Exponential Stability of a Transmission Problem with History and Delay

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Abstract In this paper, we consider a transmission problem in the presence of history and delay terms. Under appropriate assumptions, we prove well-posedness by using the semigroup theory. Our stability estimate proves that the unique dissipation given by the history term is strong enough to stabilize exponentially the system in presence of delay by introducing a suitable Lyaponov functional.

Keywords Wave equation, Transmission problem, Past history, Delay term.

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1. Introduction

In this paper we study the following transmission system with a past history and a delay term

\[
\begin{align*}
    u_{tt}(x,t) - au_{xx}(x,t) + \int_0^\infty g(s)u_{xx}(x,t-s)ds \\
    + \mu u_t(x,t-\tau) &= 0, \quad (x,t) \in \Omega \times (0, +\infty), \\
    v_{tt}(x,t) - bv_{xx}(x,t) &= 0, \quad (x,t) \in (L_1, L_2) \times (0, +\infty),
\end{align*}
\]

Under the boundary and transmission conditions

\[
\begin{align*}
    u(0,t) &= u(L_3,t) = 0, \\
    u(L_i,t) &= v(L_i,t), \quad i = 1, 2, \\
    a u_x(L_i,t) - \int_0^\infty g(s)u_x(L_i,t-s)ds &= b v_x(L_i,t), \quad i = 1, 2,
\end{align*}
\]

and the initial conditions

\[
\begin{align*}
    u(x, -t) &= u_0(x,t), \quad u_t(x, 0) = u_1(x), \quad x \in \Omega, \\
    u_t(x, t-\tau) &= f_0(x, t-\tau), \quad x \in \Omega, \quad t \in (0, \tau), \\
    v(x, 0) &= v_0(x), \quad v_t(x, 0) = v_1(x), \quad x \in (L_1, L_2),
\end{align*}
\]

where \(0 < L_1 < L_2 < L_3\), \(\Omega = [0, L_1] \cup [L_2, L_3]\), \(a, \mu, b\) are positive constants, \(u_0\) is given history, and \(\tau > 0\) is the delay.

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Transmission problems arise in several applications of physics and biology. We note that problem (1)-(3) is related to the wave propagation over a body which consists of two different type of materials: the elastic part and the viscoelastic part that has the past history and time delay effect.

For wave equations with various dissipations, many results concerning stabilization of solutions have been proved. Recently, wave equations with viscoelastic damping have been investigated by many authors, see [2, 4, 3, 9, 8, 10, 16, 18] and the references therein. It is showed that the dissipation produced by the viscoelastic part can produce the decay of the solution. For example, A. Guesmia [6] studied the equation

$$u_{tt} - Au + \int_0^\infty g(t)Au(t - s)ds + \mu u(t - \tau) = 0, \text{ in } \Omega \times (0, \infty),$$

and under the condition:

$$\exists \delta > 0, \quad g'(s) \leq -\delta g(s) \quad \forall s \in \mathbb{R}^+$$

the authors showed the exponential decay.

Messaoudi [12] investigated the following viscoelastic equation:

$$u_{tt} - \Delta u + \int_0^t g(t)\Delta u(t - s)ds = 0, \text{ in } \Omega \times (0, \infty),$$

in a bounded domain, and established a more general decay result, from which the usual exponential and polynomial decay rates are only special cases.

In [7] the authors examined a system of wave equations with a linear boundary damping term with a delay:

$$u_{tt}(x,t) - au_{xx}(x,t) + \int_0^\infty g(s)u_{xx}(x,t - s)ds + \mu_1 u_t(x,t) + \mu_2 u_t(x,t - \tau) = 0, \quad (x,t) \in \Omega \times (0, +\infty),$$

$$v_{tt}(x,t) - bv_{xx}(x,t) = 0, \quad (x,t) \in (L_1, L_2) \times (0, +\infty),$$

and under the assumption

$$\mu_2 \leq \mu_1$$

they proved that the solution is exponentially stable. On the contrary, if (5) does not hold, they found a sequence of delays for which the corresponding solution of (4) will be unstable.

In [11], authors considered the equation

$$u_{tt}(x,t) - \Delta_x u(x,t) - \mu_1 \Delta_x u_t(x,t) - \mu_2 \Delta_x u_t(x,t - \tau) = 0,$$

and under the assumption

$$|\mu_2| \leq \mu_1,$$

they proved the well-posedness and the exponential decay of energy.

Recently, in [19] Yadav and Jiwari considered Burgers’-Fisher equation:

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} + au \frac{\partial u}{\partial x} + bu(1 - u) = 0, \quad (x,t) \in (0, T) \times \Omega,$$

the authors proved existence and uniqueness of solution. Furthermore, they also presented finite element analysis and approximation.

The paper is organized as follows. The well-posedness of the problem is analyzed in Section 2 using the semigroup theory. In Section 3, we prove the exponential decay of the energy when time goes to infinity.
2. Preliminaries and assumptions

We assume that the function $g$ satisfies the following:

**A1:** We assume that the function $g : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is of class $C^1$ satisfying:

$$g(0) > 0, \quad a - \int_0^\infty g(t)dt = a - g_0 = t > 0.$$  \hfill (7)

**A2:** There exists a positive constant $\delta$,

$$g'(s) \leq -\delta g(s) \quad \forall s \in \mathbb{R}^+,$$

\hfill (8)

As in [14], we introduce the variable

$$z(x, \rho, t) = u_t(x, t - \tau \rho), \quad (x, \rho, t) \in \Omega \times (0, 1) \times (0, \infty).$$

Then

$$\tau z_t(x, \rho, t) + z_{\rho}(x, \rho, t) = 0, \quad (x, \rho, t) \in \Omega \times (0, 1) \times (0, \infty).$$

Following the ideal in [5], we set

$$\eta^I(x, s) = u(x, t) - u(x, t - s), \quad (x, t, s) \in \Omega \times \mathbb{R}_+ \times \mathbb{R}_+.$$  \hfill (9)

Then

$$\eta^I_t(x, s) + \eta^I_s(x, s) = u_t(x, t), \quad (x, t, s) \in \Omega \times \mathbb{R}_+ \times \mathbb{R}_+.$$

