Solving Fractional Variational Problem Via an Orthonormal Function

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Abstract In the present paper, a direct numerical technique for solving fractional optimal control problems based on an orthonormal wavelet, is introduced. First we approximate the involved functions by Sine-Cosine wavelet basis; then, an operational matrix is used to transform the given problem into a linear system of algebraic equations, which is easier. In fact operational matrix of Riemann-Liouville fractional integration and derivative of Sine-Cosine wavelet are employed to achieve a linear algebraic equation. The mentioned matrices are derived via hat functions. The solution of transformed system, gives us the solution of original problem. Two numerical examples are also given. Finally, the paper is ended with conclusion.

Keywords Fractional Optimal Control, Sine-Cosine Wavelet, Operational Matrix, Hat Function.

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1. Introduction

In recent years, fractional calculus is one of the interesting issues that attract many scientists. Many realistic models of engineering and physical phenomena can be uttered with fractional calculus. For example they can be applied in nonlinear oscillations of earthquakes [20], fluid-dynamic traffic [21], frequency dependent damping behavior of various viscoelastic materials [4], solid mechanics [35], economics [5], signal processing [32], and control theory [10]. Niels Henrik Abel, in 1823, was probably the first to give an application of fractional calculus. Abel applied the fractional calculus in the solution of an integral equation which arises in the formulation of the problem of finding the shape of a frictionless wire lying in a vertical plane such that the time of a bead placed on the wire slides to the lowest point of the wire in the same time regardless of where the bead is placed [6].

Besides modeling , the solution techniques and their reliabilities are most important. Analytical solutions of fractional differential equations are not always available, therefore, it is important to obtain numerical solutions for these equations. The most commonly techniques proposed to solve fractional problem are Adomian Decomposition Method (ADM) [37, 29], Variational Iteration Method (VIM) [12], operational matrix method [36], homotopy analysis method [1, 13, 14], homotopy perturbation methods [18], collocation method [17], Galerkin method [28], Fractional Difference Method (FDM) [30], finite difference method [27], Fractional Differential Transform Method (FDTM) [3, 16] and power series method [31], in addition, there are some classical techniques, e.g. Laplace transform method [33]. For further study on recent papers in the area of fractional differential equations and their applications, referred to [9, 42, 43].

A fractional optimal control problem (FOCP) is an optimal control problem in which the performance index or the differential equations in the dynamics of the system or both contain at least one fractional order derivative

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term [39]. Tricaud and Chen have solved a large class of FOCPs (linear, nonlinear, time-invariant, time-variant, SISO, MIMO, state or input constrained, etc.) by converting them into a general and rational form of optimal control problem [40]. In addition to the above methods, orthogonal function method is also applicable to solve the fractional order systems and as a result, FOCPs.

Approximation by orthogonal families of basis functions is widely used in science and engineering. The main idea of applying an orthogonal basis is reduction of considered problem into a system of algebraic equations, by truncating series of orthogonal basis functions for the solution of the problem and applying operational matrix of integration and differentiation to eliminate the integral and derivative operations whenever needed, thus greatly simplifying the problem. These matrices can be uniquely determined based on the particular orthogonal functions.

The orthogonal functions are classified into three main categories [38], the first one is sets of piecewise constant orthogonal functions such as the Walsh functions and block pulse functions. The second one is orthogonal polynomials such as the Laguerre, Legendre and Chebyshev functions [26], and the last one is sine-cosine functions. On one hand, approximating a continuous function with piecewise constant basis functions results in a piecewise constant approximation, on the other hand, if we approximate a discontinuous function with continuous basis the resulting approximation is continuous which is not proper for modelling the discontinuities. So, neither continuous basis functions nor piecewise constant basis functions, can efficiently model both continuity and discontinuity of phenomena at the same time. In the case that the function under approximation is not analytic, wavelet functions will be more effective.

The operational matrix of fractional integrals has been derived for many types of orthogonal polynomials such as Legendre polynomials [2, 15], Jacobi polynomials [8], Laguerre polynomials [7] and so on. In this paper, two new operational matrices are introduced and a direct method based on Sine-Cosine wavelet with their fractional integration and differentiation operational matrix is proposed to solve a FOCP and a variational problem. The main idea is to reduce the problem under consideration into a system of algebraic equations. To this end, we expand the fractional derivative of the state and control variables using the Sine-Cosine wavelet with unknown coefficients. There are many numerical methods to solve the transformed problem. The proposed method can also be applied to systems with time varying coefficients by using operational matrix of production. This matrix could easily obtained by using the sin and cos multiplication properties.

