boldsymbol{I}_{S_1} & 0 & \dots & 0 \\ 0 & (S_2+1)\,\boldsymbol{I}_{S_2} & & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & \dots & \dots & (S_p+1)\,\boldsymbol{I}_{S_p} \end{pmatrix} \mathbf{\Delta}^{(n)}_{\mathcal{N},2}(\vartheta)$$

and

$$\mathbf{\Gamma}^{(n)*}_{\mathcal{N},g_1,44}(\vartheta) = T \begin{pmatrix} \frac{\mu_2(g_1)-\mu_1^2(g_1)}{S_p+1} & 0 & \dots & 0 \\ 0 & \frac{1}{S_1+1} & & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & \dots & \dots & \frac{1}{S_p+1} \end{pmatrix}$$

$$- \frac{T}{n^2} \begin{pmatrix} 1 & \mathbf{0} \\ \mathbf{0} & \mathbf{D}^{(n)} \end{pmatrix} \mathbf{L}^{'} \mathbf{K}^{(n)} \begin{pmatrix} (S_1+1)\,\boldsymbol{I}_{S_1} & 0 & \dots & 0 \\ 0 & (S_2+1)\,\boldsymbol{I}_{S_2} & & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & \dots & \dots & (S_p+1)\,\boldsymbol{I}_{S_p} \end{pmatrix} \mathbf{K}^{(n)} \mathbf{L} \begin{pmatrix} 1 & \mathbf{0} \\ \mathbf{0} & \mathbf{D}^{(n)} \end{pmatrix}.$$

The pseudo-Gaussian test $T^{\bullet(n)}_{\mathcal{N},g_1}(\vartheta)$ depends on ϑ, $\mu_1(g_1)$, and $\mu_2(g_1)$, which are unknown. Therefore, we replace them with their estimations $\widehat{\vartheta}_n$, $m_1^{(n)}$, and $m_2^{(n)}$, respectively.

We now consider the periodic multiple regression model with period S_j. Its matrix representation is

$$Y = X\vartheta + \epsilon, \tag{7}$$

where $X = \left(X_1^{'}, ..., X_T^{'}\right)^{'}$, $X_t = \begin{bmatrix} & \mathbf{X}^{(1)}_{1,t} & & & \mathbf{X}^{(p)}_{1,t} \\ & \mathbf{X}^{(1)}_{2,t} & \dots & \dots & \mathbf{X}^{(p)}_{2,t} \\ \mathbf{1}_{m_p} \otimes \mathbf{I}_{S_p \times S_p} & \vdots & \dots & \dots & \vdots \\ & \mathbf{X}^{(1)}_{m_1,t} & & & \mathbf{X}^{(p)}_{m_p,t} \end{bmatrix}$ for $t = 1, ..., T$,

$\mathbf{X}^{(j)}_{k,t} = diag\left(x^{(j)}_{(k-1)S_j+1}, \dots, x^{(j)}_{kS_j}\right)$ for $j = 1, \dots, p$ and $k = 1, \dots, m_j$, $Y = (\mathbf{Y}_1^{'}, ..., \mathbf{Y}_T^{'})^{'}$, $\mathbf{Y}_t = (y_{1,t}, \dots, y_{n,t})^{'}$, $\epsilon = (\epsilon_1^{'}, ..., \epsilon_T^{'})^{'}$, $\epsilon_t = (\varepsilon_{1,t}, ..., \varepsilon_{n,t})^{'}$, $\mathbf{1}_{m_p} = (1, ..., 1)^{'}$, $\mathbf{I}_{S_p \times S_p}$ is the identity matrix, and $\vartheta = \left(\boldsymbol{\mu}^{'}, \boldsymbol{\beta}^{'}\right)^{'}$ with $\boldsymbol{\mu} = (\mu_1, ..., \mu_{S_p})^{'}$ and $\boldsymbol{\beta} = \left(\beta_1^{(1)}, ..., \beta_{S_1}^{(1)}; ...; \beta_1^{(p)}, ..., \beta_{S_p}^{(p)}\right)^{'}$. The least squares estimator

(LSE) is given by

$$\widehat{\vartheta}_n = \left(X^{'}X\right)^{-1} X^{'}Y. \tag{8}$$

Proposition 3. *Assume that assumptions (A) and (B) hold. Then, we have*
(i) $T_{\mathcal{N}}^{\bullet(n)}\left(\widehat{\vartheta}_n\right)$ *is asymptotically chi-square with* $p+1$ *degrees of freedom under* $\mathbf{P}^{(n)}_{\theta;f_1}$. *Under* $\mathbf{P}^{(n)}_{\theta+n^{\frac{-1}{2}}\nu^{(n)}\boldsymbol{\tau}^{(n)};f_1}$, *it is asymptotically noncentral chi-square with* $p+1$ *degrees of freedom and noncentrality parameter*

$$\lambda^{\bullet}_{\mathcal{N}} = \boldsymbol{\tau}_4^{(n)'}\boldsymbol{\Gamma}^{(n)*}_{\mathcal{N},44}(\boldsymbol{\vartheta})\boldsymbol{\tau}_4^{(n)};$$

(ii) The sequence of the statistic test has asymptotic power $1-F(\chi^2_{p+1,1-\alpha};\lambda^{\bullet}_{\mathcal{N}})$ *under* $\mathbf{P}^{(n)}_{\theta+n^{\frac{-1}{2}}\nu^{(n)}\boldsymbol{\tau}^{(n)};f_1}$, *where* $F(.;\lambda^{\bullet}_{\mathcal{N}})$ *denotes the noncentral chi-square distribution function with* $p+1$ *degrees of freedom and noncentrality parameter* $\lambda^{\bullet}_{\mathcal{N}}$.

4.2. Sequential Procedure for Period Selection

Although the proposed test is designed to compare two consecutive periods, it can be naturally extended to identify an appropriate period within a given range. Suppose that, for a given variable j, the period is suspected to lie in the interval $[S_{\min}, S_{\max}]$. A simple sequential procedure can then be applied. Starting from $S_{\min}$, the test is performed between S_{current} and $S_{\text{current}}+1$. If the null hypothesis is rejected, the current period is deemed inadequate and increased by one. This process is repeated until the null hypothesis is no longer rejected, in which case the current period is retained as an acceptable choice. This procedure requires at most $(S_{\max}-S_{\min})$ tests, making it computationally efficient.

4.3. Computational Considerations

When several variables are involved, a naive exhaustive search over all possible combinations of periods quickly becomes computationally infeasible, as the number of candidate models grows exponentially. To address this issue, we adopt a simple and practical strategy: the sequential procedure described above is applied independently to each variable. This yields a candidate period for each component without requiring a combinatorial exploration of all possible configurations. The total computational cost is therefore linear in both the number of variables p and the size of the period range, i.e., of order $p\times(S_{\max}-S_{\min})$. This makes the approach feasible even in moderately high-dimensional settings.

5. Simulation

This section presents a Monte Carlo simulation study based on 2500 replications of samples of size $N = T\times n$, generated from model (1), to assess both the validity and the power of the proposed pseudo-Gaussian test $T_{\mathcal{N}}^{\bullet(n)}$, as well as the efficiency of the proposed LSE under the null hypothesis, i.e., the estimation of parameters in periodic coefficient regression models with period S_j, $j=1,\dots,p$. For this purpose, We consider model (1) where

i) $T = 5, 10,\ 20$, $p=2$, $S_1=2$, $S_2=4$;
ii) $\mu=(2,4,3.7,8,\lambda_1)$, $\beta^{(1)}=(3,7,\lambda_2)$, and $\beta^{(2)}=(4,5.2,6,2,\lambda_3)$ with $\lambda_i\in\mathcal{R}$ for $i=1,2,3$;
iii) the $x^{(1)}$'s are i.i.d. uniform $(0,10)$;
iv) the $x^{(2)}$'s are i.i.d. uniform $(0,20)$;
v) the $\varepsilon_{i,t}$'s are i.i.d. with density Gaussian $\mathcal{N}$, double exponential $\mathcal{D}e$, logistic $\mathcal{L}$, student t_v with degrees of freedom $v=3,5$, skew normal $s\mathcal{N}(5)$, $s\mathcal{N}(10)$, skew student $st_3(5)$, $st_5(7)$ see [2], skew double exponential $s\mathcal{D}e(5)$, $s\mathcal{D}e(10)$, skew logistic $s\mathcal{L}(5)$, $s\mathcal{L}(7)$, or mixture of two normal distributions k_ω where

$$k_\omega(x)=\frac{1}{\sqrt{2\pi}}\left(\frac{\omega}{\sqrt{1.1}}e^{-\frac{x^2}{2\times1.1}}+\frac{1-\omega}{\sqrt{0.95}}e^{-\frac{x^2}{2\times0.95}}\right),\ 0<\omega<1.$$

Under the alternative hypothesis H_1, the periods are given by $(S_1+1, S_2+1) = (3, 5)$. The parameters $\lambda_1, \lambda_2, \lambda_3$ are introduced to control the additional components of the model. When $\lambda_1 = \lambda_2 = \lambda_3 = 0$, the alternative model reduces to the null model. Note that random samples from the considered distributions were generated in **R** using the following packages: the *sn* package `https://CRAN.R-project.org/package=sn` for skew normal and skew student densities; the *rmutil* package `https://CRAN.R-project.org/package=rmutil` for the skew Laplace density; the *glogis* package `https://CRAN.R-project.org/package=glogis` for the skew logistic density; and the *EnvStats* package `https://CRAN.R-project.org/package=EnvStats` for mixtures of two normal distributions. All simulations and real-data analyses were carried out using the **R** statistical software.

Under the null hypothesis, $\lambda_1 = \lambda_2 = \lambda_3 = 0$, the parameters of model (1) are estimated using the proposed estimator. Tables 1, 2, and 4 report the rejection frequencies for the null hypothesis $\lambda_1 = \lambda_2 = \lambda_3 = 0$, as well as for increasing alternatives $\lambda_1 = \lambda_2 = \lambda_3 = 0.1$, 0.3, 0.5, 0.7, and 0.9 using the pseudo-Gaussian test $T_{\mathcal{N}}^{\bullet(n)}$. Results are presented for different sample sizes n ($n = 30$, $n = 45$), time dimension T ($T = 5$, $T = 10$, and $T = 20$), and for both symmetric and asymmetric error distributions, including: Gaussian $\mathcal{N}$, double exponential $\mathcal{D}e$, logistic $\mathcal{L}$, student t_v with degrees of freedom $v = 3, 5$, skew normal $s\mathcal{N}(5)$, $s\mathcal{N}(10)$, skew student $st_3(5)$, $st_5(7)$, skew double exponential $s\mathcal{D}e(5)$, $s\mathcal{D}e(10)$, skew logistic $s\mathcal{L}(5)$, $s\mathcal{L}(7)$, and mixtures of two normal distributions k_ω with $\omega = 0.1$, 0.2, 0.3, 0.5, 0.7, 0.8, and 0.9.