Thus, system (1) becomes

$$u_{tt}(x, t) - lu_{xx}(x, t) - \int_0^\infty g(s)\eta^I(x, s)ds + \mu z(x, 1, t) = 0, \quad (x, t) \in \Omega \times (0, +\infty),$$

$$v_{tt}(x, t) - bv_{xx}(x, t) = 0, \quad (x, t) \in (L_1, L_2) \times (0, +\infty),$$

$$\tau z_t(x, \rho, t) + z_{\rho}(x, \rho, t) = 0, \quad (x, \rho, t) \in \Omega \times (0, 1) \times (0, +\infty),$$

$$\eta^I_t(x, s) + \eta^I_s(x, s) = u_t(x, t), \quad (x, t, s) \in \Omega \times (0, +\infty) \times (0, +\infty),$$

the boundary and transmission conditions (2) become

$$u(0, t) = u(L_3, t) = 0,$$

$$u(L_i, t) = v(L_i, t), \quad i = 1, 2, \ t \in (0, +\infty),$$

$$lu_x(L_i, t) + \int_0^\infty g(s)\eta^I_x(L_i, s)ds = bv_x(L_i, t), \quad i = 1, 2, \ t \in (0, +\infty),$$

\hfill (11)

and the initial conditions (3) become

$$u(x, -t) = u_0(x, t), \quad u_t(x, 0) = u_1(x), \quad x \in \Omega,$$

$$z(x, 0, t) = u_t(x, t), \quad z(x, 1, t) = f_0(x, t - \tau), \quad (x, t) \in \Omega \times (0, +\infty),$$

$$v(x, 0) = v_0(x), \quad v_t(x, 0) = v_1(x), \quad x \in (L_1, L_2),$$

\hfill (12)

It is clear that

$$\eta^I(x, 0) = 0, \quad \text{for all } x > 0,$$

$$\eta^I(0, s) = \eta^I(L_3, s) = 0, \quad \text{for all } s > 0,$$

$$\eta^I(x, s) = \eta_0(s), \quad \text{for all } s > 0.$$  \hfill (13)

Let $V := (u, v, \varphi, \psi, z, \eta^I)^T$, then $V$ satisfies the problem

$$V_t = (\mathcal{A} + \mathcal{B})V(t), \quad t > 0,$$

$$V(0) = V_0,$$

\hfill (14)
where $V_0 := (u_0(\cdot, 0), v_0, u_1, v_1, f_0(\cdot, -\tau), \eta_0)^T$. The operator $\mathcal{A}$ and $\mathcal{B}$ are linear and defined by

$$
\mathcal{A} \begin{pmatrix} u \\ v \\ \varphi \\ \psi \\ z \\ w \end{pmatrix} = \begin{pmatrix} 
\varphi \\
\psi \\
lu_{xx} + \int_0^{+\infty} g(s)w_{xx}(s)ds - \mu \varphi - \mu z(\cdot, 1) \\
-bv_{xx} \\
-\frac{1}{\tau}z_{\rho} \\
-w_{\tau} + \varphi
\end{pmatrix}
$$

and

$$
\mathcal{B}(u, v, \varphi, \psi, z, \eta^T) = \mu(0, 0, 0, 0, 0)^T
$$

where

$$
X_* = \left\{ (u, v) \in H^1(\Omega) \times H^1(L_1, L_2) : u(0, t) = u(L_3, t) = 0, u(L_i, t) = v(L_i, t), 
lu_x(L_i, t) + \int_0^{+\infty} g(s)\eta_x^2(L_i, s)ds = bv_x(L_i, t), i = 1, 2 \right\}
$$

and $L^2_g(\mathbb{R}_+, H^1(\Omega))$ denotes the Hilbert space of $H^1$-valued functions on $\mathbb{R}_+$, endowed with the inner product

$$
(\varphi, \psi)_{L^2_g(\mathbb{R}_+, H^1(\Omega))} = \int_\Omega \int_0^{+\infty} g(s)\varphi_x(s)\psi_x(s)dsdx.
$$

Set

$$
V = (u, v, \varphi, \psi, z, w)^T, \quad \bar{V} = (\bar{u}, \bar{v}, \bar{\varphi}, \bar{\psi}, \bar{z}, \bar{w})^T.
$$

We define the inner product in the energy space $\mathcal{H}$,

$$
\langle V, V \rangle_\mathcal{H} = \int_\Omega \varphi \bar{\varphi}dx + \int_{L_1}^{L_2} \psi \bar{\psi}dx + \int_\Omega lu_x \bar{u}_x dx + \int_{L_1}^{L_2} bv_x \bar{v}_x dx + \int_0^{+\infty} g(s)w_x(s)\bar{w}_x(s)dsdx + \tau \mu \int_0^1 \bar{z} \bar{d} \rho dx.
$$

The domain of $\mathcal{A}$ is

$$
D(\mathcal{A}) = \left\{ (u, v, \varphi, \psi, z, w)^T \in \mathcal{H} : (u, v) \in \left( H^2(\Omega) \times H^2(L_1, L_2) \right) \cap X_* \right\},
$$

$$
\varphi \in H^1(\Omega), \psi \in H^1(L_1, L_2), w \in L^2_g(\mathbb{R}_+, H^2(\Omega)) \cap H^1(\Omega),
$$

$$
z_{\rho} \in L^2((0, 1), L^2(\Omega)), w(x, 0) = 0, z(x, 0) = \varphi(x).
$$

and $D(\mathcal{B}) = \mathcal{H}$ The well-posedness of problem (10)-(11) is ensured by the following theorem.

**Theorem 1**

Assume that (A1),(A2) hold. Let $V_0 \in \mathcal{H}$, then there exists a unique weak solution $V \in C(\mathbb{R}_+, \mathcal{H})$ of problem (14). Moreover, if $V_0 \in D(\mathcal{A})$, then

$$
V \in C(\mathbb{R}_+, D(\mathcal{A})) \cap C^1(\mathbb{R}_+, \mathcal{H})
$$

**Proof**

We use the semigroup approach. So, first, we prove that the operator $\mathcal{A}$ is dissipative. In fact, for $(u, v, \varphi, \psi, z, w)^T \in D(\mathcal{A})$, where $\varphi(L_i) = \psi(L_i), i = 1, 2$, we have
that is, \( \mathcal{A} \) is dissipative.

