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On the small-time behavior of stochastic logistic models

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Abstract In this paper we investigate the small-time behaviors of the solution to a stochastic logistic model. The obtained results allow us to estimate the number of individuals in the population and can be used to study stochastic prey-predator systems.

Keywords Stochastic logistic model, small-time behavior, predator-prey systems.

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1. Introduction

It is known that the logistic equation is one of the most important models in mathematical ecology. The aim of this paper is to provide some new contributions to stochastic logistic equations of the form

$$dX_t = \left(a(t)X_t - b(t)X_t^2\right) dt + \sigma(t)X_t dB_t,\ t \in [0, T], \tag{1}$$

with the initial condition $X_0 = x_0 > 0$, where a, b, σ are deterministic continuous functions and B is a standard Brownian motion.

The study of the model (1) has a long history. When a, b, σ are constants, the stability of solutions to (1) was studied by May in [11], the optimal harvesting plan was discussed by Alvarez and Shepp in [1], etc. In recent years, various aspects of stochastic logistic models have been continued studying by many authors (see [2, 3, 4, 5, 9, 10, 13] and references therein).

From ecological point of view, if the intensity of noises is large enough, the population size can be changed even when the time is small. Motivations of this paper come from the following important and interesting question: If the number of individuals in the population at the present time ($t_0 = 0$) is x_0, what can we say about the number of individuals at time $t \simeq t_0$? In order to answer this question, one need to investigate the small-time behavior of solutions to (1). However, in the most of papers related to logistic models, the authors only focus on the long-time behaviors of solutions.

Since the solution is continuous, we always have $X_t \to x_0$ as $t \to 0$. Our purpose in the present paper is to exactly describe the rate of this convergence and hence, give an answer to the above question. More specifically, we obtain the following new contributions

(i) $\frac{E[X_t^n] - x_0^n}{t} = n[a(0) + \frac{1}{2}(n-1)\sigma^2(0)]x_0^n - b(0)x_0^{n+1} + O(t),$

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(ii) $\frac{X_t^n - x_0^n}{\sqrt{t}}$ converges in distribution to a centered Gaussian random variable, where n is a real number and $n > 1$.

In order to prove the property (i), we will use the technique of measure transformations via Girsanov's theorem. Meanwhile, to obtain Gaussian convergence (property (ii)) we will introduce an interpolation method which allows us to estimate the distance between $\frac{X_t^n - x_0^n}{\sqrt{t}}$ and a centered Gaussian random variable. Then, the convergence follows from the classical results in probability theory.

The rest of this paper is organized as follows. Section 2 contains the main results of this paper and an application to predator-prey systems. The conclusion and some remarks are given in Section 3.

2. The main results

Throughout this section, we assume that

$$b(t) \geq 0 \text{ for all } t \in [0, T],$$

which ensures the existence and uniqueness of solutions. In fact, the explicit solution of (1) is given by (see, e.g. page 125 in [7])

$$X_t = \frac{e^{\int_0^t [a(s) - \frac{1}{2}\sigma^2(s)]ds + \int_0^t \sigma(s)dB_s}}{\left(x_0^{-1} + \int_0^t b(s) e^{\int_0^s [a(u) - \frac{1}{2}\sigma^2(u)]du + \int_0^s \sigma(u)dB_u} ds\right)}, \quad t \in [0, T]. \tag{2}$$

We first study the small-time behavior for the moments of the solution.

Theorem 2.1
For any real number $n > 1$, we have

$$E[X_t^n] = x_0^n + c_1 t + O(t^2)$$

as $t \to 0$, where $c_1 := n[a(0) + \frac{1}{2}(n-1)\sigma^2(0)]x_0^n - b(0)x_0^{n+1}$. Consequently,

$$\lim_{t \to 0} \frac{E[X_t^n] - x_0^n}{t} = c_1.$$

Proof
We have

$$X_t^n = \frac{e^{\int_0^t n[a(s) - \frac{1}{2}\sigma^2(s)]ds + \int_0^t n\sigma(s)dB_s}}{\left(x_0^{-1} + \int_0^t b(s) e^{\int_0^s [a(u) - \frac{1}{2}\sigma^2(u)]du + \int_0^s \sigma(u)dB_u} ds\right)^n}. \tag{3}$$