In addition, table 3 displays the rejection frequencies for various non-zero parameter values $(\lambda_1, \lambda_2, \lambda_3)$, namely $(0.1, 0.2, 0.15)$, $(0.2, 0.15, 0.3)$, $(0.4, 0.3, 0.2)$, and $(0.5, 0.4, 0.3)$, for the pseudo-Gaussian test with sizes $n = 30$ and $n = 45$, again across symmetric and asymmetric error distributions.

Table 5 presents the rejection frequencies for $\lambda_1 = \lambda_2 = \lambda_3 = 0$ (null hypothesis) and for increasing alternatives: 0.5, 0.7, 0.9, and 1.1, for the pseudo-Gaussian test $T_{\mathcal{N}}^{\bullet(n)}$ and likelihood ratio (LR) test.

The simulation results indicate that the proposed pseudo-Gaussian test remains valid under a wide range of distributions, including symmetric ones (normal, double exponential, logistic, Student) as well as their asymmetric counterparts (skew-normal, skew-double-exponential, skew-logistic, skew-Student, and mixtures of normals with varying ω). Furthermore, the results highlight that the test achieves the highest power under Gaussian errors. Notably, the power of the pseudo-Gaussian test increases with larger sample sizes, reflecting improved sensitivity as n grows. Moreover, the pseudo-Gaussian test consistently outperforms the likelihood ratio test in terms of power under a range of error densities.

Tables 6, 7, and 8 present, based on 2500 replications of samples of size N, the parameter estimates of model (1) under the null hypothesis, for both symmetric and asymmetric error distributions: Gaussian $\mathcal{N}$, double exponential $\mathcal{D}e$, skew-normal $s\mathcal{N}(10)$, and skew-Student $st_5(10)$. The specifications are as follows: $p = 2$, $x^{(1)} \sim \mathcal{U}(0, 3)$, $x^{(2)} \sim \mathcal{U}(0, 5)$, $S_1 = 2$, $S_2 = 4$, $n = 24$, $T = 10$ for Table 6; $p = 3$, $x^{(1)} \sim \mathcal{U}(0, 3)$, $x^{(2)} \sim \mathcal{U}(0, 5)$, $x^{(3)} \sim \mathcal{U}(0, 7)$, $S_1 = 2$, $S_2 = 3$, $S_3 = 5$, $n = 30$, $T = 4$ for Table 7; and $p = 4$, $x^{(1)} \sim \mathcal{U}(0, 3)$, $x^{(2)} \sim \mathcal{U}(0, 5)$, $x^{(3)} \sim \mathcal{U}(0, 7)$, $x^{(4)} \sim \mathcal{U}(0, 9)$, $S_1 = 2$, $S_2 = 3$, $S_3 = 5$, $S_4 = 7$, $n = 42$, $T = 5$ for Table 8.

The results show that the Gaussian error distribution consistently yields the smallest RMSE compared to the double exponential, skew-normal, and skew-Student distributions. Therefore, the LSE is most efficient when the errors follow a Gaussian distribution.

Tables 9 and 10 report the empirical performance of the proposed pseudo-Gaussian test, which compares a periodic regression model with variable-specific periods $(S_1, \ldots, S_p)$ against a model with periods $(S_1 + 1, \ldots, S_p + 1)$. Alongside the p-values of the test statistic $T_{\mathcal{N}}^{\bullet(n)}$, we present the root mean squared error (RMSE) of the corresponding parameter estimates. In Table 9, where the true model has $S_1 = 3$, and $S_2 = 5$ with $p = 2$, the test consistently yields non-significant p-values under the correct null hypothesis ($S_1 = 3, S_2 = 5$), with relatively small RMSE values close to 1. In contrast, when incorrect periods are imposed (e.g., $S_1 = 2, S_2 = 4$), the test strongly rejects the null as the sample size increases (p-values approaching zero), while the RMSE becomes substantially larger (around 6). This confirms both the *validity under the null* and the *power against alternatives* of the pseudo-Gaussian test.

Table 10 extends the analysis to the case $p = 3$, with true periods $S_1 = 3$, $S_2 = 4$, and $S_3 = 5$. A similar pattern emerges: the correct specification leads to high p-values and low RMSE, while alternative specifications (e.g.,

$S_1 = 2, S_2 = 3, S_3 = 4$) are systematically rejected with extremely small p-values (down to 10^{-15}), accompanied by large RMSE values exceeding 12. This further highlights the robustness of the proposed procedure in identifying the correct periodic structure.

Overall, the results in Tables 9 and 10 demonstrate that the pseudo-Gaussian test is well-suited under the null hypothesis and has strong discriminatory power with respect to poorly specified periodic models, with the root mean square error (RMSE) providing further evidence of the accuracy of the parameter estimation.

Table 11 reports the rejection rates of the pseudo-Gaussian test along with the corresponding values of the AIC and BIC criteria for different candidate periods (S_1, S_2), while the true period is $(4, 6)$. The results show that the highest rejection rates are obtained for the candidate periods $(3, 5)$ and $(2, 4)$, with values of 82.4% and 76.2%, respectively. This result highlights the strong sensitivity of the proposed test to model misspecification when the candidate periods are close to the true period, thereby leading to a high rejection frequency of the null hypothesis. In contrast, for candidate periods further from the true one, such as $(6, 8)$, $(6, 12)$, and $(5, 11)$, the rejection rates are substantially lower, taking values of 17.7%, 34.3%, and 37.2%, respectively. This suggests a decrease in the power of the test when the deviation from the true periodic structure increases. Regarding the information criteria, the smallest values of AIC and BIC are obtained for the period $(6, 8)$, indicating that this model provides the best fit among the candidates according to these criteria, despite not corresponding to the true period. Overall, these results highlight a discrepancy between the testing procedure and the model selection criteria: while the pseudo-Gaussian test effectively detects certain types of misspecification, the AIC and BIC criteria may favor models that do not reflect the true underlying periodic structure. This underlines the importance of combining testing procedures with model selection criteria in periodic regression analysis.

Table 1. Rejection frequencies at asymptotic level $\alpha = 5\%$ for various values of $\lambda_1 = \lambda_2 = \lambda_3$ with error density g_1 of the pseudo-Gaussian test $T_{\mathcal{N}}^{\bullet(n)}$.

	$\lambda_1 = \lambda_2 = \lambda_3$ with $n = 30$ and $T = 5$						$\lambda_1 = \lambda_2 = \lambda_3$ with $n = 45$ and $T = 5$					
g_1	0	0.1	0.3	0.5	0.7	0.9	0	0.1	0.3	0.5	0.7	0.9
$\mathcal{N}$	0.0648	0.1104	0.2924	0.5211	0.6728	0.784	0.068	0.1956	0.4552	0.6728	0.828	0.9344
$\mathcal{D}e$	0.0658	0.0996	0.2868	0.4928	0.654	0.7284	0.064	0.1844	0.4516	0.6722	0.8268	0.9064
$\mathcal{L}$	0.0632	0.1064	0.2904	0.5032	0.6528	0.7256	0.064	0.1952	0.4388	0.6676	0.818	0.9292
t_3	0.0624	0.1052	0.2906	0.4788	0.6352	0.738	0.066	0.1848	0.4496	0.6722	0.8272	0.9136
t_5	0.068	0.1096	0.2916	0.5076	0.6492	0.7356	0.069	0.1948	0.444	0.6724	0.8128	0.9092
$s\mathcal{N}(5)$	0.0674	0.102	0.2901	0.466	0.6308	0.7264	0.066	0.1952	0.4504	0.6712	0.8172	0.9172
$s\mathcal{N}(10)$	0.0642	0.11	0.276	0.5012	0.6202	0.7296	0.068	0.182	0.4504	0.6714	0.8176	0.9112
$st_3(5)$	0.0682	0.1056	0.2712	0.508	0.64	0.7284	0.0606	0.192	0.4344	0.6696	0.822	0.9148
$st_5(7)$	0.0676	0.1064	0.2884	0.5084	0.6336	0.7222	0.0618	0.1916	0.4396	0.67	0.8156	0.9104
$s\mathcal{D}e(5)$	0.0536	0.0916	0.2692	0.484	0.6432	0.7216	0.068	0.1816	0.4376	0.6656	0.8208	0.9212
$s\mathcal{D}e(10)$	0.0488	0.1028	0.272	0.498	0.6352	0.736	0.069	0.1704	0.4308	0.6648	0.8212	0.912
$s\mathcal{L}(5)$	0.0681	0.1076	0.2914	0.49	0.6384	0.726	0.068	0.1908	0.4472	0.6644	0.8176	0.9056
$s\mathcal{L}(7)$	0.069	0.102	0.2792	0.4988	0.6444	0.726	0.068	0.188	0.4504	0.674	0.816	0.9188

Table 2. Rejection frequencies at asymptotic level $\alpha = 5\%$ for various values of $\lambda_1 = \lambda_2 = \lambda_3$ under the mixture of two normal distributions k_ω, of the pseudo-Gaussian test $T_{\mathcal{N}}^{\bullet(n)}$.

	$\lambda_1 = \lambda_2 = \lambda_3$ with $n = 30$ and $T = 5$						$\lambda_1 = \lambda_2 = \lambda_3$ with $n = 45$ and $T = 5$					
ω	0	0.1	0.3	0.5	0.7	0.9	0	0.1	0.3	0.5	0.7	0.9
0.1	0.064	0.106	0.2816	0.508	0.6408	0.728	0.064	0.1952	0.4504	0.6648	0.8176	0.9112
0.2	0.065	0.1102	0.2908	0.5112	0.65	0.7504	0.062	0.1948	0.4404	0.6702	0.8104	0.914
0.3	0.0652	0.1121	0.2922	0.5092	0.6352	0.7304	0.064	0.1952	0.4506	0.6702	0.8112	0.9108
0.5	0.068	0.1076	0.2876	0.4952	0.6348	0.7614	0.064	0.1946	0.4476	0.6788	0.8188	0.9208
0.7	0.066	0.1088	0.2824	0.5108	0.6416	0.7424	0.068	0.1906	0.442	0.6724	0.8108	0.9136
0.8	0.058	0.1156	0.2808	0.4988	0.652	0.7468	0.059	0.1908	0.4552	0.6718	0.8156	0.9212
0.9	0.068	0.1132	0.2912	0.5004	0.634	0.7204	0.058	0.1912	0.4528	0.6718	0.828	0.9214

Table 3. Rejection frequencies at asymptotic level $\alpha = 5\%$ for various values of $(\lambda_1; \lambda_2; \lambda_3)$ with error density g_1 of the pseudo-Gaussian test $T_{\mathcal{N}}^{\bullet(n)}$.