Next, we prove that \(-\mathcal{A}\) is maximal. Actually, let \( F = (f_1, f_2, f_3, f_4, f_5, f_6)^T \in \mathcal{H} \), we prove that there exists \( V = (u, v, \varphi, z, w)^T \in D(\mathcal{A}) \) satisfying
\[
(\lambda I - \mathcal{A})V = F,
\]
(16)

For the last term of the right side of (15), we obtain
\[
\mu \int \int_0^1 zz_x(x,\rho)d\rho dx = \mu \int \int_0^1 \frac{1}{2} \partial_\rho z^2(x,\rho)d\rho dx = \frac{\mu}{2} \int (z^2(x,1) - z^2(x,0)) dx.
\]

Noticing that \( z(x,0,t) = \varphi(x,t), w(x,0) = 0 \) and \( \varphi(L_i) = \psi(L_i), i = 1, 2 \), we obtain
\[
\langle \mathcal{A} V, V \rangle _\mathcal{H} = \left[ lu_x \varphi + \int_0^{+\infty} g(s)w_x(s)ds \varphi \right]_{\partial \Omega} + [b v_x \varphi]_{L_1}^{L_2} + \int_\Omega (-\mu \varphi - \mu z(.,1)) \varphi dx - \frac{1}{2} \int_\Omega \left[ g(s)|w_x(x,s)|^2 ds \right]_{0}^{+\infty} + \frac{1}{2} \int_0^1 \int_0^{+\infty} g'(s)|w_x(x,s)|^2 ds dx - \frac{\mu}{2} \int (z^2(x,1) - \varphi^2(x)) dx,
\]
where we have used that
\[
[lu_x \varphi + \int_0^{+\infty} g(s)w_x(s)ds \varphi]_{\partial \Omega} = \left( lu_x(L_1, t) + \int_0^{+\infty} g(s)w_x(L_1, s)ds \right) \varphi(L_1, t) - \left( lu_x(L_2, t) + \int_0^{+\infty} g(s)w_x(L_2, s)ds \right) \varphi(L_2, t) = -[b v_x \varphi]_{L_1}^{L_2}
\]

Using Young’s inequality, we have
\[
\langle \mathcal{A} V, V \rangle _\mathcal{H} = \frac{1}{2} \int \int_0^{+\infty} g'(s)|w_x(x,s)|^2 ds dx.
\]

Consequently, taking (A2) into account, we conclude that
\[
\langle \mathcal{A} V, V \rangle _\mathcal{H} \leq 0;
\]
that is, \( \mathcal{A} \) is dissipative.
which is equivalent to
\[
\begin{align*}
\lambda u - \varphi &= f_1, \\
\lambda v - \psi &= f_2, \\
\lambda \varphi - lu_{xx} - \int_0^\infty g(s)w_{xx}(s)ds + \mu \varphi + \mu z(., t) &= f_3, \\
\lambda \psi - bv_{xx} &= f_4, \\
\lambda z + \frac{1}{\tau} z_\rho &= f_5, \\
\lambda w + w_s - \varphi &= f_6.
\end{align*}
\] (17)

Assume that with the suitable regularity we have found \( u \) and \( v \), then
\[
\begin{align*}
\varphi &= \lambda u - f_1, \\
\psi &= \lambda v - f_2.
\end{align*}
\] (18)

So we have \( \varphi \in H^1(\Omega) \) and \( \psi \in H^1(L_1, L_2) \). Moreover, we can find \( z \) with \( z(x, 0) = \varphi(x) \), for \( x \in \Omega \).

Using the equation in (17), we obtain
\[
z(x, \rho) = \varphi(x)e^{-\lambda \rho} + \tau e^{-\lambda \rho} \int_0^\rho f_5(x, \sigma)e^{\lambda \sigma}d\sigma.
\]

From (18), we obtain
\[
z(x, \rho) = \lambda u e^{-\lambda \rho} - f_1 e^{-\lambda \rho} + \tau e^{-\lambda \rho} \int_0^\rho f_5(x, \sigma)e^{\lambda \sigma}d\sigma.
\] (19)

It is easy to see that the last equation in (17) with \( w(x, 0) = 0 \) has a unique solution
\[
w(x, s) = \left( \int_0^s e^{\lambda y}(f_6(x, y) + \varphi(x))dy \right)e^{-\lambda s}
\]
\[
= \left( \int_0^s e^{\lambda y}(f_6(x, y) + \lambda u(x) - f_1(x))dy \right)e^{-\lambda s}.
\] (20)

By using (17), (18) and (20), the functions \( u \) and \( v \) satisfy
\[
\left( \lambda^2 + \mu \lambda + \mu \lambda e^{-\lambda} \right) u - \tilde{l} u_{xx} = \tilde{f},
\]
\[
\lambda^2 v - bv_{xx} = f_4 + \lambda f_2,
\] (21)

where
\[
\tilde{l} = l + \lambda \int_0^\infty g(s)e^{-\lambda s} \left( \int_0^s e^{\lambda y}dy \right)ds
\]
and
\[
\tilde{f} = \int_0^\infty g(s)e^{-\lambda s} \left( \int_0^s e^{\lambda y}(f_6(x, y) - f_1(x, y))_{xx}dy \right)ds
\]
\[
- \mu \tau e^{-\lambda \tau} \int_0^\rho f_5(x, \sigma)e^{\lambda \sigma}d\sigma + (\lambda + \mu + \mu e^{-\lambda}) f_1 + f_3.
\]

We just need to prove that (21) has a solution \((u, v) \in X_* \) and replace in (18), (19) and (20) to get \( V = (u, v, \varphi, \psi, z, w)^T \in D(\mathcal{A}) \) satisfying (16). Consequently, problem (21) is equivalent to the problem
\[
\Phi((u, v), (\omega_1, \omega_2)) = l(\omega_1, \omega_2),
\] (22)
where the bilinear form \( \Phi : (X_*, X_*) \to \mathbb{R} \) and the linear form \( l : X_* \to \mathbb{R} \) are defined by

\[
\Phi((u, v), (\omega_1, \omega_2)) = \int_{\Omega} \left[ (\lambda^2 + \mu \lambda + \mu \lambda e^{-\lambda \gamma}) u \omega_1 + \tilde{f} u_x(\omega)_x \right] dx - [l u_x \omega_1]|_{\partial \Omega}
+ \int_{L_1} (\lambda^2 \nu \omega_2 + b v_x(\omega)_x) dx - [b v_x \omega_2]_{L_1}^2
\]

and

\[
l(\omega_1, \omega_2) = \int_{\Omega} \tilde{f} \omega_1 dx + \int_{L_1} (f_4 + \lambda f_2) \omega_2 dx.
\]