By Girsanov's theorem (see, e.g. [6]), the stochastic process

$$W_t := B_t - \int_0^t n\sigma(s)ds$$

is a standard Brownian motion under the probability measure Q, where Q is defined as

$$\frac{dQ}{dP} = \exp\left(-\frac{1}{2}\int_0^T n^2\sigma^2(s)ds + \int_0^T n\sigma(s)dB_s\right).$$

We have

$$\begin{aligned} E[X_t^n] &= E_Q\left[\frac{e^{\int_0^t n[a(s)+\frac{1}{2}(n-1)\sigma^2(s)]ds}}{\left(x_0^{-1}+\int_0^t b(s)e^{\int_0^s [a(u)+(n-\frac{1}{2})\sigma^2(u)]du+\int_0^s \sigma(u)dW_u}ds\right)^n}\right] \\ &= e^{\int_0^t n[a(s)+\frac{1}{2}(n-1)\sigma^2(s)]ds} F(n,t), \end{aligned} \tag{4}$$

where the function F defined on $(1,\infty)\times[0,\infty)$ by

$$F(p,t) = E_Q\left[\frac{1}{\left(x_0^{-1}+\int_0^t b(s)e^{\int_0^s [a(u)+(p-\frac{1}{2})\sigma^2(u)]du+\int_0^s \sigma(u)dW_u}ds\right)^p}\right].$$

By the straightforward computations we obtain

$$\begin{aligned} &\int_0^t \frac{b(v)e^{\int_0^v [a(u)+(p-\frac{1}{2})\sigma^2(u)]du+\int_0^v \sigma(u)dW_u}}{\left(x_0^{-1}+\int_0^v b(s)e^{\int_0^s [a(u)+(p-\frac{1}{2})\sigma^2(u)]du+\int_0^s \sigma(u)dW_u}ds\right)^{p+1}}dv \\ &\qquad = x_0^p - \frac{1}{\left(x_0^{-1}+\int_0^t b(s)e^{\int_0^s [a(u)+(p-\frac{1}{2})\sigma^2(s)]du+\int_0^s \sigma(u)dW_u}ds\right)^p}. \end{aligned}$$

As a consequence,

$$F(p,t) = x_0^p - \int_0^t E_Q\left[\frac{b(v)e^{\int_0^v [a(u)+(p-\frac{1}{2})\sigma^2(u)]du+\int_0^v \sigma(u)dW_u}}{\left(x_0^{-1}+\int_0^v b(s)e^{\int_0^s [a(u)+(p-\frac{1}{2})\sigma^2(u)]du+\int_0^s \sigma(u)dW_u}ds\right)^{p+1}}\right]dv. \tag{5}$$

We now define the probability measure $\widetilde{Q}$ by

$$\frac{d\widetilde{Q}}{dQ} = \exp\left(-\frac{1}{2}\int_0^T \sigma^2(s)ds + \int_0^T \sigma(s)dW_s\right).$$

Under $\widetilde{Q}$, the stochastic process

$$\widetilde{W_t} = W_t - \int_0^t \sigma(s)ds$$

is a standard Brownian motion and we have

$$E_Q\left[\frac{b(v)e^{\int_0^v[a(u)+(p-\frac{1}{2})\sigma^2(u)]du+\int_0^v\sigma(u)dW_u}}{\left(x_0^{-1}+\int_0^v b(s)e^{\int_0^s[a(u)+(p-\frac{1}{2})\sigma^2(u)]du+\int_0^s\sigma(u)dW_u}ds\right)^{n+1}}\right]$$
$$=E_{\widetilde{Q}}\left[\frac{b(v)e^{\int_0^v[a(u)+p\sigma^2(u)]du}}{\left(x_0^{-1}+\int_0^v b(s)e^{\int_0^s[a(u)+(p+\frac{1}{2})\sigma^2(u)]du+\int_0^s\sigma(u)d\widetilde{W}_u}ds\right)^{p+1}}\right]$$
$$=b(v)e^{\int_0^v[a(u)+p\sigma^2(u)]du}F(p+1,v).$$