	$\begin{pmatrix}\lambda_1\\ \lambda_2\\ \lambda_3\end{pmatrix}$ with $n = 30$ and $T = 5$				$\begin{pmatrix}\lambda_1\\ \lambda_2\\ \lambda_3\end{pmatrix}$ with $n = 45$ and $T = 5$			
g_1	$\begin{pmatrix}0.1\\ 0.2\\ 0.15\end{pmatrix}$	$\begin{pmatrix}0.2\\ 0.15\\ 0.3\end{pmatrix}$	$\begin{pmatrix}0.4\\ 0.3\\ 0.2\end{pmatrix}$	$\begin{pmatrix}0.5\\ 0.4\\ 0.3\end{pmatrix}$	$\begin{pmatrix}0.1\\ 0.2\\ 0.15\end{pmatrix}$	$\begin{pmatrix}0.2\\ 0.15\\ 0.3\end{pmatrix}$	$\begin{pmatrix}0.4\\ 0.3\\ 0.2\end{pmatrix}$	$\begin{pmatrix}0.5\\ 0.4\\ 0.3\end{pmatrix}$
$\mathcal{N}$	0.1496	0.2524	0.2156	0.3216	0.28	0.4172	0.386	0.5044
$\mathcal{D}e$	0.1488	0.264	0.2172	0.316	0.27	0.3968	0.378	0.482
$\mathcal{L}$	0.1492	0.264	0.2108	0.316	0.28	0.4056	0.3744	0.5012
t_5	0.1436	0.2634	0.2104	0.3192	0.2744	0.3992	0.3788	0.4884
$s\mathcal{N}(10)$	0.1422	0.2576	0.2012	0.3076	0.27	0.4044	0.376	0.4964
$st_5(10)$	0.142	0.2812	0.2112	0.312	0.268	0.3972	0.3552	0.4764
$s\mathcal{D}e(10)$	0.1348	0.2528	0.1964	0.2948	0.2724	0.3896	0.374	0.4776
$s\mathcal{L}(10)$	0.142	0.2452	0.1952	0.3202	0.272	0.398	0.358	0.4836
$k_{0.3}$	0.1492	0.2728	0.2122	0.3204	0.28	0.3976	0.3628	0.4968

Table 4. Rejection frequencies at asymptotic level $\alpha = 5\%$ for various values of $\lambda_1 = \lambda_2 = \lambda_3$ with error density g_1 of the pseudo-Gaussian test $T_{\mathcal{N}}^{\bullet(n)}$.

	$\lambda_1 = \lambda_2 = \lambda_3$ with $n = 30$ and $T = 10$					$\lambda_1 = \lambda_2 = \lambda_3$ with $n = 30$ and $T = 20$			
g_1	0	0.1	0.3	0.5	0.7	0	0.1	0.3	0.5
$\mathcal{N}$	0.048	0.2764	0.5524	0.7716	0.9188	0.056	0.4792	0.7764	0.9612
$\mathcal{D}e$	0.046	0.2562	0.5472	0.7796	0.9172	0.054	0.4614	0.7614	0.9514
$\mathcal{L}$	0.052	0.2764	0.5476	0.7876	0.9172	0.056	0.4718	0.7608	0.9522
t_3	0.052	0.254	0.5412	0.786	0.9104	0.048	0.4658	0.7714	0.9602
$s\mathcal{N}(5)$	0.054	0.272	0.5616	0.776	0.9112	0.056	0.466	0.7618	0.9618
$st_3(5)$	0.048	0.2672	0.5764	0.7856	0.9136	0.054	0.464	0.7588	0.9544
$s\mathcal{D}e(5)$	0.046	0.2616	0.558	0.7648	0.9162	0.054	0.464	0.7588	0.9524
$s\mathcal{L}(7)$	0.052	0.2514	0.556	0.7644	0.9014	0.056	0.4604	0.7744	0.9598

Table 5. Rejection frequencies at asymptotic level $\alpha = 5\%$ for various values of $\lambda_1 = \lambda_2 = \lambda_3$ with error density g_1 of the pseudo-Gaussian test $T_{\mathcal{N}}^{\bullet(n)}$ and likelihood ratio (LR) test.

		$\lambda_1 = \lambda_2 = \lambda_3$ with $n = 30$ and $T = 5$			
g_1	Test	0.5	0.7	0.9	1.1
$\mathcal{N}$	$T_{\mathcal{N}}^{\bullet(n)}$	0.5211	0.6728	0.784	0.9888
	LR	0.134	0.4224	0.692	0.852
$\mathcal{D}e$	$T_{\mathcal{N}}^{\bullet(n)}$	0.4928	0.6528	0.7284	0.9612
	LR	0.1314	0.4214	0.698	0.853
$\mathcal{L}$	$T_{\mathcal{N}}^{\bullet(n)}$	0.5032	0.6528	0.738	0.914
	LR	0.1214	0.4144	0.688	0.834
t_3	$T_{\mathcal{N}}^{\bullet(n)}$	0.4788	0.6352	0.738	0.9414
	LR	0.1228	0.4224	0.668	0.8214
$s\mathcal{N}(5)$	$T_{\mathcal{N}}^{\bullet(n)}$	0.466	0.6308	0.7264	0.9342
	LR	0.1308	0.4114	0.6882	0.8432
$st_3(5)$	$T_{\mathcal{N}}^{\bullet(n)}$	0.508	0.64	0.7284	0.9316
	LR	0.1214	0.3918	0.6678	0.8414
$s\mathcal{D}e(5)$	$T_{\mathcal{N}}^{\bullet(n)}$	0.484	0.6432	0.7216	0.9442
	LR	0.1224	0.3888	0.6658	0.8378
$s\mathcal{L}(7)$	$T_{\mathcal{N}}^{\bullet(n)}$	0.4988	0.6444	0.7256	0.9312
	LR	0.1288	0.4214	0.6814	0.8354

Table 6. $p = 2$, $S_1 = 2$, $S_2 = 4$, $n = 24$, $T = 10$, $\mu = (1.5; 2.3; -1.2; 3)$, $\beta^{(1)} = (3.1; 5)$, and $\beta^{(2)} = (2.6; -4; -2; 5)$.

f_1	ϑ/period	1	2	3	4	RMSE
$\mathcal{N}$	μ	1.6048	2.4009	-1.2966	2.9819	$\mathbf{0.0556}$
	$\beta^{(1)}$	3.1007	5.0008			
	$\beta^{(2)}$	2.6004	-4.0016	-2.0083	5.001	
$\mathcal{D}e$	μ	1.6103	2.4083	-1.3016	2.9824	0.0588
	$\beta^{(1)}$	3.0983	4.9989			
	$\beta^{(2)}$	2.6017	-4.0037	-1.9975	5.0053	
$s\mathcal{N}(10)$	μ	1.429	2.4411	-1.1795	2.8936	0.0627
	$\beta^{(1)}$	3.1012	5.0014			
	$\beta^{(2)}$	2.6070	-3.9977	-1.9971	5.0069	
$st_5(10)$	μ	1.4086	2.4319	-1.005	2.7801	0.1058
	$\beta^{(1)}$	3.0986	5.0005			
	$\beta^{(2)}$	2.5984	-4.0011	-2.0071	4.9982	

Table 7. $p = 3$, $S_1 = 2$, $S_2 = 3$, $S_3 = 5$, $n = 30$, $T = 4$, $\mu = (2; -3.2; -5; 4; 1.3)$, $\beta^{(1)} = (3; 4)$, $\beta^{(2)} = (2; 5; 1)$, and $\beta^{(3)} = (-3; -1; 4; 3.2; 5.2)$.

f_1	ϑ/period	1	2	3	4	5	RMSE
$\mathcal{N}$	μ	2.24	-3.28	-5.04	4.07	1.09	$\mathbf{0.0879}$
	$\beta^{(1)}$	2.99	4.002				
	$\beta^{(2)}$	1.96	5.03	0.98			
	$\beta^{(3)}$	- 2.99	-0.99	4.01	3.19	5.19	
$\mathcal{D}e$	μ	2.09	-2.81	-4.83	4.03	1.22	0.125
	$\beta^{(1)}$	3.03	3.97				
	$\beta^{(2)}$	1.83	5.09	1.07			
	$\beta^{(3)}$	- 2.99	-0.99	4.01	3.21	5.21	
$s\mathcal{N}(10)$	μ	2.13	-3.52	-4.99	4.33	1.17	0.1323
	$\beta^{(1)}$	3.01	3.97				
	$\beta^{(2)}$	1.99	4.94	1.07			
	$\beta^{(3)}$	- 3.01	-1.01	3.99	3.21	5.19	
$st_5(10)$	μ	2.07	-2.97	-5.24	3.63	1.63	0.1571
	$\beta^{(1)}$	3.03	3.96				
	$\beta^{(2)}$	1.96	5.04	0.99			
	$\beta^{(3)}$	- 2.99	-1.01	3.99	3.18	5.19	

Table 8. $p=4$, $S_1=2$, $S_2=3$, $S_3=5$, $S_4=7$, $n=42$, $T=5$, $\mu=(3.2;4.1;5;-2;-5;2;1.5)$, $\beta^{(1)}=(3.4;2.1)$, $\beta^{(2)}=(-3;2.1;4)$, $\beta^{(3)}=(2.4;5;6.2;4.3;-2.5)$, and $\beta^{(4)}=(-1;5;-2;3.5;4;-5;7)$.

f_1	ϑ/period	1	2	3	4	5	6	7	RMSE
$\mathcal{N}$	μ	3.098	4.098	5.041	-1.823	-4.775	2.181	1.386	**0.0934**
	$\beta^{(1)}$	3.366	4.131						
	$\beta^{(2)}$	-2.891	2.072	3.917					
	$\beta^{(3)}$	2.321	4.929	6.373	4.229	-2.452			
	$\beta^{(4)}$	- 1.001	5.001	-1.997	3.498	3.997	-5.001	6.999	
$\mathcal{D}e$	μ	3.161	4.343	4.942	-1.767	-4.741	2.384	1.535	0.1261
	$\beta^{(1)}$	3.363	4.137						
	$\beta^{(2)}$	-2.846	2.076	3.871					
	$\beta^{(3)}$	2.254	5.076	6.184	4.313	-2.446			
	$\beta^{(4)}$	- 0.998	5.003	-1.998	3.497	3.999	-4.999	6.999	
$s\mathcal{N}(10)$	μ	3.501	4.313	5.002	-2.288	-4.977	2.021	1.218	0.1292
	$\beta^{(1)}$	3.406	4.093						
	$\beta^{(2)}$	-3.018	2.189	3.929					
	$\beta^{(3)}$	2.494	5.033	6.263	4.181	-2.573			
	$\beta^{(4)}$	- 1.001	5.001	-2.001	3.499	4.001	-4.998	7.001	
$st_5(10)$	μ	3.331	4.147	4.615	-1.925	-4.851	2.157	1.338	0.1306
	$\beta^{(1)}$	3.391	2.109						
	$\beta^{(2)}$	-2.792	2.036	3.846					
	$\beta^{(3)}$	2.346	4.909	6.224	4.276	-2.357			
	$\beta^{(4)}$	- 0.999	4.998	-1.998	3.501	4.001	-5.001	7.001	