The paper is organized as follows. In next section we will give the preliminaries of fractional calculus, then in section 3 express a brief review of Hat function and the related fractional operational matrix. In section 4, we describe Sine-Cosine wavelets and its application in function approximation. In section 5, operational matrices of fractional integration and differentiation for considered wavelet is given. In section 6, the proposed method is described for solving the underlying FOCP. The proposed method is applied for solving numerical examples, in section 7. Finally, the paper is ended with conclusion.

2. Preliminaries of fractional calculus

Fractional order calculus deals with the non-integer order differentiation and integration. Now, we give necessary definitions of the fractional calculus theory. The most commonly used definitions for fractional integral and derivative are the Riemann–Liouville and Caputo definitions. The Riemann–Liouville fractional integration operator of a function \( f \) of order \( \alpha \geq 0 \) is defined in [34] as

\[
(I^\alpha f)(t) = \begin{cases} 
  \frac{1}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha-1} f(\tau) d\tau & \alpha > 0, \\
  f(t) & \alpha = 0,
\end{cases}
\]  

(1)

as two properties of Riemann-Liouville fractional integration we have

\[
I^\alpha t^k = \frac{\Gamma(k + 1)}{\Gamma(k + 1 + \alpha)} t^{\alpha + k} \quad k \in \mathbb{N} \cup \{0\}, \ t > 0,
\]

\[
I^\alpha I^\beta t^k = I^{\alpha + \beta} t^k.
\]
The fractional derivative operator of order $\alpha > 0$ in the Caputo sense is defined in [34] as:

$$D^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} f^{(n)}(\tau)d\tau = I^{n-\alpha} f^{(n)}(t) \quad n-1 < \alpha \leq n.$$  \hspace{1cm} (2)

Two useful relation between the Riemann-Liouville and Caputo operators is as follow

$$D^\alpha I^\alpha f(t) = f(t),$$
$$I^\alpha D^\alpha f(t) = f(t) - \sum_{k=0}^{n-1} f^{(k)}(0) \frac{t^k}{k!} \quad t > 0, \quad n-1 < \alpha \leq n.$$

3. Review of Hat functions and the related fractional operational matrix

In this section, we introduce Hat functions (HFs) and its operational matrix of fractional integration.

3.1. Definition of HFs

A $\tilde{m}$ – set of HFs basis functions is defined by Tripathi et al. [41] as follows:

$$\phi_0(t) = \begin{cases} \frac{h-t}{h} & 0 \leq t < h, \\ 0 & \text{O.W.,} \end{cases}$$ \hspace{1cm} (3)

$$\phi_i(t) = \begin{cases} \frac{t-(i-1)h}{h} & (i-1)h \leq t < ih, \\ \frac{(i+1)h-t}{h} & ih \leq t < (i+1)h \quad i = 1, 2, \cdots, \tilde{m} - 2, \\ 0 & \text{O.W.,} \end{cases}$$ \hspace{1cm} (4)

$$\phi_{\tilde{m}-1}(t) = \begin{cases} \frac{t-(1-h)}{h} & 1-h \leq t \leq 1, \\ 0 & \text{O.W.,} \end{cases}$$ \hspace{1cm} (5)

where $h = \frac{1}{\tilde{m} - 1}$. An arbitrary function like $f(t) \in L^2[0, 1]$ can be expanded by HFs as:

$$f(t) = \sum_{i=0}^{\tilde{m}-1} f_i \phi_i(t) = F^T \Phi(t) = \Phi^T(t) F,$$ \hspace{1cm} (6)

where

$$F = [f_0, f_1, \cdots, F_{\tilde{m}-1}]^T, \quad \Phi = [\phi_0, \phi_1, \cdots, \phi_{\tilde{m}-1}]^T.$$

An important property of HFs in approximating the function $f(t)$ is that the coefficients $f_i$ in the above equation are stated by:

$$f_i = f(ih), \quad i = 0, 1, \cdots, \tilde{m} - 1.$$ \hspace{1cm} (7)

3.2. Operational matrix of fractional integration for HFs

Operational matrix of fractional integration of order $\alpha$ for HFs, which given in [41] is as:

$$(I^\alpha \Phi)(t) \simeq Q^\alpha \Phi(t),$$
\[ Q^\alpha = \frac{h^\alpha}{\Gamma(\alpha + 2)} \begin{pmatrix} 0 & \zeta_1 & \cdots & \zeta_{\hat{m}-2} & \zeta_{\hat{m}-1} \\ 0 & 1 & \zeta_1 & \cdots & \zeta_{\hat{m}-3} & \zeta_{\hat{m}-2} \\ 0 & 0 & 1 & \cdots & \zeta_{\hat{m}-4} & \zeta_{\hat{m}-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 1 & \xi_1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \]

and

\[ \zeta_i = i^\alpha (\alpha - i + 1) + (i - 1)^{\alpha+1}, \quad i = 1, 2, \cdots, \hat{m} - 1, \]
\[ \xi_i = (i + 1)^{\alpha+1} - 2i^{\alpha+1} + (i - 1)^{\alpha+1}, \quad i = 1, 2, \cdots, \hat{m} - 2. \]