Using the properties of the space \( X_* \), it is easy to see that \( \Phi \) is continuous and coercive, and \( l \) is continuous. Applying the Lax-Milgram theorem, we infer that for all \( (\omega_1, \omega_2) \in X_* \), problem (22) has a unique solution \( (u, v) \in X_* \). It follows from (21) that \( (u, v) \in \{(H^2(\Omega) \times H^2(L_1, L_2)) \cap X_*\} \). Thence, the operator \( \lambda - \mathcal{A} \) is surjective for any \( \lambda > 0 \). That means \( \mathcal{A} \) is maximal monotone operator. Then, using Lumer-Phillips theorem [15], we deduce that \( \mathcal{A} \) is an infinitesimal generator of a linear \( C_0 \)-semigroup on \( \mathcal{H} \).

On the other hand, it is clear that the linear operator \( \mathcal{B} \) is Lipschitz continuous. Finally, also \( \mathcal{A} + \mathcal{B} \) is an infinitesimal generator of a linear \( C_0 \)-semigroup on \( \mathcal{H} \). Consequently (14) is well-posed in the sense of Theorem 1 (see [15]).

\[\square\]

3. Exponential stability

In this section, we consider a decay result of problem (1)-(3). In fact using the energy method to produce a suitable Lyapunov functional

**Theorem 2**

Let \((u, v)\) be the solution of (1)-(3). Assume that (A1), (A2) hold, and that

\[
a > \frac{8(L_2 - L_1)}{L_1 + L_3 - L_2}, \quad b > \frac{8(L_2 - L_1)}{L_1 + L_3 - L_2},
\]

(23)

then there exist two constants \( \gamma_1, \gamma_2 > 0 \) such that,

\[
E(t) \leq \gamma_2 e^{-\gamma_1 t}, \quad \forall t \in \mathbb{R}_+
\]

(24)

For the proof of Theorem 2, we need some lemmas.

For a solution of (1)-(3), we define the energy

\[
E(t) = \frac{1}{2} \int_{\Omega} |u(t, x)|^2 + l(u_x(t, x))|dx + \frac{1}{2} \int_{L_1} |v^2(t, x) + b v_x(x, t)| dx
+ \frac{1}{2} \int_{\Omega} \int_0^\infty g(s)|\eta_x(t, x, s)|^2 \text{d}s \text{d}x + \frac{\tau \mu}{2} \int_{\Omega} \int_0^1 z^2(x, \rho, t) \text{d}\rho \text{d}x.
\]

(25)

**Lemma 1**

Let \((u, v, \eta, z)\) be the solution of (10)-(11). Then we have the inequality

\[
\frac{d}{dt} E(t) \leq \mu \int_{\Omega} u_x^2(t, x) dx + \frac{1}{2} \int_{\Omega} \int_0^\infty g'(s)|\eta_x(t, x, s)|^2 \text{d}s \text{d}x.
\]

(26)
Proof
We have

\[
\frac{d}{dt} E(t) \int_{\Omega} \left( u_t u_{tt} + l u_x u_{xt} + \int_0^\infty g(s) \eta_x^t(x,s) \eta_{xt}^t ds \right) dx \\
+ \int_{L_1} \left( v_t v_{tt} + b v_x v_{xt} \right) dx + \tau |\mu| \int_{\Omega} \int_{0}^{1} z_t(x, \rho, t) z(x, \rho, t) d\rho dx \\
= \left( \left( l u_x + \int_0^\infty g(s) \eta_x^t(x,s) ds \right) u_t \right)_{\partial \Omega} - [b v_x v_t]_{L_1}^{L_2} \\
- \int_{\Omega} \int_0^\infty g(s) \eta_x^t(x,s) \eta_{xt}^t(x,s) ds dx \\
- \mu \int_{\Omega} u_t z(x, 1, t) dx + \frac{\mu}{2} \int_{\Omega} u_t^2(x, t) dx - \frac{\mu}{2} \int_{\Omega} z^2(x, 1, t) dx \\
= \frac{1}{2} \int_{\Omega} \int_0^\infty g'(s) \eta_x^t(x,s)^2 ds dx - \mu \int_{\Omega} u_t z(x, 1, t) dx + \frac{\mu}{2} \int_{\Omega} u_t^2(x, t) dx \\
- \frac{\mu}{2} \int_{\Omega} z^2(x, 1, t) dx
\]

where we have used that

\[
\left[ \left( l u_x + \int_0^\infty g(s) \eta_x^t(x,s) ds \right) u_t \right]_{\partial \Omega} \\
= \left( l u_x(L_1, t) + \int_0^\infty g(s) \eta_x^t(L_1, s) ds \right) u_t(L_1, t) \\
- \left( l u_x(L_2, t) + \int_0^\infty g(s) \eta_x^t(L_2, s) ds \right) u_t(L_2, t) \\
= - [b v_x v_t]_{L_1}^{L_2}.
\]

and

\[
\left[ \frac{1}{2} \int_{\Omega} g(s) \eta_x^t(x,s)^2 ds \right]_0^\infty = 0,
\]

and

\[
\frac{\tau \mu}{2} \frac{d}{dt} \int_{\Omega} \int_0^1 z^2(x, \rho, t) d\rho dx = - \frac{\mu}{2\tau} \int_{\Omega} (z^2(x, 1) - z^2(x, 0)) dx.
\]

Young’s inequality gives us

\[
\frac{d}{dt} E(t) \leq \mu \int_{\Omega} u_t^2(x, t) dx + \frac{1}{2} \int_{\Omega} \int_0^\infty g'(s) \eta_x^t(x,s)^2 ds dx.
\]

Now, we define the functional

\[
\mathcal{D}(t) = \int_{\Omega} u u_t dx + \int_{L_1} v v_t dx,
\]

then we have the following lemma.