Table 9. p-values of the test $T_{\mathcal{N}}^{\bullet(n)}$ and the RMSE for a simulated model with $p=2$, $S_1=3$, and $S_2=5$.

size n	Null hypothesis $\mathcal{H}_0^{(n)}$	p-value	RMSE
$n=60$	$\mathcal{H}_0^{(n)}: S_1=3; S_2=5$	0.39207	**0.8869**
	$\mathcal{H}_0^{(n)}: S_1=2; S_2=4$	0.0143	5.977
	$\mathcal{H}_0^{(n)}: S_1=2; S_2=3$	0.1654	5.7734
	$\mathcal{H}_0^{(n)}: S_1=2; S_2=2$	0.1125	6.0723
$n=120$	$\mathcal{H}_0^{(n)}: S_1=3; S_2=5$	0.2547	**0.91603**
	$\mathcal{H}_0^{(n)}: S_1=2; S_2=4$	0.00096	6.6723
	$\mathcal{H}_0^{(n)}: S_1=2; S_2=3$	0.6093	6.5696
	$\mathcal{H}_0^{(n)}: S_1=2; S_2=2$	0.13601	6.07176
$n=180$	$\mathcal{H}_0^{(n)}: S_1=3; S_2=5$	0.0719	**0.9933**
	$\mathcal{H}_0^{(n)}: S_1=2; S_2=4$	4.0459×10^{-6}	6.6927
	$\mathcal{H}_0^{(n)}: S_1=2; S_2=3$	0.4785	6.5161
	$\mathcal{H}_0^{(n)}: S_1=2; S_2=2$	0.2227	6.7315

6. Real Data Example

This section presents an empirical application of the proposed pseudo-Gaussian test using a publicly available meteorological dataset `https://www.kaggle.com/datasets/rafunlearnhub/weatherhistory`, which contains 96453 observations of the response variable **Temperature (°C)** and two predictors: **Humidity** and

Table 10. p-values of the test $T_{\mathcal{N}}^{\bullet(n)}$ and the RMSE for a simulated model with $p = 3$, $S_1 = 3$, $S_2 = 4$, and $S_5 = 5$.

size n	Null hypothesis $\mathcal{H}_0^{(n)}$	p-value	RMSE
$n = 60$	$\mathcal{H}_0^{(n)} : S_1 = 3; S_2 = 4; S_3 = 5$	0.97	**0.8121**
	$\mathcal{H}_0^{(n)} : S_1 = 2; S_2 = 3; S_3 = 4$	0	12.856
	$\mathcal{H}_0^{(n)} : S_1 = 2; S_2 = 3; S_3 = 3$	0.91	14.93
$n = 120$	$\mathcal{H}_0^{(n)} : S_1 = 3; S_2 = 4; S_3 = 5$	0.94	**0.9084**
	$\mathcal{H}_0^{(n)} : S_1 = 2; S_2 = 3; S_3 = 4$	1.07×10^{-11}	15.676
	$\mathcal{H}_0^{(n)} : S_1 = 2; S_2 = 3; S_3 = 3$	1.11×10^{-16}	17.0874
$n = 180$	$\mathcal{H}_0^{(n)} : S_1 = 3; S_2 = 4; S_3 = 5$	0.1556	**1.0498**
	$\mathcal{H}_0^{(n)} : S_1 = 2; S_2 = 3; S_3 = 4$	8.2156×10^{-15}	13.8058
	$\mathcal{H}_0^{(n)} : S_1 = 2; S_2 = 3; S_3 = 3$	3.6834×10^{-11}	15.0318

Table 11. Rejection rates of the pseudo-Gaussian test $T_{\mathcal{N}}^{\bullet(n)}$ and values of the AIC and BIC criteria for different candidate periods (S_1, S_2), with true period $(4, 6)$.

(S_1, S_2)	(2,4)	(3,5)	(6,8)	(6,12)	(5,11)
Rejection rate (%)	76.2	82.4	17.7	34.3	37.2
AIC	1545.74	1361.91	1200.37	1423.11	1617.38
BIC	1590.99	1396.72	1276.94	1520.57	1711.35

Precipitation. Meteorological data exhibit complex periodic structures that pose unique challenges for standard harmonic analysis (a common period across all independent variables). First, different meteorological variables operate on distinct periodic scales: temperature typically exhibits a daily (24-hour) or a monthly (12-month) cycle, while humidity often exhibits semi-diurnal patterns (12-hour) due to atmospheric tides, and precipitation may show weekly and monthly cycles. Standard periodic regression models, which assume a common period across all variables, cannot capture these differential periodic behaviors. Second, meteorological processes are subject to gradual period shifts due to climate change and seasonal transitions; for instance, the timing of seasonal temperature extremes has shifted over recent decades. The proposed S_j vs $S_j + 1$ test provides a framework for detecting such subtle period changes. Testing consecutive periods helps determine whether an observed periodicity reflects a fundamental frequency or an aliased harmonic.

The role of the pseudo-Gaussian test as a diagnostic tool for *nested* model validation is illustrated in Table 14. The null hypotheses associated with misspecified periods, such as $(S_1, S_2) = (3, 11)$, $(2, 11)$ and $(4, 11)$, are strongly rejected at the 5% level, with p-values equal to zero. This systematic rejection indicates that these specifications fail to capture the underlying periodic structure of the data.

In contrast, models involving $S_2 = 12$ are not rejected in several cases, most notably $(S_1, S_2) = (12, 12)$, which yields a large p-value (0.6537), indicating an adequate fit. This sequential elimination of incorrect hypotheses provides a *blind* identification strategy, whereby incompatible periodic structures are ruled out without relying on prior RMSE-based selection.

Importantly, the test is not used to compare non-nested models (e.g., $(4, 12)$ vs $(4, 4)$), but only to assess nested hypotheses, in line with its theoretical justification.

For model comparison across non-nested specifications, Table 13 reports the AIC and BIC values. Both criteria consistently favor models with a second period equal to 12, with AIC selecting $(12, 12)$ and BIC selecting $(3, 12)$. These results reinforce the conclusion that $S_2 = 12$ is a key structural component of the data.

Finally, the RMSE values reported in Table 12 provide a complementary descriptive assessment of in-sample fit, showing that models close to $(4, 12)$ achieve low prediction errors. However, as RMSE tends to improve with model complexity, it is not used for formal selection.

Overall, the combination of information criteria for model selection and pseudo-Gaussian testing for nested hypothesis validation provides a coherent and statistically sound framework. The convergence of these approaches toward models with $S_2 = 12$ highlights the relevance of the identified periodic structure and demonstrates the practical usefulness of the proposed methodology for analyzing data with complex seasonal patterns.

Table 12. In-sample predictive accuracy (RMSE) for candidate periodic regression models with different period combinations (S_1, S_2).

(S_1, S_2)	(3,11)	(4,12)	(12,12)	(4,4)	(3,12)
RMSE	4.4955	**1.1712**	**1.3871**	4.4864	**1.1936**
(S_1, S_2)	(3,4)	(4,11)	(3,3)	(2,11)	(11,11)
RMSE	4.4608	4.4867	4.5424	4.4918	4.2481

Table 13. Model selection using information criteria (AIC and BIC) across non-nested periodic regression models.

(S_1, S_2)	(3,11)	(4,12)	(12,12)	(4,4)	(3,12)
AIC	446.8181	97.7291	**91.5292**	420.2803	100.7322
BIC	518.8881	178.5676	195.3105	454.87339	**178.4476**
(S_1, S_2)	(3,4)	(4,11)	(3,3)	(2,11)	(11,11)
AIC	416.7679	448.3001	417.5526	444.5999	447.8673
BIC	448.4787	523.2529	443.4978	513.7872	542.9998

Table 14. Sequential pseudo-Gaussian tests for nested hypotheses on candidate periods (S_1, S_2).

Null hypothesis $\mathcal{H}_0^{(n)}$	(3,11)	(4,12)	(12,12)	(4,4)	(3,12)
Observed value	656.9707	307.8057	1.6251	-0.1861	243.1248
p-value	0	0	0.6537	1	0
Null hypothesis $\mathcal{H}_0^{(n)}$	(3,4)	(4,11)	(3,3)	(2,11)	(11,11)
Observed value	-0.6911	1787.92	-41.3538	4638.98	-59.35
p-value	1	0	1	0	1

7. Conclusion and discussion

In this paper, we derived a pseudo-Gaussian test for comparing periodic multiple regression models with consecutive periods. The proposed test is straightforward to implement and exhibits superior power performance across various sample sizes under Gaussian distributions. We also constructed a least squares estimator (LSE) for the proposed model, which coincides with the function `per_lm()` in the **PerRegMod** package when $S_1 = \ldots = S_p$.

Simulation studies further confirm the strong performance of the pseudo-Gaussian tests in the Gaussian setting, as well as their superiority compared to the likelihood ration test. Using real data and AIC and BIC as performance criteria, we demonstrate the relevance and effectiveness of periodic regression models across different periods.

The main contributions of this work can be summarized as follows: (1) we extend periodic regression modeling to allow variable-specific periods, (2) we develop a locally optimal test for comparing consecutive periods within the ULAN framework, (3) we construct a pseudo-Gaussian version of the test that remains valid under general error distributions, (4) we provide simulation evidence of the test's good size and power properties, (5) we demonstrate the test's utility for sequential period selection, and (6) we compare its performance favorably against AIC and BIC in period identification.

While our analysis has focused on periodic multiple linear regression models with distinct periods, several extensions remain open for future research. In particular, developing alternative estimation approaches beyond LSE would be valuable, since LSE achieves optimal results primarily under Gaussian distributions. Such extensions would improve model robustness in practical situations where normality assumptions may not hold. Additionally, extending the test to handle non-consecutive period comparisons (e.g., S_j versus $S_j + k$ for $k > 1$) would broaden its applicability. The methodology developed in this paper will be incorporated into the `PerRegMod` R package [17], which is currently available on CRAN at `https://CRAN.R-project.org/package=PerRegMod`. These developments would enhance the applicability and versatility of periodic regression models in modern statistical practice.

Code availability

The code used in this article is available from the corresponding author.