4. Description of Sine-Cosine wavelets and its application in function approximation

Wavelets have been very successful in approximate solution of different types of systems. They constitute a family of functions constructed from dilation and translation of a single function called the mother wavelet. When the dilation parameter \( a \) and the translation parameter \( b \) vary continuously, we have the following family of continuous wavelets [19].

\[ \psi_{a,b} = |a|^{-\frac{1}{2}} \psi\left(\frac{t-b}{a}\right), \quad a, b \in \mathbb{R}, \quad a \neq 0, \]

if we restrict the parameters \( a \) and \( b \) to discrete values \( a = a_0^{-k}, b = nb_0a_0^{-k}, \) where \( a_0 > 1, b_0 > 0, n \) and \( k \) are positive integers, we have the following family of discrete wavelets:

\[ \psi_{k,n} = |a_0|^{\frac{k}{2}} \psi(a_0^K t - nb_0), \]

which are a wavelet basis for \( L^2(\mathbb{R}) \). Sine-Cosine wavelets are defined as follows [23]

\[ \psi_{n,m}(t) = \begin{cases} 2^{\frac{k}{2}+1} f_m(2^k t - n) & \frac{n}{2^k} \leq t < \frac{n+1}{2^k}, \\
0 & \text{O.W.} \end{cases}, \]

with

\[ f_m(t) = \begin{cases} \frac{1}{\sqrt{2}} & m = 0, \\
\cos(2m\pi t) & m = 1, 2, \cdots, l, \\
\sin(2(m-l)\pi t) & m = l + 1, \cdots, 2l, \end{cases} \]

\( n = 0, 1, \cdots, 2^k - 1, k = 0, 1, \cdots \) and \( l \) is any positive integer. A function \( f(t) \in L^2[0,1] \) can be approximated as:

\[ f(t) = \sum_{n=0}^{2^k-1} \sum_{m=0}^{2l} c_{n,m} \psi_{n,m}(t) = C^T \Psi(t) = \Psi^T(t) C, \]

where \( c_{n,m} = \langle f(t), \psi_{n,m} \rangle \) and \( \langle , , \rangle \) denotes the inner product as:

\[ c_{n,m} = \int_{-\infty}^{+\infty} f(t) \psi_{n,m}(t) dt. \]

\( \Psi(t) \) represent the vector of considered wavelet. \( C \) and \( \Psi(t) \) are \( 2^k(2l + 1) \times 1 \) matrices which are given by:

\[ C^T = [c_{0,0}, c_{0,1}, \cdots, c_{0,2l}, c_{1,0}, \cdots, c_{1,2l}, \cdots, c_{2^k-1,0}, \cdots, c_{2^k-1,2l}], \]
\[ \Psi^T = [\psi_{0,0}, \psi_{0,1}, \cdots, \psi_{0,2l}, \psi_{1,0}, \cdots, \psi_{1,2l}, \cdots, \psi_{2^k-1,0}, \cdots, \psi_{2^k-1,2l}]. \]
5. Operational matrix of fractional calculus for Sine-Cosine wavelet

In this section, we derive the operational matrix of fractional derivative for the considered wavelet using the operational matrix of fractional integration for HFs.

5.1. Express \( \Psi(t) \) in terms of HFs

Each \( \psi_{n,m}(t) \) as a function, can be expanded in terms of HFs function, thus for \( \Psi(t) \) we will have

\[
\Psi(t) \simeq \Phi_{\hat{m} \times \hat{m}} \Phi(t).
\] (11)

In the above equation \( \hat{m} = 2^k (2l + 1) \) and \( \Phi_{\hat{m} \times \hat{m}} \) obtain as follow. We choose \( h = \frac{1}{\hat{m}-1} \), then by considering the property which is given in Eq. (6) we have

\[
\psi_{n,m}(t) = \sum_{i=0}^{\hat{m}-1} c_i^{n,m} \phi_i(t) = C_{n,m} \Phi(t),
\]

\[
c_i^{n,m} = \psi_{n,m}(ih),
\]

\[
C_{n,m} = [c_0^{n,m}, c_1^{n,m}, \ldots, c_{\hat{m}-1}^{n,m}] = [\psi_{n,m}(0), \psi_{n,m}(h), \ldots, \psi_{n,m}(1)],
\]

therefore

\[
\psi_{n,m}(t) = [\psi_{n,m}(0), \psi_{n,m}(h), \ldots, \psi_{n,m}(1)] \Phi(t).
\] (12)

The vector \( C_{n,m} \) represent a row of matrix \( \Phi_{\hat{m} \times \hat{m}} \).