Recalling (5), we get the following relation

$$F(p,t)=x_0^p-\int_0^t b(v)e^{\int_0^v[a(u)+p\sigma^2(u)]du}F(p+1,v)dv. \tag{6}$$

By its definition, the function F is continuous and

$$F(p,0)=x_0^p \text{ for all } p>1.$$

Moreover, it follows from (6) that F is differentiable in t. We have

$$F_t'(p,t)=-b(t)e^{\int_0^t[a(u)+p\sigma^2(u)]du}F(p+1,t)$$

and

$$F_t'(p,0)=-b(0)F(p+1,0)=-b(0)x_0^{p+1} \text{ for all } p>1.$$

From (4) we deduce

$$\frac{dE[X_t^n]}{dt}=n[a(t)+\frac{1}{2}(n-1)\sigma^2(t)]e^{\int_0^t n[a(s)+\frac{1}{2}(n-1)\sigma^2(s)]ds}F(n,t)+e^{\int_0^t n[a(s)+\frac{1}{2}(n-1)\sigma^2(s)]ds}F_t'(n,t),$$

which gives us

$$\frac{dE[X_t^n]}{dt}\Big|_{t=0}=n[a(0)+\frac{1}{2}(n-1)\sigma^2(0)]x_0^n-b(0)x_0^{n+1}.$$

By using the Taylor expansion, we can obtain

$$E[X_t^n]=x_0^n+\frac{dE[X_t^n]}{dt}\Big|_{t=0}t+O(t^2)$$
$$=x_0^n+\left(n[a(0)+\frac{1}{2}(n-1)\sigma^2(0)]x_0^n-b(0)x_0^{n+1}\right)t+O(t^2).$$

So we can finish the proof. □

Remark 2.1. If the coefficients a, b and σ are differentiable functions, then F is a differentiable function of second order. By simple calculations we have

$$F_t''(p,0)=-b'(0)x_0^{p+1}-b(0)[a(0)+p\sigma^2(0)]x_0^{p+1}+b^2(0)x_0^{p+2} \text{ for all } p>1$$

and

$$\frac{d^2E[X_t^n]}{dt^2}\Big|_{t=0} = n[a'(0) + \frac{1}{2}(n-1)\sigma'^2(0)]x_0^p + n^2[a(0) + \frac{1}{2}(n-1)\sigma^2(0)]^2 x_0^p$$
$$- 2n[a(0) + \frac{1}{2}(n-1)\sigma^2(0)]b(0)x_0^{p+1} - b'(0)x_0^{p+1} - b(0)[a(0) + p\sigma^2(0)]x_0^{p+1} + b^2(0)x_0^{p+2}$$
$$:= c_2.$$

Hence, we can obtain the Taylor expansion of second order as follows

$$E[X_t^n] = x_0^n + c_1 t + c_2 t^2 + O(t^3).$$

Denote by $\mathcal{C}_b^2$ the set of all real-valued bounded functions with bounded derivatives up to second order. We need the following fundamental result (see, e.g. Remark 2.16 in [12]) to establish the small-time behavior of the solution in distribution.

Lemma 2.1
If the sequence $\{F_n\}_{n\geq 1}$ is such that $|Eh(F_n) - Eh(F)| \to 0$ as $n \to \infty$ for every $h \in \mathcal{C}_b^2$, then $F_n \to F$ in distribution.

Theorem 2.2
Let X_t be the solution to the equation (1). It holds that

$$\frac{X_t^n - x_0^n}{\sqrt{t}} \to \mathcal{N}(0, n^2\sigma^2(0)x_0^{2n}),$$

in distribution as $t \to 0$, where $\mathcal{N}(0, n^2\sigma^2(0)x_0^{2n})$ is a normal random variable with mean 0 and variance $n^2\sigma^2(0)x_0^{2n}$.