Appendix

Proof of Proposition 1

The proof of this proposition relies on conditions 2 to 7 from [20], with condition 2 being the most delicate. The main goal is to show that $q^{\frac{1}{2}}_{\boldsymbol{\mu},\boldsymbol{\beta},\sigma^2,\boldsymbol{\lambda};f_1}$ is quadratic mean differentiability at any $\left(\boldsymbol{\mu}^{'},\boldsymbol{\beta}^{'},\sigma^2,\mathbf{0}\right)^{'}$. We have:

$$q^{\frac{1}{2}}_{\boldsymbol{\mu},\boldsymbol{\beta},\sigma^2,\boldsymbol{\lambda};f_1}(y) = \left(\frac{1}{\sigma^{nT}}\prod_{i=1}^{n}\prod_{t=1}^{T} f_1\left(\sigma^{-1}\left(y-\mu_{l(i,S_p)}-\sum_{j=1}^{p}\beta^{(j)}_{l(i,S_j)}x^{(j)}_{i,t}-\lambda_1-\sum_{j=1}^{p}\lambda_{j+1}x^{(j)}_{i,t}\right)\right)\right)^{1/2}.$$

The quadratic mean differentiability of $q^{\frac{1}{2}}_{\boldsymbol{\mu},\boldsymbol{\beta},\sigma^2,\boldsymbol{\lambda};f_1}$ at $\left(\boldsymbol{\mu}^{'},\boldsymbol{\beta}^{'},\sigma^2,\mathbf{0}\right)^{'}$ implies that, for $\boldsymbol{w}$, $\boldsymbol{\eta}$, v, and $\boldsymbol{\gamma} \to \mathbf{0}$, we have:

$$\int_{\mathcal{R}}\Bigg[q^{\frac{1}{2}}_{\boldsymbol{\mu}+\boldsymbol{w},\boldsymbol{\beta}+\boldsymbol{\eta},\sigma^2+v,\boldsymbol{\gamma};f_1}(y)-q^{\frac{1}{2}}_{\boldsymbol{\mu},\boldsymbol{\beta},\sigma^2,\mathbf{0};f_1}(y)$$
$$-\left(\boldsymbol{w}^{'},\boldsymbol{\eta}^{'},v,\boldsymbol{\gamma}^{'}\right)\times\begin{pmatrix}\frac{\partial}{\partial\boldsymbol{\mu}}q^{\frac{1}{2}}_{\boldsymbol{\mu},\boldsymbol{\beta},\sigma^2,\mathbf{0};f_1}(y)\\ \frac{\partial}{\partial\boldsymbol{\beta}}q^{\frac{1}{2}}_{\boldsymbol{\mu},\boldsymbol{\beta},\sigma^2,\mathbf{0};f_1}(y)\\ \frac{\partial}{\partial\sigma^2}q^{\frac{1}{2}}_{\boldsymbol{\mu},\boldsymbol{\beta},\sigma^2,\mathbf{0};f_1}(y)\\ \frac{\partial}{\partial\boldsymbol{\lambda}}q^{\frac{1}{2}}_{\boldsymbol{\mu},\boldsymbol{\beta},\sigma^2,\boldsymbol{\lambda};f_1}(y)\,|_{\boldsymbol{\lambda}=\mathbf{0}}\end{pmatrix}\Bigg]^2 dy = o\left(\left\|\begin{pmatrix}\boldsymbol{w}\\ \boldsymbol{\eta}\\ v\\ \boldsymbol{\gamma}\end{pmatrix}\right\|^2\right), \tag{9}$$

where

$$\frac{\partial}{\partial \boldsymbol{\mu}} q^{\frac{1}{2}}_{\boldsymbol{\mu},\boldsymbol{\beta},\sigma^2,\mathbf{0};f_1}(y) = \frac{1}{2\sigma} q^{\frac{1}{2}}_{\boldsymbol{\mu},\boldsymbol{\beta},\sigma^2,\mathbf{0};f_1}(y) \sum_{t=1}^{T} \sum_{r=0}^{m_p-1} \boldsymbol{\phi}_{f_1}(\boldsymbol{Z}_{r,t}),$$

$$\frac{\partial}{\partial \boldsymbol{\beta}} q^{\frac{1}{2}}_{\boldsymbol{\mu},\boldsymbol{\beta},\sigma^2,\mathbf{0};f_1}(y) = \frac{1}{2\sigma} q^{\frac{1}{2}}_{\boldsymbol{\mu},\boldsymbol{\beta},\sigma^2,\mathbf{0};f_1}(y)\, \mathbf{K}^{(n)} \begin{pmatrix} \sum_{t=1}^{T} \sum_{r=0}^{m_1-1} \boldsymbol{\phi}^{x^{(1)}}_{f_1}(\boldsymbol{Z}_{r,t}) \\ \vdots \\ \sum_{t=1}^{T} \sum_{r=0}^{m_p-1} \boldsymbol{\phi}^{x^{(p)}}_{f_1}(\boldsymbol{Z}_{r,t}) \end{pmatrix},$$

$$\frac{\partial}{\partial \sigma^2} q^{\frac{1}{2}}_{\boldsymbol{\mu},\boldsymbol{\beta},\sigma^2,\mathbf{0};f_1}(y) = \frac{1}{4\sigma^2} q^{\frac{1}{2}}_{\boldsymbol{\mu},\boldsymbol{\beta},\sigma^2,\mathbf{0};f_1}(y) \sum_{t=1}^{T} \sum_{s=1}^{S_p+1} \sum_{r=0}^{m_p-1} \varphi_{f_1}(Z_{s+(S_p+1)r,t}),$$

and

$$\begin{aligned} \frac{\partial}{\partial \boldsymbol{\lambda}} q^{\frac{1}{2}}_{\boldsymbol{\mu},\boldsymbol{\beta},\sigma^2,\boldsymbol{\lambda};f_1}(y)\,|_{\boldsymbol{\lambda}=\mathbf{0}} &= \frac{1}{2\sigma} q^{\frac{1}{2}}_{\boldsymbol{\mu},\boldsymbol{\beta},\sigma^2,\mathbf{0};f_1}(y) \sum_{t=1}^{T} \boldsymbol{\psi}^{x}_{f_1}(\boldsymbol{Z}_t) \\ &= \frac{1}{2\sigma} q^{\frac{1}{2}}_{\boldsymbol{\mu},\boldsymbol{\beta},\sigma^2,\mathbf{0};f_1}(y) \sum_{t=1}^{T} \begin{pmatrix} \sum_{r=0}^{m_p-1} \phi_{f_1}(Z_{(S_p+1)(r+1),t}) \\ \mathbf{D}^{(n)} \begin{pmatrix} \sum_{r=0}^{m_1-1} \phi_{f_1}(Z_{(S_1+1)(r+1),t}) x^{(1)}_{(S_1+1)(r+1),t} \\ \vdots \\ \sum_{r=0}^{m_p-1} \phi_{f_1}(Z_{(S_p+1)(r+1),t}) x^{(p)}_{(S_p+1)(r+1),t} \end{pmatrix} \end{pmatrix}. \end{aligned}$$

In order to prove (9), it is sufficient to demonstrate the following three parts:

(i)

$$\begin{aligned} &\int_{\mathcal{R}} \left[q^{\frac{1}{2}}_{\boldsymbol{\mu}+\boldsymbol{w},\boldsymbol{\beta}+\boldsymbol{\eta},\sigma^2+v,\boldsymbol{\gamma};f_1}(y) - q^{\frac{1}{2}}_{\boldsymbol{\mu}+\boldsymbol{w},\boldsymbol{\beta}+\boldsymbol{\eta},\sigma^2+v,\mathbf{0};f_1}(y) - \boldsymbol{\gamma}^{'} \frac{\partial}{\partial \boldsymbol{\lambda}} q^{\frac{1}{2}}_{\boldsymbol{\mu}+\boldsymbol{w},\boldsymbol{\beta}+\boldsymbol{\eta},\sigma^2+v,\mathbf{0};f_1}(y)\,|_{\boldsymbol{\lambda}=\mathbf{0}} \right]^2 dy \\ &= o\left(\|\boldsymbol{\gamma}\|^2 \right); \end{aligned} \tag{10}$$

(ii)

$$\begin{aligned} &\int_{\mathcal{R}} \Bigg[q^{\frac{1}{2}}_{\boldsymbol{\mu}+\boldsymbol{w},\boldsymbol{\beta}+\boldsymbol{\eta},\sigma^2+v,\mathbf{0};f_1}(y) - q^{\frac{1}{2}}_{\boldsymbol{\mu},\boldsymbol{\beta},\sigma^2,\mathbf{0};f_1}(y) \\ &\quad - \left(\boldsymbol{w}^{'}, \boldsymbol{\eta}^{'}, v \right) \times \begin{pmatrix} \frac{\partial}{\partial \boldsymbol{\mu}} q^{\frac{1}{2}}_{\boldsymbol{\mu},\boldsymbol{\beta},\sigma^2,\mathbf{0};f_1}(y) \\ \frac{\partial}{\partial \boldsymbol{\beta}} q^{\frac{1}{2}}_{\boldsymbol{\mu},\boldsymbol{\beta},\sigma^2,\mathbf{0};f_1}(y) \\ \frac{\partial}{\partial \sigma^2} q^{\frac{1}{2}}_{\boldsymbol{\mu},\boldsymbol{\beta},\sigma^2,\mathbf{0};f_1}(y) \end{pmatrix} \Bigg]^2 dy = o\left(\left\| \begin{pmatrix} \boldsymbol{w} \\ \boldsymbol{\eta} \\ v \end{pmatrix} \right\|^2 \right); \end{aligned} \tag{11}$$

(iii)

$$\int_{\mathcal{R}} \left[\frac{\partial}{\partial \boldsymbol{\lambda}} q^{\frac{1}{2}}_{\boldsymbol{\mu}+\boldsymbol{w},\boldsymbol{\beta}+\boldsymbol{\eta},\sigma^2+v,\boldsymbol{\lambda};f_1}(y)\,|_{\boldsymbol{\lambda}=\mathbf{0}} - \frac{\partial}{\partial \boldsymbol{\lambda}} q^{\frac{1}{2}}_{\boldsymbol{\mu},\boldsymbol{\beta},\sigma^2,\boldsymbol{\lambda};f_1}(y)\,|_{\boldsymbol{\lambda}=\mathbf{0}} \right]^2 dy = o\,(1)\,. \tag{12}$$

The left hand-side in (i) takes the form:

$$\begin{aligned} \int_{\mathcal{R}} \frac{1}{(\sigma^2+v)^{\frac{nT}{4}}} & \left[\left(\prod_{i=1}^{n}\prod_{t=1}^{T} f_1 \left(z_{i,t} - \frac{\gamma_1 + \sum_{j=1}^{p} \gamma_{j+1} x_{i,t}^{(j)}}{\sqrt{\sigma^2+v}} \right) \right)^{\frac{1}{2}} - \left(\prod_{i=1}^{n}\prod_{t=1}^{T} f_1\left(z_{i,t}\right) \right)^{\frac{1}{2}} \right. \\ & \left. - \frac{1}{2(\sigma^2+v)^{\frac{1}{2}}} \boldsymbol{\gamma}^{'} \left(\prod_{i=1}^{n}\prod_{t=1}^{T} f_1\left(z_{i,t}\right) \right)^{\frac{1}{2}} \sum_{t=1}^{T} \boldsymbol{\psi}^{x}_{f_1}(\boldsymbol{Z}_t) \right]^2 dz. \end{aligned} \tag{13}$$