Lemma 2

The functional \( \mathcal{D}(t) \) satisfies

\[
\frac{d}{dt} \mathcal{D}(t) \leq \int_{\Omega} u_t^2 dx + \int_{L_1} v_t^2 dx + (L^2 \varepsilon + \varepsilon - l) \int_{\Omega} u_t^2 dx - \int_{L_1} b v_t^2 dx \\
+ \frac{9\mu}{4\varepsilon} \int_{\Omega} \int_0^\infty g(s) \eta_x^t(x,s)^2 ds dx + \frac{\mu^2}{4\varepsilon} \int_{\Omega} z^2(x, 1, t) dx.
\]
Proof
Taking the derivative of $\mathcal{D}(t)$ with respect to $t$ and using (10), we have
\[
\frac{d}{dt} \mathcal{D}(t) = \int_\Omega u_t^2 dx - l \int_\Omega u_x^2 dx - \mu \int_\Omega z(x,1,t)udx + [bv_x v]_{L_1}^L + \int_{L_1}^{L_2} v_t^2 dx
\]
\[
\quad + \left[ \left( lu_x + \int_0^\infty g(s)\eta_x^t(x,s)ds \right) u \right]_{\partial\Omega}
\]
\[
\quad - \int_\Omega u_x(x,t) \int_0^\infty g(s)\eta_x^t(x,s)ds dx - \int_{L_1}^{L_2} be_x^2 dx
\]
\[
= \int_\Omega u_t^2 dx - l \int_\Omega u_x^2 dx - \mu \int_\Omega z(x,1,t)udx + \int_{L_1}^{L_2} v_t^2 dx
\]
\[
\quad - \int_{L_1}^{L_2} be_x^2 dx - \int_\Omega u_x(x,t) \int_0^\infty g(s)\eta_x^t(x,s)ds dx,
\]
where we used that
\[
\left[ \left( lu_x + \int_0^\infty g(s)\eta_x^t(x,s)ds \right) u \right]_{\partial\Omega} = \left( lu_x(L_1,t) + \int_0^\infty g(s)\eta_x^t(L_1,s)ds \right) u(L_1,t)
\]
\[
\quad - \left( lu_x(L_2,t) + \int_0^\infty g(s)\eta_x^t(L_2,s)ds \right) u(L_2,t)
\]
\[
= \left[ [bv_x u]_{L_1}^{L_2} \right].
\]

By the boundary conditions (2), we have
\[
u^2(x,t) = \left( \int_0^x u_x(x,t)dx \right)^2 \leq L_1 \int_0^{L_1} u_x^2(x,t)dx, \quad x \in [0,L_1],
\]
\[
u^2(x,t) \leq (L_3 - L_2) \int_{L_2}^{L_3} u_x^2(x,t)dx, \quad x \in [L_2,L_3],
\]
which implies
\[
\int_\Omega u^2(x,t)dx \leq L^2 \int_\Omega u_x^2 dx, \quad x \in \Omega,
\]
where $L = \max\{L_1, L_3 - L_2\}$. By making use of Young’s inequality and (31), for any $\varepsilon > 0$, we obtain
\[
\mu \int_\Omega z(x,1,t)udx \leq \frac{\mu^2}{4\varepsilon} \int_\Omega z^2(x,1,t)dx + L^2 \varepsilon \int_\Omega u_x^2 dx.
\]

Young’s inequality, Hölder’s inequality and (A2) imply that
\[
\int_\Omega u_x(x,t) \int_0^\infty g(s)\eta_x^t(x,s)ds dx \leq \varepsilon \int_\Omega u^2(x,t)dx + \frac{\gamma_0}{4\varepsilon} \int_\Omega \int_0^\infty g(s)|\eta_x^t(x,s)|^2 ds dx.
\]

Inserting the estimates (32) and (33) into (30), then (29) is fulfilled.

Next, enlightened by [13], we introduce the functional
\[
q(x) = \begin{cases}
\frac{x - L_1}{2}, & x \in [0,L_1), \\
\frac{L_1 - L_2 + L_3 - L_2}{2(L_2 - L_1)}(x - L_1), & x \in (L_1,L_2), \\
\frac{x}{2} - \frac{L_2 + L_3}{4}, & x \in [L_2,L_3].
\end{cases}
\]
We pay attention to the derivative of the functional $\mathcal{F}_1(t)$ and $\mathcal{F}_2(t)$.

Lemma 3

The functionals $\mathcal{F}_1(t)$ and $\mathcal{F}_2(t)$ satisfy

$$\frac{d}{dt}\mathcal{F}_1(t)$$

and

$$\frac{d}{dt}\mathcal{F}_2(t)$$

Proof

Taking the derivative of $\mathcal{F}_1(t)$ with respect to $t$ and using (10), we obtain

$$\frac{d}{dt}\mathcal{F}_1(t)$$

We pay attention to

$$\frac{1}{2} \int q(x)(lu_x + \int_0^\infty g(s)\eta^*_x(x,s)ds)^2 dx$$

and

$$\int q(x)(lu_x + \int_0^\infty g(s)\eta^*_x(x,s)ds)x^2 dx = \left[ \frac{q(x)}{2} \right]_{\partial \Omega}.$$