Proof
We set $Y_t = X_t^n - x_0^n$ and use Itô's formula to get

$$dY_t = n\left(\left(a(t) + \frac{1}{2}(n-1)\sigma^2(t)\right)X_t^n - b(t)X_t^{n+1}\right)dt + n\sigma(t)X_t^n dB_t,\ t \in [0,T].$$

Thanks to Lemma 2.1, we need to show that

$$\left|Eh\left(\frac{Y_t}{\sqrt{t}}\right) - Eh\left(\mathcal{N}(0, n^2\sigma^2(0)x_0^{2n})\right)\right| \to 0 \tag{7}$$

as $t \to 0$ for every $h \in \mathcal{C}_b^2$.

Let $\varphi(z)$ be the density function of standard normal random variable

$$\varphi(z) = \frac{1}{\sqrt{2\pi}}e^{-\frac{z^2}{2}}.$$

Given a function $h \in \mathcal{C}_b^2$, consider the interpolation function $H : [0,t] \times \mathbb{R} \longrightarrow \mathbb{R}$ which is defined by

$$H(s,y) = \int_{-\infty}^{\infty} h\left(\frac{y}{\sqrt{t}} + n\sigma(0)x_0^n\sqrt{1 - \frac{s}{t}}\,z\right)\varphi(z)dz.$$

Obviously, we have

$$EH(t, Y_t) = Eh\left(\frac{Y_t}{\sqrt{t}}\right) \text{ and } EH(0, Y_0) = Eh\left(\mathcal{N}(0, n^2\sigma^2(0)x_0^{2n})\right). \tag{8}$$

By straightforward calculations we obtain

$$\frac{\partial}{\partial y}H(s,y) = \frac{1}{\sqrt{t}}\int_{-\infty}^{\infty} h'\left(\frac{y}{\sqrt{t}} + n\sigma(0)x_0^n\sqrt{1-\frac{s}{t}}\,z\right)\varphi(z)dz$$

and

$$\frac{\partial^2}{\partial y^2}H(s,y) = \frac{1}{t}\int_{-\infty}^{\infty} h''\left(\frac{y}{\sqrt{t}} + n\sigma(0)x_0^n\sqrt{1-\frac{s}{t}}\,z\right)\varphi(z)dz$$

As a consequence, we obtain the following estimates

$$\frac{\partial}{\partial y}H(s,y) \leq \frac{\|h'\|}{\sqrt{t}} \text{ and } \frac{\partial^2}{\partial y^2}H(s,y) \leq \frac{\|h''\|}{t}, \tag{9}$$

where $\|h'\| = \sup_{x\in\mathbb{R}}|h'(x)|$ and $\|h''\| = \sup_{x\in\mathbb{R}}|h''(x)|$. On the other hand, by the integration by parts formula combined with the fact $\varphi'(z) = -z\varphi(z)$ we get

$$\begin{aligned}
\frac{\partial}{\partial s}H(s,y) &= -\frac{n\sigma(0)x_0^n}{2t\sqrt{1-\frac{s}{t}}}\int_{-\infty}^{\infty} h'\left(\frac{y}{\sqrt{t}} + n\sigma(0)x_0^n\sqrt{1-\frac{s}{t}}\,z\right)z\varphi(z)dz \\
&= \frac{n\sigma(0)x_0^n}{2t\sqrt{1-\frac{s}{t}}}\int_{-\infty}^{\infty} h'\left(\frac{y}{\sqrt{t}} + n\sigma(0)x_0^n\sqrt{1-\frac{s}{t}}\,z\right)\varphi'(z)dz \\
&= \frac{n\sigma(0)x_0^n}{2t\sqrt{1-\frac{s}{t}}}\int_{-\infty}^{\infty} h'\left(\frac{y}{\sqrt{t}} + n\sigma(0)x_0^n\sqrt{1-\frac{s}{t}}\,z\right)d\varphi(z) \\
&= -\frac{n^2\sigma^2(0)x_0^{2n}}{2t}\int_{-\infty}^{\infty} h''\left(\frac{y}{\sqrt{t}} + n\sigma(0)x_0^n\sqrt{1-\frac{s}{t}}\,z\right)\varphi(z)dz \\
&= -\frac{n^2\sigma^2(0)x_0^{2n}}{2}\frac{\partial^2}{\partial y^2}H(s,y).
\end{aligned} \tag{10}$$