Let $\boldsymbol{\gamma} = h\boldsymbol{k}$ and $\zeta_{i,t}(\boldsymbol{\gamma}) = \boldsymbol{\gamma}_1 + \sum_{j=1}^{p} \boldsymbol{\gamma}_{j+1} x_{i,t}^{(j)}$ with $h \in \mathcal{R}$. Therefore, (13) becomes:

$$\begin{aligned} \int_{\mathcal{R}} \frac{1}{(\sigma^2+v)^{\frac{nT}{4}}} & \left[\left(\prod_{i=1}^{n}\prod_{t=1}^{T} f_1 \left(z_{i,t} - \frac{\zeta_{i,t}(h\boldsymbol{k})}{\sqrt{\sigma^2+v}} \right) \right)^{\frac{1}{2}} - \left(\prod_{i=1}^{n}\prod_{t=1}^{T} f_1\left(z_{i,t}\right) \right)^{\frac{1}{2}} \right. \\ & \left. - \frac{h}{2(\sigma^2+v)^{\frac{1}{2}}} \boldsymbol{k}^{'} \left(\prod_{i=1}^{n}\prod_{t=1}^{T} f_1\left(z_{i,t}\right) \right)^{\frac{1}{2}} \sum_{t=1}^{T} \boldsymbol{\psi}^{x}_{f_1}(\boldsymbol{Z}_t) \right]^2 dz. \end{aligned} \tag{14}$$

The part (i) is equivalent to

$$\begin{aligned} \int_{\mathcal{R}} & \left[\frac{1}{h} \left(\left(\prod_{i=1}^{n}\prod_{t=1}^{T} f_1 \left(z_{i,t} - \frac{\zeta_{i,t}(h\boldsymbol{k})}{\sqrt{\sigma^2+v}} \right) \right)^{\frac{1}{2}} - \left(\prod_{i=1}^{n}\prod_{t=1}^{T} f_1\left(z_{i,t}\right) \right)^{\frac{1}{2}} \right) \right. \\ & \left. - \frac{1}{2(\sigma^2+v)^{\frac{1}{2}}} \boldsymbol{k}^{'} \left(\prod_{i=1}^{n}\prod_{t=1}^{T} f_1\left(z_{i,t}\right) \right)^{\frac{1}{2}} \sum_{t=1}^{T} \boldsymbol{\psi}^{x}_{f_1}(\boldsymbol{Z}_t) \right]^2 dz = o(1). \end{aligned} \tag{15}$$

We have

$$\begin{aligned}
&\lim_{h\to 0}\frac{1}{h}\left(\left(\prod_{i=1}^{n}\prod_{t=1}^{T}f_1\left(z_{i,t}-\frac{\zeta_{i,t}(h\boldsymbol{k})}{\sqrt{\sigma^2+v}}\right)\right)^{\frac{1}{2}}-\left(\prod_{i=1}^{n}\prod_{t=1}^{T}f_1\left(z_{i,t}\right)\right)^{\frac{1}{2}}\right)\\
&=\lim_{h\to 0}\frac{1}{\frac{\zeta_{i,t}(h\boldsymbol{k})}{\sqrt{\sigma^2+v}}}\left(\left(\prod_{i=1}^{n}\prod_{t=1}^{T}f_1\left(z_{i,t}-\frac{\zeta_{i,t}(h\boldsymbol{k})}{\sqrt{\sigma^2+v}}\right)\right)^{\frac{1}{2}}-\left(\prod_{i=1}^{n}\prod_{t=1}^{T}f_1\left(z_{i,t}\right)\right)^{\frac{1}{2}}\right)\times\frac{\zeta_{i,t}(h\boldsymbol{k})}{h\sqrt{\sigma^2+v}}\\
&=\left(\left(\prod_{i=1}^{n}\prod_{t=1}^{T}f_1\left(z_{i,t}\right)\right)^{\frac{1}{2}}\right)^{'}\times\frac{1}{\sqrt{\sigma^2+v}}\boldsymbol{k}^{'}\begin{pmatrix}1\\ x_{i,t}^{(1)}\\ \vdots\\ x_{i,t}^{(p)}\end{pmatrix}\\
&=\frac{1}{2(\sigma^2+v)^{\frac{1}{2}}}\boldsymbol{k}^{'}\left(\prod_{i=1}^{n}\prod_{t=1}^{T}f_1\left(z_{i,t}\right)\right)^{\frac{1}{2}}\sum_{t=1}^{T}\boldsymbol{\psi}_{f_1}^{x}(\boldsymbol{Z}_t),
\end{aligned}$$

and just show

$$\begin{aligned}
&\int_{\mathcal{R}}\left[\frac{1}{h}\left(\left(\prod_{i=1}^{n}\prod_{t=1}^{T}f_1\left(z_{i,t}-\frac{\zeta_{i,t}(h\boldsymbol{k})}{\sqrt{\sigma^2+v}}\right)\right)^{\frac{1}{2}}-\left(\prod_{i=1}^{n}\prod_{t=1}^{T}f_1\left(z_{i,t}\right)\right)^{\frac{1}{2}}\right)\right]^2 dz\\
&\leq\int_{\mathcal{R}}\left[\frac{1}{2(\sigma^2+v)^{\frac{1}{2}}}\boldsymbol{k}^{'}\left(\prod_{i=1}^{n}\prod_{t=1}^{T}f_1\left(z_{i,t}\right)\right)^{\frac{1}{2}}\sum_{t=1}^{T}\boldsymbol{\psi}_{f_1}^{x}(\boldsymbol{Z}_t)\right]^2 dz<\infty.
\end{aligned}$$

We know that

$$\left(\prod_{i=1}^{n}\prod_{t=1}^{T}f_1\left(z_{i,t}-u(h\boldsymbol{k})\right)\right)^{\frac{1}{2}}-\left(\prod_{i=1}^{n}\prod_{t=1}^{T}f_1\left(z_{i,t}\right)\right)^{\frac{1}{2}}=\int_{z_{i,t}}^{z_{i,t}-u(h\boldsymbol{k})}\left(\left(\prod_{i=1}^{n}\prod_{t=1}^{T}f_1\left(v\right)\right)^{\frac{1}{2}}\right)^{'}dv,$$

where $u(h\boldsymbol{k}) = \frac{\zeta_{i,t}(h\boldsymbol{k})}{\sqrt{\sigma^2+v}}$. Then

$$\begin{aligned}
&\int\limits_{\mathcal{R}} \left[\frac{1}{h} \left(\left(\prod_{i=1}^{n} \prod_{t=1}^{T} f_1 \left(z_{i,t} - \frac{\zeta_{i,t}(h\boldsymbol{k})}{\sqrt{\sigma^2+v}} \right) \right)^{\frac{1}{2}} - \left(\prod_{i=1}^{n} \prod_{t=1}^{T} f_1 \left(z_{i,t} \right) \right)^{\frac{1}{2}} \right) \right]^2 dz \\
&= \frac{1}{h^2} \int\limits_{\mathcal{R}} \left[\int_{z_{i,t}}^{z_{i,t}-u(h\boldsymbol{k})} \left(\left(\prod_{i=1}^{n} \prod_{t=1}^{T} f_1 \left(w \right) \right)^{\frac{1}{2}} \right)' dw \right]^2 dz \\
&\leq \frac{u(h\boldsymbol{k})}{h^2} \int\limits_{\mathcal{R}} \int_{z_{i,t}}^{z_{i,t}-u(h\boldsymbol{k})} \left[\left(\left(\prod_{i=1}^{n} \prod_{t=1}^{T} f_1 \left(w \right) \right)^{\frac{1}{2}} \right)' \right]^2 dwdz \\
&\leq \frac{u^2(h\boldsymbol{k})}{h^2} \int\limits_{\mathcal{R}} \left[\left(\left(\prod_{i=1}^{n} \prod_{t=1}^{T} f_1 \left(w \right) \right)^{\frac{1}{2}} \right)' \right]^2 dw \\
&\leq \left(\frac{1}{\sqrt{\sigma^2+v}} \boldsymbol{k}' \begin{pmatrix} 1 \\ x_{i,t}^{(1)} \\ \vdots \\ x_{i,t}^{(p)} \end{pmatrix} \right)^2 \int\limits_{\mathcal{R}} \left[\left(\left(\prod_{i=1}^{n} \prod_{t=1}^{T} f_1 \left(w \right) \right)^{\frac{1}{2}} \right)' \right]^2 dw \\
&\leq \int\limits_{\mathcal{R}} \left[\frac{1}{2(\sigma^2+v)^{\frac{1}{2}}} \boldsymbol{k}' \left(\prod_{i=1}^{n} \prod_{t=1}^{T} f_1 \left(z_{i,t} \right) \right)^{\frac{1}{2}} \sum_{t=1}^{T} \boldsymbol{\psi}_{f_1}^{x} (\boldsymbol{Z}_t) \right]^2 dz.
\end{aligned}$$

This completes the proof of part (i).

(ii) By setting $\boldsymbol{\lambda} = \boldsymbol{0}$ in $q^{\frac{1}{2}}_{\boldsymbol{\mu},\boldsymbol{\beta},\sigma^2,\boldsymbol{\lambda};f_1}$, the model specializes to a periodic-coefficients regression model. In this case, the resulting formulation coincides with the framework studied in [19] and [18].