It is easy to see that $q(x)$ is bounded: $|q(x)| \leq M$, where $M = \max\{\frac{L_1}{2}, \frac{L_3 - L_2}{2}\}$.
The last term in (36) can be treated as follows

\[- \int_\Omega q(x) u_t \left( l u_x + \int_0^\infty g(s) \eta^t_{\text{ex}}(x, s) ds \right) dx\]

\[= - l \int_\Omega q(x) u_t u_x dx - \int_\Omega q(x) u_t \int_0^\infty g(s) \eta^t_{\text{ex}}(x, s) ds dx\]

\[= \left[ - \frac{l}{2} q(x) u_t^2 \right]_{\partial \Omega} + \frac{l}{2} \int_\Omega q'(x) u_t^2 dx\]

\[- \int_\Omega q(x) u_t \int_0^\infty g(s) \left( u_t - \eta^t_{\text{ex}}(x, s) \right) ds dx\]

\[= \left[ - \frac{l}{2} q(x) u_t^2 \right]_{\partial \Omega} + \frac{l}{2} \int_\Omega q'(x) u_t^2 dx\]

\[- \int_\Omega q(x) u_t \int_0^\infty g(s) \eta^t_{\text{ex}}(x, s) ds dx\]

\[= \left[ - \frac{l + g_0}{2} q(x) u_t^2 \right]_{\partial \Omega} + \frac{l + g_0}{2} \int_\Omega q'(x) u_t^2 dx\]

\[- \int_\Omega q(x) u_t \int_0^\infty g'(s) \eta^t_{\text{ex}}(x, s) ds dx,\]

where we used that

\[- \left[ \int_\Omega q(x) u_t g(s) \eta^t_{\text{ex}}(x, s) ds \right]_0^\infty = 0.\]

Inserting (37) and (38) in (36), we arrive at

\[\frac{d}{dt} \mathcal{F}_1(t)\]

\[= \left[ \frac{q(x)}{2} \left( l u_x + \int_0^\infty g(s) \eta^t_{\text{ex}}(x, s) ds \right) \right]_{\partial \Omega} - \left[ \frac{l + g_0}{2} q(x) u_t^2 \right]_{\partial \Omega}\]

\[+ \frac{1}{2} \int_\Omega q'(x) \left( l u_x + \int_0^\infty g(s) \eta^t_{\text{ex}}(x, s) ds \right)^2 dx\]

\[+ \mu \int_\Omega q(x) z(x, 1, t) \left( l u_x + \int_0^\infty g(s) \eta^t_{\text{ex}}(x, s) ds \right) dx\]

\[+ \frac{l + g_0}{2} \int_\Omega q'(x) u_t^2 dx - \int_\Omega q(x) u_t \int_0^\infty g'(s) \eta^t_{\text{ex}}(x, s) ds dx.\]

Using Minkowski and Young’s inequalities, we have

\[\frac{1}{2} \int_\Omega \left( l u_x + \int_0^\infty g(s) \eta^t_{\text{ex}}(x, s) ds \right)^2 dx\]

\[\leq l^2 \int_\Omega u_x^2 dx + g_0 \int_\Omega \int_0^\infty g(s) |\eta^t_{\text{ex}}(x, s)|^2 ds dx.\]

Young’s inequality gives us that for any \( \varepsilon_1 > 0, \)

\[\left| \mu \int_\Omega q(x) z(x, 1, t) \left( l u_x + \int_0^\infty g(s) \eta^t_{\text{ex}}(x, s) ds \right) dx \right|\]

\[\leq \frac{M^2 \mu^2}{4 \varepsilon_1} \int_\Omega z^2(x, 1, t) dx + l^2 \varepsilon_1 \int_\Omega u_x^2(x, t) dx\]

\[+ g_0 \varepsilon_1 \int_\Omega \int_0^\infty g(s) |\eta^t_{\text{ex}}(x, s)|^2 ds dx.\]
It is clear that
\[\left| \int_{\Omega} q(x) u_t \int_{0}^{\infty} g'(s) u_x^t ds dx \right| \leq \varepsilon_1 M^2 \int_{\Omega} u_t^2 dx - \frac{g(0)}{4\varepsilon_1} \int_{\Omega} \int_{0}^{\infty} g'(s) |u_x^t(x, s)|^2 ds dx.\] (42)

Inserting (40)-(42) into (39), we obtain (34).

By the same method, taking the derivative of \(\mathcal{F}_1(t)\) with respect to \(t\), we obtain
\[
\frac{d}{dt} \mathcal{F}_2(t) = - \int_{L_1}^{L_2} q(x) v_{x\xi} v_t dx - \int_{L_1}^{L_2} q(x) v_x v_{t\xi} dx
\]
\[
\quad \left[ - \frac{1}{2} q(x) v_t^2 \right]_{L_1}^{L_2} + \frac{1}{2} \int_{L_1}^{L_2} q'(x) v_t^2 dx + \frac{1}{2} \int_{L_1}^{L_2} b q'(x) v_x^2 dx
\]
\[
\quad + \left[ - \frac{b}{2} q(x) v_x^2 \right]_{L_1}^{L_2} \leq - \frac{L_1 + L_3 - L_2}{4(L_2 - L_1)} \left( \int_{L_1}^{L_2} v_t^2 dx + \int_{L_1}^{L_2} b v_x^2 dx \right) + \frac{L_1}{4} v_x^2(L_1)
\]
\[
\quad + \frac{L_3 - L_2}{4} v_x^2(L_2) + \frac{b}{4} (L_3 - L_2) v_x^2(L_2, t) + L_1 v_x^2(L_1, t).
\]

Thus, the proof of Lemma 3 is complete. \(\square\)

We define the functional
\[\mathcal{F}_3(t) = \tau \int_{\Omega} \int_{0}^{1} e^{-\tau p} z^2(x, \rho, t) d\rho dx,\]
then we have the following estimate.

**Lemma 4**
The functionals \(\mathcal{F}_3(t)\) satisfies
\[
\frac{d}{dt} \mathcal{F}_3(t) \leq -c_2 \left( \int_{\Omega} z^2(x, 1, t) dx + \tau \int_{\Omega} \int_{0}^{1} z^2(x, \rho, t) d\rho dx \right) + \int_{\Omega} u_t^2(x, t) dx.
\]

**Proof**
\[
\frac{d}{dt} \mathcal{F}_3(t) = 2\tau \int_{0}^{1} \int_{\Omega} e^{-\tau p} z_t(x, \rho, t) z(x, \rho, t) d\rho dx
\]
\[
= -2 \int_{0}^{1} \int_{\Omega} e^{-\tau p} z(x, \rho, t) z(x, \rho, t) d\rho dx
\]
\[
= -\int_{0}^{1} \int_{\Omega} e^{-\tau p} \frac{\partial}{\partial \rho} \left( z^2(x, \rho, t) \right) d\rho dx
\]
\[
= -\tau \int_{0}^{1} \int_{\Omega} e^{-\tau p} z^2(x, \rho, t) d\rho dx + \int_{\Omega} u_t^2(x, t) dx - e^{-\tau} \int_{\Omega} z^2(x, 1, t) dx
\]
\[
\leq -e^{-\tau} \left( \tau \int_{0}^{1} \int_{\Omega} z^2(x, \rho, t) d\rho dx + \int_{\Omega} z^2(x, 1, t) dx \right) + \int_{\Omega} u_t^2(x, t) dx.
\]
\(\square\)
We define the functional

\[ \mathcal{F}_4(t) = - \int_\Omega u_t \int_0^\infty g(s)(u(t) - u(t-s))dsdx, \]

then we have the following estimate.