Applying Itô's formula to $H(s,Y_s)$ yields, for all $s\in[0,t]$

$$\begin{aligned}
H(s,Y_s) - H(0,Y_0) &= \int_0^s\left(\frac{\partial}{\partial u}H(u,Y_u) + \frac{\partial}{\partial y}H(u,Y_u)f(u,X_u)\right)du \\
&\quad + \int_0^s \frac{\partial}{\partial y}H(u,Y_u)n\sigma(u)X_u^n dB_u + \frac{1}{2}\int_0^s\frac{\partial^2}{\partial y^2}H(u,Y_u)n^2\sigma^2(u)X_u^{2n}du,
\end{aligned} \tag{11}$$

where, for the simplicity, we put

$$f(u,X_u) := n\left(\left(a(u) + \frac{1}{2}(n-1)\sigma^2(u)\right)X_u^n - b(u)X_u^{n+1}\right).$$

Inserting the relation (10) into (11) we deduce

$$\begin{aligned}
EH(t,Y_t) - EH(0,Y_0) &= E\int_0^t \frac{\partial}{\partial y}H(u,Y_u)f(u,X_u)du \\
&\quad + \frac{1}{2}E\int_0^t\frac{\partial^2}{\partial y^2}H(u,Y_u)n^2[\sigma^2(u)X_u^{2n} - \sigma^2(0)x_0^{2n}]du.
\end{aligned}$$

This, togherther with (8) and the estimates (9), gives us

$$\left|Eh\left(\frac{Y_t}{\sqrt{t}}\right) - Eh\left(\mathcal{N}(0, n^2\sigma^2(0)x_0^{2n})\right)\right| \leq \frac{\|h'\|}{\sqrt{t}}\int_0^t E|f(u,X_u)|du + \frac{n^2\|h''\|}{2t}\int_0^t E|\sigma^2(u)X_u^{2n} - \sigma^2(0)x_0^{2n}|du. \quad (12)$$

It is easy to see from (2) that, for any $p > 0$

$$\begin{aligned} E|X_t|^p &\leq x_0^p E\left[e^{\int_0^t p[a(s)-\frac{1}{2}\sigma^2(s)]ds + \int_0^t p\sigma(s)dB_s}\right] \\ &= x_0^p e^{\int_0^t p[a(s)-\frac{1}{2}\sigma^2(s)]ds + \frac{1}{2}\int_0^t p^2\sigma^2(s)ds} \\ &\leq x_0^p e^{\int_0^T p|a(s)-\frac{1}{2}\sigma^2(s)|ds + \frac{1}{2}\int_0^T p^2\sigma^2(s)ds}, \ t \in [0,T], \end{aligned}$$

and hence, there exists a positive constant C_T^1 such that

$$\sup_{0\leq u\leq T} E|f(u,X_u)| \leq C_T^1.$$

The above estimate points out that

$$\lim_{t\to 0}\frac{\|h'\|}{\sqrt{t}}\int_0^t E|f(u,X_u)|du = 0. \quad (13)$$

Moreover, by the triangle inequality

$$\frac{\|h'\|n^2}{2t}\int_0^t E|\sigma^2(u)X_u^{2n} - \sigma^2(0)x_0^{2n}|du \leq \frac{\|h'\|n^2}{2t}\int_0^t \sigma^2(u)E|X_u^{2n} - x_0^{2n}|du + \frac{\|h'\|n^2}{2t}\int_0^t |\sigma^2(u) - \sigma^2(0)|x_0^{2n}du. \quad (14)$$

Since σ is a continuous function, this implies

$$\lim_{t\to 0}\frac{\|h'\|n^2}{t}\int_0^t |\sigma^2(u) - \sigma^2(0)|x_0^{2n}du = \|h'\|n^2x_0^{2n}\frac{d\left(\int_0^t|\sigma^2(u)-\sigma^2(0)|du\right)}{dt}\Big|_{t=0} = 0. \quad (15)$$