(iii) For the perturbations of $\boldsymbol{\mu}$ and $\boldsymbol{\beta}$, we have, as $\boldsymbol{w}, \boldsymbol{\eta} \to \boldsymbol{0}$,

$$S_{\boldsymbol{u},\boldsymbol{\beta}} = \int\limits_{\mathcal{R}} \left[\frac{\partial}{\partial \boldsymbol{\lambda}} q^{\frac{1}{2}}_{\boldsymbol{\mu}+\boldsymbol{w},\boldsymbol{\beta}+\boldsymbol{\eta},\sigma^2,\boldsymbol{\lambda};f_1} \left(y \right) |_{\boldsymbol{\lambda}=\boldsymbol{0}} - \frac{\partial}{\partial \boldsymbol{\lambda}} q^{\frac{1}{2}}_{\boldsymbol{\mu},\boldsymbol{\beta},\sigma^2,\boldsymbol{\lambda};f_1} \left(y \right) |_{\boldsymbol{\lambda}=\boldsymbol{0}} \right]^2 dy = o\left(1\right). \tag{16}$$

Indeed,

$$S_{\boldsymbol{u},\boldsymbol{\beta}} = \frac{1}{4\sigma^{\frac{nT+4}{2}}} \int\limits_{\mathcal{R}} \left[\left\{ \prod_{i=1}^{n} \prod_{t=1}^{T} f_1 \left(\underline{z}_{i,t} \right) \right\}^{\frac{1}{2}} \sum_{t=1}^{T} \boldsymbol{\psi}_{f_1}^{x} (\underline{\boldsymbol{Z}}_t) - \left\{ \prod_{i=1}^{n} \prod_{t=1}^{T} f_1 \left(z_{i,t} \right) \right\}^{\frac{1}{2}} \sum_{t=1}^{T} \boldsymbol{\psi}_{f_1}^{x} (\boldsymbol{Z}_t) \right]^2 dz, \tag{17}$$

where $\underline{z}_{i,t} = z_{i,t} - w_{l(i,S_p)} - \sum_{j=1}^{p} \eta^{(j)}_{l(i,S_j)} x^{(j)}_{i,t}$. Therefore

$$S_{\boldsymbol{u},\boldsymbol{\beta}} = o(1) \text{ as } \boldsymbol{w},\ \boldsymbol{\eta} \to \mathbf{0}.$$

Letting $z_{i,t} = y_{i,t} - \mu_{l(i,S_p)} - \sum_{j=1}^{p} \beta^{(j)}_{l(i,S_j)} x^{(j)}_{i,t}$. As for the perturbations of σ^2, we then have

$$\begin{aligned}
S_{\sigma^2} &= \int_{\mathcal{R}} \left[\frac{\partial}{\partial \boldsymbol{\lambda}} q^{\frac{1}{2}}_{\boldsymbol{\mu},\boldsymbol{\beta},\sigma^2+v,\boldsymbol{\lambda};f_1}(y)\,|_{\boldsymbol{\lambda}=\mathbf{0}} - \frac{\partial}{\partial \boldsymbol{\lambda}} q^{\frac{1}{2}}_{\boldsymbol{\mu},\boldsymbol{\beta},\sigma^2,\boldsymbol{\lambda};f_1}(y)\,|_{\boldsymbol{\lambda}=\mathbf{0}} \right]^2 dy \\
&= \frac{1}{4} \int_{\mathcal{R}} \left[\frac{1}{(\sigma^2+v)^{\frac{1}{2}}} \left\{ \frac{1}{(\sigma^2+v)^{\frac{nT}{2}}} \prod_{i=1}^{n} \prod_{t=1}^{T} f_1\left(\frac{z_{i,t}}{(\sigma^2+v)^{\frac{1}{2}}} \right) \right\}^{\frac{1}{2}} \sum_{t=1}^{T} \boldsymbol{\psi}^x_{f_1}\left(\frac{\boldsymbol{Z}_t}{(\sigma^2+v)^{1/2}} \right) \right. \\
&\left. - \frac{1}{\sigma} \left\{ \frac{1}{\sigma^{nT}} \prod_{i=1}^{n} \prod_{t=1}^{T} f_1\left(\frac{z_{i,t}}{\sigma} \right) \right\}^{\frac{1}{2}} \sum_{t=1}^{T} \boldsymbol{\psi}^x_{f_1}\left(\frac{\boldsymbol{Z}_{i,t}}{\sigma} \right) \right]^2 dz.
\end{aligned} \tag{18}$$

Then,

$$\begin{aligned}
S_{\sigma^2} = \frac{1}{4} \int_{\mathcal{R}} & \left[\left(\frac{1}{(\sigma^2+v)^{\frac{1}{2}}} - \frac{1}{\sigma} \right) \left\{ \frac{1}{(\sigma^2+v)^{\frac{nT}{2}}} \prod_{i=1}^{n} \prod_{t=1}^{T} f_1\left(\frac{z_{i,t}}{(\sigma^2+v)^{\frac{1}{2}}} \right) \right\}^{\frac{1}{2}} \sum_{t=1}^{T} \boldsymbol{\psi}^x_{f_1}\left(\frac{\boldsymbol{Z}_t}{(\sigma^2+v)^{1/2}} \right) \right. \\
& + \left(\frac{1}{\sigma} \left\{ \frac{1}{(\sigma^2+v)^{\frac{nT}{2}}} \prod_{i=1}^{n} \prod_{t=1}^{T} f_1\left(\frac{z_{i,t}}{(\sigma^2+v)^{\frac{1}{2}}} \right) \right\}^{\frac{1}{2}} - \frac{1}{\sigma} \left\{ \frac{1}{\sigma^{nT}} \prod_{i=1}^{n} \prod_{t=1}^{T} f_1\left(\frac{z_{i,t}}{\sigma} \right) \right\}^{\frac{1}{2}} \right) \sum_{t=1}^{T} \boldsymbol{\psi}^x_{f_1}\left(\frac{\boldsymbol{Z}_t}{(\sigma^2+v)^{1/2}} \right) \\
& \left. + \frac{1}{\sigma} \left\{ \frac{1}{\sigma^{nT}} \prod_{i=1}^{n} \prod_{t=1}^{T} f_1\left(\frac{z_{i,t}}{\sigma} \right) \right\}^{\frac{1}{2}} \sum_{t=1}^{T} \left\{ \boldsymbol{\psi}^x_{f_1}\left(\frac{\boldsymbol{Z}_t}{(\sigma^2+v)^{1/2}} \right) - \boldsymbol{\psi}^x_{f_1}\left(\frac{\boldsymbol{Z}_t}{\sigma} \right) \right\} \right]^2 dz \\
\leq{} & C(A_1 + A_2 + A_3),
\end{aligned}$$

where

$$\begin{aligned}
A_1 &= \int_{\mathcal{R}} \left[\left(\frac{1}{(\sigma^2+v)^{\frac{1}{2}}} - \frac{1}{\sigma} \right) \left\{ \frac{1}{(\sigma^2+v)^{\frac{nT}{2}}} \prod_{i=1}^{n} \prod_{t=1}^{T} f_1\left(\frac{z_{i,t}}{(\sigma^2+v)^{\frac{1}{2}}} \right) \right\}^{\frac{1}{2}} \sum_{t=1}^{T} \boldsymbol{\psi}^x_{f_1}\left(\frac{\boldsymbol{Z}_t}{(\sigma^2+v)^{1/2}} \right) \right]^2 dz, \\
A_2 &= \int_{\mathcal{R}} \left[\left(\frac{1}{\sigma} \left\{ \frac{1}{(\sigma^2+v)^{\frac{nT}{2}}} \prod_{i=1}^{n} \prod_{t=1}^{T} f_1\left(\frac{z_{i,t}}{(\sigma^2+v)^{\frac{1}{2}}} \right) \right\}^{\frac{1}{2}} - \frac{1}{\sigma} \left\{ \frac{1}{\sigma^{nT}} \prod_{i=1}^{n} \prod_{t=1}^{T} f_1\left(\frac{z_{i,t}}{\sigma} \right) \right\}^{\frac{1}{2}} \right) \times \right. \\
& \left. \sum_{t=1}^{T} \boldsymbol{\psi}^x_{f_1}\left(\frac{\boldsymbol{Z}_t}{(\sigma^2+v)^{1/2}} \right) \right]^2 dz,
\end{aligned}$$

and

$$A_3 = \frac{1}{\sigma^2} \int\limits_{\mathcal{R}} \left[\left\{ \frac{1}{\sigma^{nT}} \prod_{i=1}^{n} \prod_{t=1}^{T} f_1 \left(\frac{z_{i,t}}{\sigma} \right) \right\}^{\frac{1}{2}} \sum_{t=1}^{T} \left\{ \boldsymbol{\psi}_{f_1}^{x} \left(\frac{\boldsymbol{Z}_t}{(\sigma^2+v)^{1/2}} \right) - \boldsymbol{\psi}_{f_1}^{x} \left(\frac{\boldsymbol{Z}_t}{\sigma} \right) \right\} \right]^2 dz.$$

For A_1, we have $A_1 = D_1 \times O\left(\left[\frac{1}{(\sigma^2+v)^{\frac{1}{2}}} - \frac{1}{\sigma} \right]^2 \right)$. Hence $A_1 = o(1)$ as $v \to 0$. For A_2, we have

$$\begin{aligned} A_2 = \frac{1}{\sigma^2} \int\limits_{\mathcal{R}} \Bigg[& \left(\left\{ \frac{1}{(\sigma^2+v)^{\frac{nT}{2}}} \prod_{i=1}^{n} \prod_{t=1}^{T} f_1 \left(\frac{z_{i,t}}{(\sigma^2+v)^{\frac{1}{2}}} \right) \right\}^{\frac{1}{2}} \sum_{t=1}^{T} \boldsymbol{\psi}_{f_1}^{x} \left(\frac{\boldsymbol{Z}_t}{(\sigma^2+v)^{1/2}} \right) \right). \\ & - \left\{ \frac{1}{\sigma^{nT}} \prod_{i=1}^{n} \prod_{t=1}^{T} f_1 \left(\frac{z_{i,t}}{\sigma} \right) \right\}^{\frac{1}{2}} \sum_{t=1}^{T} \boldsymbol{\psi}_{f_1}^{x} \left(\frac{\boldsymbol{Z}_t}{\sigma} \right). \\ & + \left\{ \frac{1}{\sigma^{nT}} \prod_{i=1}^{n} \prod_{t=1}^{T} f_1 \left(\frac{z_{i,t}}{\sigma} \right) \right\}^{\frac{1}{2}} \sum_{t=1}^{T} \left\{ \boldsymbol{\psi}_{f_1}^{x} \left(\frac{\boldsymbol{Z}_t}{\sigma} \right) - \boldsymbol{\psi}_{f_1}^{x} \left(\frac{\boldsymbol{Z}_t}{(\sigma^2+v)^{1/2}} \right) \right\} \Bigg]^2 dz \\ & \leq D_2 \left(A_{21} + A_{22} \right), \end{aligned} \tag{19}$$

where

$$\begin{aligned} A_{21} = \int\limits_{\mathcal{R}} \Bigg[& \Bigg(\left(\left\{ \frac{1}{(\sigma^2+v)^{\frac{nT}{2}}} \prod_{i=1}^{n} \prod_{t=1}^{T} f_1 \left(\frac{z_{i,t}}{(\sigma^2+v)^{\frac{1}{2}}} \right) \right\}^{\frac{1}{2}} \sum_{t=1}^{T} \boldsymbol{\psi}_{f_1}^{x} \left(\frac{\boldsymbol{Z}_t}{(\sigma^2+v)^{1/2}} \right) \right. \\ & \left. - \left\{ \frac{1}{\sigma^{nT}} \prod_{i=1}^{n} \prod_{t=1}^{T} f_1 \left(\frac{z_{i,t}}{\sigma} \right) \right\}^{\frac{1}{2}} \sum_{t=1}^{T} \boldsymbol{\psi}_{f_1}^{x} \left(\frac{\boldsymbol{Z}_t}{\sigma} \right) \right) \Bigg) \Bigg]^2 dz, \end{aligned}$$

and

$$A_{22} = \int\limits_{\mathcal{R}} \left[\left\{ \frac{1}{\sigma^{nT}} \prod_{i=1}^{n} \prod_{t=1}^{T} f_1 \left(\frac{z_{i,t}}{\sigma} \right) \right\}^{\frac{1}{2}} \sum_{t=1}^{T} \left\{ \boldsymbol{\psi}_{f_1}^{x} \left(\frac{\boldsymbol{Z}_t}{\sigma} \right) - \boldsymbol{\psi}_{f_1}^{x} \left(\frac{\boldsymbol{Z}_t}{(\sigma^2+v)^{1/2}} \right) \right\} \right]^2 dz.$$