**Lemma 5**

The functional \( \mathcal{F}_4(t) \) satisfies

\[
\frac{d}{dt} \mathcal{F}_4(t) \leq -(g_0 - \delta_2) \int_\Omega u_t^2 dx + \delta_2 t^2 \int_\Omega u_x^2 dx + \delta_2 \mu \int_\Omega z^2(x, 1, t)dx \\
+ \left( g_0 + \frac{g_0}{4\delta_2} + \frac{\mu g_0 L^2}{2\delta_2} \right) \int_\Omega \int_0^\infty g(s)|\eta_x^l(x, s)|^2 dsdx \\
- \frac{g(0)L^2}{\delta_2} \int_\Omega \int_0^\infty g'(s)|\eta_x^l(x, s)|^2 dsdx. 
\] (43)

**Proof**

Taking the derivative of \( \mathcal{F}_4(t) \) with respect to \( t \) and using (10), we have

\[
\frac{d}{dt} \mathcal{F}_4(t) \\
= - \int_\Omega \left( t u_{xx} + \int_0^\infty g(s)\eta_{xx}^l(x, s)ds - \mu z(x, 1, t) \right) \\
\times \int_0^\infty g(s)(u(t) - u(t-s))dsdx - \int_\Omega u_t \int_0^\infty g(s)(u_t(t) - u_t(t-s))dsdx \\
= \int_\Omega u_x \int_0^\infty g(s)(u_x(t) - u_x(t-s))dsdx - g_0 \int_\Omega u_t^2 dx \\
+ \int_\Omega u_t \int_0^\infty g(s)\eta_x^l(s)dsdx + \int_\Omega \left( \int_0^\infty g(s)(u_x(t) - u_x(t-s))ds \right)^2 dx \\
+ \int_\Omega \mu z(x, 1, t) \int_0^\infty g(s)(u(t) - u(t-s))dsdx. 
\] (44)

Using Young’s inequality and (31), we obtain for any \( \delta_2 > 0 \),

\[
\int_\Omega u_x \int_0^\infty g(s)(u_x(t) - u_x(t-s))dsdx \\
\leq \delta_2 t^2 \int_\Omega u_x^2 dx + \frac{g_0}{4\delta_2} \int_\Omega \int_0^\infty g(s)|\eta_x^l(x, s)|^2 dsdx, 
\]

\[
\int_\Omega \mu z(x, 1, t) \int_0^\infty g(s)(u(t) - u(t-s))dsdx \\
\leq \delta_2 \mu \int_\Omega z^2(x, 1, t)dx + \frac{\mu g_0 L^2}{4\delta_2} \int_\Omega \int_0^\infty g(s)|\eta_x^l(x, s)|^2 dsdx. 
\] (46)
We notice that
\[
\int_{\Omega} \left( \int_0^\infty g(s)(u_x(t) - u_x(t - s))ds \right)^2 dx \\
= \int_{\Omega} \left( \int_0^\infty \sqrt{g(s)}\sqrt{g(s)}(u_x(t) - u_x(t - s))ds \right)^2 dx \\
\leq \int_{\Omega} \int_0^\infty g(s)ds \left( \int_0^\infty g(s)|\eta_x|^2 ds \right) dx \\
\leq g_0 \int_{\Omega} \int_0^\infty g(s)|\eta_x|^2 ds dx
\]
and
\[
\int_{\Omega} u_i \int_0^\infty g(s)|\eta_x|^2 ds dx = - \int_{\Omega} u_i \int_0^\infty g'(s)|\eta_x|^2 ds dx \\
\leq \delta_2 \int_{\Omega} u_i^2 dx - \frac{g(0) L^2}{4\delta_2} \int_{\Omega} \int_0^\infty g'(s)|\eta_x|^2 ds dx.
\] (48)

Inserting the estimates (45)-(48) into (44), we obtain (43). The proof is complete.

\[\square\]

**Proof**

We define the Lyapunov functional
\[
\mathcal{L}(t) = N_1 E(t) + N_2 \mathcal{D}(t) + \mathcal{F}_1(t) + N_4 \mathcal{F}_2(t) + N_5 \mathcal{F}_3(t) + N_6 \mathcal{F}_4(t),
\] (49)

where $N_1, N_2, N_4, N_5$ and $N_6$ are positive constants that will be fixed later.

Taking the derivative of (49) with respect to $t$ and taking advantage of the above lemmas, we have
\[
\frac{d}{dt} \mathcal{L}(t) \leq - \left\{ N_6(g_0 - \delta_2) - N_2 \left( \frac{l + g_0}{2} + \epsilon_1 M^2 \right) \right. \\
- N_5 - N_1 \mu \right\} \int_{\Omega} u_i^2 dx \\
- \left\{ N_5 \epsilon_2 - \frac{N_2 \mu^2}{4\epsilon_1} - \frac{M^2 \mu^2}{4\epsilon_1} - N_6 \delta_2 \mu \right\} \int_{\Omega} \xi^2(x, 1, t) dx \\
- \left\{ N_2((l - L^2 \xi - \epsilon) - (l^2 + \xi^2 \epsilon_1)) - N_6 \delta_2 \right\} \int_{\Omega} u_i^2 dx \\
- \left\{ \frac{b(L_1 + L_3 - L_2)}{4(L_2 - L_1)} N_4 + N_2 b \right\} \int_{L_1}^{L_2} v_x^2 dx \\
- \left\{ \frac{L_1 + L_3 - L_2}{4(L_2 - L_1)} N_4 - N_2 \right\} \int_{L_1}^{L_2} v_t^2 dx \\
- \frac{b - N_4}{4} \left( (L_3 - L_2) v_x^2(L_2, t) + L_1 v_x^2(L_1, t) \right) \\
- (a - N_4) \left[ \frac{L_1}{4} v_t^2(L_1, t) + \frac{L_3 - L_2}{4} v_t^2(L_2, t) \right] \\
+ c(N_2, N_6) \int_{\Omega} \int_0^\infty g(s)|\eta_x|^2 ds dx \\
+ \left( \frac{N_1}{2} - \frac{g(0)}{4\epsilon_1} - \frac{N_6 g(0) L^2}{4\delta_2} \right) \int_{\Omega} \int_0^\infty g'(s)|\eta_x|^2 ds dx.
\]