Using Itô's formula and Burkholder-Davis-Gundy inequality we get

$$X_t^{2n} = x_0^{2n} + \int_0^t 2n\left(\left(a(s) + \frac{1}{2}(2n-1)\sigma^2(s)\right)X_s^n - b(s)X_s^{2n+1}\right)ds + \int_0^t 2n\sigma(s)X_s^{2n}dB_s$$

and

$$\begin{aligned} E|X_t^{2n} - x_0^{2n}| &\leq \int_0^t 2nE\left|\left(a(s) + \frac{1}{2}(2n-1)\sigma^2(s)\right)X_s^n - b(s)X_s^{2n+1}\right|ds \\ &\quad + \left(\int_0^t 4n^2\sigma^2(s)E|X_s^{4n}|ds\right)^{\frac{1}{2}} \\ &\leq C_T^2 t + \left(C_T^3 t\right)^{\frac{1}{2}} \leq C_T^4\sqrt{t}, \ t\in[0,T], \end{aligned}$$

where $C_T^i, i = 2, 3, 4$ are finite positive constants. We therefore obtain

$$\frac{\|h'\|n^2}{2t}\int_0^t \sigma^2(u)E|X_u^{2n} - x_0^{2n}|du \le \frac{\|h'\|n^2\|\sigma\|^2 C_T^4}{3}\sqrt{t},$$

where $\|\sigma\| = \sup_{t\in[0,T]} |\sigma(t)|$. So it holds that

$$\lim_{t\to 0}\frac{\|h'\|n^2}{2t}\int_0^t \sigma^2(u)E|X_u^{2n} - x_0^{2n}|du = 0. \tag{16}$$

Combining (12)-(16) yields the claim (7). So we can finish the proof. □

Now we are in a position to give an answer to the question mentioned in introduction.

Corollary 2.1
Let X_t be the solution to the equation (1) and $\alpha \in (0, 1)$. When t is small, we have

$$X_t \in \left(x_0 - z_{\alpha/2}\sigma(0)x_0\sqrt{t}, x_0 + z_{\alpha/2}\sigma(0)x_0\sqrt{t}\right), \tag{17}$$

with confidence level of approximately $1 - \alpha$, where $z_{\alpha/2}$ is defined by

$$P(\mathcal{N}(0, 1) < z_{\alpha/2}) = 1 - \alpha/2.$$

Proof
It follows from Theorem 2.2 with $n = 1$ that, when t is small, the random variable $\frac{X_t - x_0}{\sqrt{\sigma^2(0)x_0^2 t}}$ is approximated by the standard normal random variable $\mathcal{N}(0, 1)$. Hence,

$$\begin{aligned}
&P\left(x_0 - z_{\alpha/2}\sigma(0)x_0\sqrt{t} < X_t < x_0 + z_{\alpha/2}\sigma(0)x_0\sqrt{t}\right)\\
&= P\left(-z_{\alpha/2} < \frac{X_t - x_0}{\sqrt{\sigma^2(0)x_0^2 t}} < z_{\alpha/2}\right)\\
&\simeq P\left(-z_{\alpha/2} < \mathcal{N}(0, 1) < z_{\alpha/2}\right)\\
&= 1 - \alpha.
\end{aligned}$$

So the proof is complete. □

Remark 2.2. If we make an observation about the system at t_0 and find out that the number of individuals in the population is x_{t_0}. Then, by repeating the proof of Theorem 2.2 and Corollary 2.1, we can obtain

$$\frac{X_t^n - x_{t_0}^n}{\sqrt{t - t_0}} \to \mathcal{N}(0, n^2\sigma^2(0)x_{t_0}^{2n}),\ t \to t_0$$

and

$$X_t \in \left(x_{t_0} - z_{\alpha/2}\sigma(t_0)x_{t_0}\sqrt{t - t_0}, x_{t_0} + z_{\alpha/2}\sigma(t_0)x_{t_0}\sqrt{t - t_0}\right),\ t \simeq t_0. \tag{18}$$

Thus, to estimate the number of individuals in a near future, one only need to know the information about the number of individuals and intensity of noises at the presence time t_0.