Let $e^u = \frac{z}{\sigma}$, then A_{21} and A_{22} take the form

$$\begin{aligned} A_{21} = \int \Bigg[& e^{nT \times \left(\frac{u}{2} - \frac{1}{4} ln\left(1+\frac{v}{\sigma^2}\right) \right)} f_1^{\frac{1}{2}} \left(e^{u - \frac{1}{2} ln\left(1+\frac{v}{\sigma^2}\right)} \right) \sum_{t=1}^{T} \boldsymbol{\psi}_{f_1}^{x} \left(e^{\boldsymbol{u}_t - \frac{1}{2} ln\left(1+\frac{v}{\sigma^2}\right)} \right) \\ & - e^{\frac{nTu}{2}} f_1^{\frac{1}{2}} \left(e^{u} \right) \sum_{t=1}^{T} \boldsymbol{\psi}_{f_1}^{x} \left(e^{\boldsymbol{u}_t} \right) \Bigg]^2 d\boldsymbol{u}, \end{aligned}$$

and

$$A_{22} = \int \left[e^{\frac{nTu}{2}} f_1^{\frac{1}{2}}\left(e^{u}\right) \sum_{t=1}^{T} \left\{ \boldsymbol{\psi}_{f_1}^{x}\left(e^{\boldsymbol{u}_t}\right) - \boldsymbol{\psi}_{f_1}^{x}\left(e^{\boldsymbol{u}_t - \frac{1}{2} ln\left(1+\frac{v}{\sigma^2}\right)}\right) \right\} \right]^2 d\boldsymbol{u},$$

respectively. Turning to A_3, we obtain

$$A_3 = \frac{1}{\sigma^2} \int e^{nTu} f_1\left(e^{u}\right) \left[\sum_{t=1}^{T} \left\{ \boldsymbol{\psi}_{f_1}^{x}\left(e^{\boldsymbol{u}_t - \frac{1}{2} ln\left(1+\frac{v}{\sigma^2}\right)}\right) - \boldsymbol{\psi}_{f_1}^{x}\left(e^{\boldsymbol{u}_t}\right) \right\} \right]^2 d\boldsymbol{u}.$$

Since $e^{nTu/2}$, $e^{nTu/2} f_1^{1/2}(e^u) \boldsymbol{\psi}_{f_1}^{x}(e^u)$, and $\boldsymbol{\psi}_{f_1}^{x}(e^u)$ are square integrable, A_{21}, A_{22}, hence A_2 and A_3, hence also S_{σ^2}, are $o(1)$ as $v \to 0$. This completes the proof of part (iii). Therefore the proof of Proposition 1. ☐

Proof of Proposition 2
i) We have

$$\begin{aligned}
\boldsymbol{\Delta}_{f_1,4}^{(n)*}(\widehat{\boldsymbol{\theta}}_n) - \boldsymbol{\Delta}_{f_1,4}^{(n)*}(\boldsymbol{\theta}) = {} & \boldsymbol{\Delta}_{f_1,4}^{(n)}(\widehat{\boldsymbol{\theta}}_n) - \boldsymbol{\Delta}_{f_1,4}^{(n)}(\boldsymbol{\theta}) \\
& - \left(\mathbf{0}, \boldsymbol{\Gamma}_{f_1,24}^{(n)'}(\widehat{\boldsymbol{\theta}}_n), \boldsymbol{\Gamma}_{f_1,34}^{(n)'}(\widehat{\boldsymbol{\theta}}_n)\right) \begin{pmatrix} \boldsymbol{\Gamma}_{f_1,11}^{(n)}(\widehat{\boldsymbol{\theta}}_n) & \mathbf{0} & \boldsymbol{\Gamma}_{f_1,13}^{(n)}(\widehat{\boldsymbol{\theta}}_n) \\ \mathbf{0} & \boldsymbol{\Gamma}_{f_1,22}^{(n)}(\widehat{\boldsymbol{\theta}}_n) & \mathbf{0} \\ \boldsymbol{\Gamma}_{f_1,13}^{(n)'}(\widehat{\boldsymbol{\theta}}_n) & \mathbf{0} & \Gamma_{f_1,33}^{(n)}(\widehat{\boldsymbol{\theta}}_n) \end{pmatrix}^{-1} \times \\
& \begin{pmatrix} \boldsymbol{\Delta}_{f_1,1}^{(n)}(\widehat{\boldsymbol{\theta}}_n) - \boldsymbol{\Delta}_{f_1,1}^{(n)}(\boldsymbol{\theta}) \\ \boldsymbol{\Delta}_{f_1,2}^{(n)}(\widehat{\boldsymbol{\theta}}_n) - \boldsymbol{\Delta}_{f_1,2}^{(n)}(\boldsymbol{\theta}) \\ \Delta_{f_1,3}^{(n)}(\widehat{\boldsymbol{\theta}}_n) - \Delta_{f_1,3}^{(n)}(\boldsymbol{\theta}) \end{pmatrix} + \left[\left(\mathbf{0}, \boldsymbol{\Gamma}_{f_1,24}^{(n)'}(\widehat{\boldsymbol{\theta}}_n), \boldsymbol{\Gamma}_{f_1,34}^{(n)'}(\widehat{\boldsymbol{\theta}}_n)\right) \times \right. \\
& \begin{pmatrix} \boldsymbol{\Gamma}_{f_1,11}^{(n)}(\widehat{\boldsymbol{\theta}}_n) & \mathbf{0} & \boldsymbol{\Gamma}_{f_1,13}^{(n)}(\widehat{\boldsymbol{\theta}}_n) \\ \mathbf{0} & \boldsymbol{\Gamma}_{f_1,22}^{(n)}(\widehat{\boldsymbol{\theta}}_n) & \mathbf{0} \\ \boldsymbol{\Gamma}_{f_1,13}^{(n)'}(\widehat{\boldsymbol{\theta}}_n) & \mathbf{0} & \Gamma_{f_1,33}^{(n)}(\widehat{\boldsymbol{\theta}}_n) \end{pmatrix}^{-1} \\
& \left. - \left(\mathbf{0}, \boldsymbol{\Gamma}_{f_1,24}^{(n)'}(\boldsymbol{\theta}), \boldsymbol{\Gamma}_{f_1,34}^{(n)'}(\boldsymbol{\theta})\right) \begin{pmatrix} \boldsymbol{\Gamma}_{f_1,11}^{(n)}(\boldsymbol{\theta}) & \mathbf{0} & \boldsymbol{\Gamma}_{f_1,13}^{(n)}(\boldsymbol{\theta}) \\ \mathbf{0} & \boldsymbol{\Gamma}_{f_1,22}^{(n)}(\boldsymbol{\theta}) & \mathbf{0} \\ \boldsymbol{\Gamma}_{f_1,13}^{(n)'}(\boldsymbol{\theta}) & \mathbf{0} & \Gamma_{f_1,33}^{(n)}(\boldsymbol{\theta}) \end{pmatrix}^{-1} \right] \times \\
& \begin{pmatrix} \boldsymbol{\Delta}_{f_1,1}^{(n)}(\boldsymbol{\theta}) \\ \boldsymbol{\Delta}_{f_1,2}^{(n)}(\boldsymbol{\theta}) \\ \Delta_{f_1,3}^{(n)}(\boldsymbol{\theta}) \end{pmatrix}.
\end{aligned}$$

From asymptotic linearity, We have

$$\begin{aligned}
\Delta_{f_1}^{(n)}\left(\widehat{\boldsymbol{\theta}}_n\right) - \Delta_{f_1}^{(n)}(\boldsymbol{\theta}) &= -n^{\frac{1}{2}} \Gamma_{f_1}(\boldsymbol{\theta}) \nu^{(n)-1} \left(\widehat{\boldsymbol{\theta}}_n - \boldsymbol{\theta}\right) + o_{\mathbf{P}}(1) \\
&= -n^{\frac{1}{2}} \begin{bmatrix} \boldsymbol{\Gamma}_{f_1,11}^{(n)}(\boldsymbol{\theta}) & \mathbf{0} & \boldsymbol{\Gamma}_{f_1,13}^{(n)}(\boldsymbol{\theta}) & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\Gamma}_{f_1,22}^{(n)}(\boldsymbol{\theta}) & \mathbf{0} & \boldsymbol{\Gamma}_{f_1,24}^{(n)}(\boldsymbol{\theta}) \\ \boldsymbol{\Gamma}_{f_1,13}^{(n)'}(\boldsymbol{\theta}) & \mathbf{0} & \Gamma_{f_1,33}^{(n)}(\boldsymbol{\theta}) & \boldsymbol{\Gamma}_{f_1,34}^{(n)}(\boldsymbol{\theta}) \\ \mathbf{0} & \boldsymbol{\Gamma}_{f_1,24}^{(n)'}(\boldsymbol{\theta}) & \boldsymbol{\Gamma}_{f_1,34}^{(n)'}(\boldsymbol{\theta}) & \boldsymbol{\Gamma}_{f_1,44}^{(n)}(\boldsymbol{\theta}) \end{bmatrix} \begin{bmatrix} (\widehat{\boldsymbol{\mu}} - \boldsymbol{\mu}) \\ \mathbf{K}^{(n)-1}\left(\widehat{\boldsymbol{\beta}} - \boldsymbol{\beta}\right) \\ \widehat{\sigma}^2 - \sigma^2 \\ \mathbf{0} \end{bmatrix} + o_{\mathbf{P}}(1).
\end{aligned}$$

We obtain that

$$\mathbf{\Delta}^{(n)*}_{f_1,4}(\widehat{\boldsymbol{\theta}}_n) - \mathbf{\Delta}^{(n)*}_{f_1,4}(\boldsymbol{\theta}) = o_{\mathbf{P}}(1).$$

Since the mapping $\boldsymbol{\theta} \to \mathbf{\Gamma}^{(n)*}_{f_1,44}(\boldsymbol{\theta})$ is continuous, it follows that $T^{(n)}_{f_1}(\widehat{\theta}_n) = T^{(n)}_{f_1}(\theta) + o_{\mathbf{P}}(1)$.
ii) This follows directly from part (i). $\square$

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