At this moment, we wish all coefficients except the last two in (50) will be negative. We want to choose \( N_2 \) and \( N_4 \) to ensure that

\[
a - N_4 \geq 0, \quad b - N_4 \geq 0, \quad \frac{L_1 + L_3 - L_2}{4(L_2 - L_1)} N_4 - N_2 > 0.
\]

. For this purpose, since \( \frac{8l(L_2 - L_1)}{L_1 + L_3 - L_2} < \min\{a, b\} \) we first choose \( N_4 \) satisfying

\[
\frac{8l(L_2 - L_1)}{L_1 + L_3 - L_2} < N_4 \leq \min\{a, b\}.
\]

Once \( N_4 \) is fixed, we pick \( N_2 \) satisfying

\[
2l < N_2 < \frac{L_1 + L_3 - L_2}{4(L_2 - L_1)} N_4.
\]

Then we take \( \varepsilon, \varepsilon_1 \) and \( \varepsilon_1 \) small enough, and \( \delta_2 < \frac{1}{2N_6} \) we have

\[
N_2(l - L^2 \varepsilon - \varepsilon) - 2l^2 \varepsilon_1 > \frac{3}{2} l^2.
\]

Once \( \varepsilon \) and \( \varepsilon_1 \) are fixed, we take \( N_5 \) satisfying

\[
N_5 > \max\left\{ \frac{2N_2 \mu^2}{\varepsilon c_2}, \frac{2M^2 \mu^2}{\varepsilon_1 c_2} \right\}
\]

and \( \delta_2 < \frac{N_5 c_2}{8 \mu N_6} \) such that

\[
N_5 c_2 - \frac{N_2 \mu^2}{4 \varepsilon} - \frac{M^2 \mu^2}{4 \varepsilon_1} > \frac{3}{8} N_5 c_2.
\]

Further, we take \( \delta_2 < \frac{\mu}{2} \) we choose \( N_6 \) satisfying

\[
N_6 > \frac{2N_2}{g_0} + \frac{l + g_0}{g_0} + \frac{2 \varepsilon_1 M^2}{g_0} + \frac{2N_5}{g_0} + \frac{2N_1 \mu}{g_0}.
\]

Then we have

\[
N_6 > \max\left\{ \frac{2N_2}{g_0}, \frac{l + g_0}{g_0} + \frac{2 \varepsilon_1 M^2}{g_0}, \frac{2N_5}{g_0}, \frac{2N_1 \mu}{g_0} \right\}.
\]

Then, we pick \( \delta_2 \) satisfying

\[
\delta_2 < \min\left\{ \frac{g_0}{2} \frac{N_5 c_2}{8 \mu N_6}, \frac{1}{2N_6} \right\},
\]

\[
\left\{ N_5 c_2 - \frac{N_2 \mu^2}{4 \varepsilon} - \frac{M^2 \mu^2}{4 \varepsilon_1} - N_6 \delta_2 \mu \right\} \geq 0.
\]

Once

\[
\{ N_2(l - L^2 \varepsilon - \varepsilon) - (l^2 + l^2 \varepsilon_1) - N_6 \delta_2 l^2 \} \geq 0.
\]

Finally, choosing \( N_1 \) large enough such that the first and the last coefficients in (50) is positive.

From the above, we deduce that there exist two positive constants \( \alpha_1 \) and \( \alpha_2 \) such that (50) becomes

\[
\frac{d}{dt} \mathcal{G}(t) \leq -\alpha_1 E(t) + \alpha_2 \int_{\Omega} \int_{0}^{\infty} g(s) |\eta_x'(x,s)|^2 ds dx
\]

\[
\leq -\alpha_1 E(t) - \frac{\alpha_2}{\delta} \int_{\Omega} \int_{0}^{\infty} g'(s) |\eta_x'(x,s)|^2 ds dx
\]

\[
\leq -\alpha_1 E(t) - \alpha_3 E'(t).
\]
That is
\[ \left( \mathcal{L}(t) + \alpha_3 E(t) \right)' \leq -\alpha_1 E(t) \quad (52) \]
where \( \alpha_3 > 0 \). Denote \( \mathcal{E}(t) = \mathcal{L}(t) + \alpha_3 E(t) \), then it is easy to see that
\[ \mathcal{E}(t) \sim E(t), \]
i.e., there exist two positive constants \( \beta_1, \beta_2 \):
\[ \beta_1 E(t) \leq \mathcal{E}(t) \leq \beta_2 E(t), \quad \forall t \geq 0. \quad (53) \]
Combining (52) and (53), we deduce that there exists \( \gamma_1 > 0 \) for which the estimate
\[ \frac{d\mathcal{E}(t)}{dt} \leq -\gamma_1 \mathcal{E}(t), \quad \forall t \geq 0, \quad (54) \]
since
\[ \mathcal{E}(t)(t) \leq \mathcal{E}(0)e^{-\gamma_1 t}, \quad \forall t \geq 0. \quad (55) \]
Consequently, using (55) and (53), we find
\[ E(t) \leq \frac{1}{\beta_1} \mathcal{E}(t) \leq \frac{1}{\beta_1} \mathcal{E}(0)e^{-\gamma_1 t}, \quad \forall t \geq 0. \quad (56) \]
Thus, the proof of Theorem 2 is complete.

4. Conclusion

In this paper we study the following transmission system with a past history and a delay term. Under assumptions on initial data and boundary conditions, past history and a delay term, we focused our study on the existence and asymptotic behavior of solutions where we obtained exponential decay of solutions for transmission problems.

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