We end up this section with an application to predator-prey systems. Let us consider a stochastic non-autonomous predator-prey system with Beddington-DeAngelis functional response of the form

$$\begin{cases} dx(t) = x(t)\left(a_1(t) - b_1(t)x(t) - \frac{c_1(t)y(t)}{m_1(t)+m_2(t)x(t)+m_3(t)y(t)}\right)dt + \sigma_1(t)x(t)dB_1(t) \\ dy(t) = y(t)\left(-a_2(t) - b_2(t)y(t) + \frac{c_2(t)x(t)}{m_1(t)+m_2(t)x(t)+m_3(t)y(t)}\right)dt + \sigma_2 y(t)dB_2(t), \end{cases} \tag{19}$$

where the coefficients are continuous bounded nonnegative functions, B_1 and B_2 are independent Brownian motions.

The existence of positive solutions to the system (19) and its long-time behaviors have been recently discussed in [8]. Our aim here is to establish the small-time behavior of the solutions. We have

$$\begin{aligned} x(t)\left(a_1(t) - b_1(t)x(t) - \frac{c_1(t)}{m_3(t)}\right) \\ \leq x(t)\left(a_1(t) - b_1(t)x(t) - \frac{c_1(t)y(t)}{m_1(t)+m_2(t)x(t)+m_3(t)y(t)}\right) \\ \leq x(t)\left(a_1(t) - b_1(t)x(t)\right) \ a.s. \end{aligned}$$

By the comparison theorem for stochastic differential equations we obtain

$$X_1(t) \leq x(t) \leq X_2(t) \ a.s.,$$

where $X_1(t), X_2(t)$ are the solutions to the following logistic equations

$$\begin{aligned} dX_1(t) &= X_1(t)\left(a_1(t) - \frac{c_1(t)}{m_3(t)} - b_1(t)X_1(t)\right)dt + \sigma_1(t)X_1(t)dB_1(t), \\ dX_2(t) &= X_2(t)\left(a_1(t) - b_1(t)X_2(t)\right)dt + \sigma_1(t)X_2(t)dB_1(t). \end{aligned}$$

Similarly, we also have

$$Y_1(t) \leq y(t) \leq Y_2(t) \ a.s,$$

where $Y_1(t), Y_2(t)$ satisfy the following equations

$$\begin{aligned} dY_1(t) &= Y_1(t)\left(-a_2(t) - b_2(t)Y_1(t)\right)dt + \sigma_2 Y_1(t)dB_2(t), \\ dY_2(t) &= Y_2(t)\left(-a_2(t) + \frac{c_2(t)}{m_2(t)} - b_2(t)Y_2(t)\right)dt + \sigma_2 Y_2(t)dB_2(t). \end{aligned}$$

The following theorem follows directly from Theorem 2.2 and Squeeze principle.

Theorem 2.3
We have the following convergence in distribution as $t \to 0$

$$\frac{x^n(t) - x_0^n}{\sqrt{t}} \to \mathcal{N}(0, n^2\sigma_1^2(0)x_0^{2n}),$$

$$\frac{y^n(t) - y_0^n}{\sqrt{t}} \to \mathcal{N}(0, n^2\sigma_2^2(0)y_0^{2n}).$$

3. Conclusion

The study of the stochastic logistic models has a long history. However, the results related to the small-time behaviors of the system are scarce. Our obtained result can be considered the first attempt to provide such behaviors. In this sense, we partly enrich the knowledge of the theory of the stochastic logistic models. In particular, the estimate formulas (17) and (18) are very useful in the circumstances where we have no any information about the trend coefficients $a(t)$ and $b(t)$.

We also note the the method used in the proof of Theorem 2.2 can be applied to the other population models. For example, the following nonlinear version of logistic models has been discussed in [13]

$$dN_t = \left(bN_t - aN_t^2\right)dt + \sigma\left(bN_t - aN_t^2\right)dB_t,$$

the initial condition $N_0 = x \in (0, \frac{b}{a})$ and a, b, σ are constants. For this model, one can verify that

$$\frac{N_t^n - x^n}{\sqrt{t}} \to \mathcal{N}(0, n^2\sigma^2(bx - ax^2)^2),\ t \to 0.$$

We leave the detailed computations to the reader.

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