5.2. Operational matrix of fractional integration and derivative for Sine-Cosine wavelet

Suppose \( \Psi(t) \) be the vector defined in (9), then, fractional integration of order \( \alpha > 0 \) in the Riemann-Liouville sense of this vector can be expressed as

\[
(I^\alpha \Psi)(t) \simeq P^\alpha \Psi(t),
\] (13)

where \( P^\alpha \) is the operational matrix of fractional integration. By considering Eq. (11), \( P^\alpha \) calculated as follows

\[
I^\alpha \Psi(t) \simeq I^\alpha \Phi_{\hat{m} \times \hat{m}} \Phi(t) = \Phi_{\hat{m} \times \hat{m}} I^\alpha \Phi(t) \simeq \Phi_{\hat{m} \times \hat{m}} Q^\alpha \Phi(t).
\] (14)

Using Eqs. (13) and (14) we obtain

\[
P^\alpha \Psi(t) \simeq P^\alpha \Phi_{\hat{m} \times \hat{m}} \Phi(t) = \Phi_{\hat{m} \times \hat{m}} Q^\alpha \Phi(t) \Rightarrow P^\alpha = \Phi_{\hat{m} \times \hat{m}} Q^\alpha \Phi_{\hat{m} \times \hat{m}}^{-1}.
\] (15)

Now we calculate operational matrix of derivative using \( P^\alpha \), suppose that \( x(t) \simeq X^T \Psi(t) \) then we have

\[
D^\alpha f(t) = I^{n-\alpha} f^{(n)}(t) \quad n-1 < \alpha \leq n, \quad n \in \mathbb{N},
\]

\[
D^\alpha x(t) \simeq D^\alpha X^T \Psi(t) = X^T D^\alpha \Psi(t) = X^T I^{n-\alpha} \Psi^{(n)}(t),
\] (16)

for \( \alpha \in (0, 1] \) we have \( n = 1 \) thus

\[
D^\alpha x(t) \simeq X^T I^{1-\alpha} D \Psi(t) = X^T D I^{1-\alpha} \Psi(t) = X^T D \Phi_{\hat{m} \times \hat{m}} Q^{1-\alpha} \Phi_{\hat{m} \times \hat{m}}^{-1} \Psi(t),
\] (17)
where $D$ is the operational matrix of derivative for $\Psi(t)$ which defined as $D = \text{diag}(W, W, \cdots, W)$, which is a $2^k(2l + 1) \times 2^k(2l + 1)$ matrix and $W$ is of size $(2l + 1) \times (2l + 1)$

$$W = 2^{k+1} \pi \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & -2 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & -l \\ 0 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 & 0 & \cdots & 0 \\ \end{pmatrix}.$$

$$X(t) = [x_1(t), x_2(t), \cdots, x_s(t)]^T \quad x_i(t) = \Psi^T(t)X_i = X_i^T \Psi(t),$$

$$X(t) = \tilde{\Psi}_s^T(t)X \quad X = [X_1^T, X_2^T, \cdots, X_s^T]^T \quad \tilde{\Psi}_s(t) = I_s \otimes \Psi(t),$$

$$U(t) = [u_1(t), u_2(t), \cdots, u_q(t)]^T \quad u_i(t) = \Psi^T(t)U_i = U_i^T \Psi(t),$$

$$U(t) = \tilde{\Psi}_q^T(t)U \quad U = [U_1^T, U_2^T, \cdots, U_q^T]^T \quad \tilde{\Psi}_q(t) = I_q \otimes \Psi(t),$$

thus

$$J = \frac{1}{2} X^T \tilde{\Psi}_s(1)SX(1) + \frac{1}{2} \int_0^1 [X^T \tilde{\Psi}_s Q \tilde{\Psi}_s^T X + U^T \tilde{\Psi}_q R \tilde{\Psi}_q^T U] dt.$$  

The considered wavelet is orthonormal, it means $\int_0^1 \Psi^T(t)\Psi(t) dt = I$, thus we can rewrite (21) as follows

$$J(X, U) = \frac{1}{2} X^T[S \otimes \tilde{\Psi}_s(1)]X + \frac{1}{2} [X^T(Q \otimes I)X + U^T(R \otimes I)U],$$

similarly, we use (17) and (20) for the dynamic system in (19)

$$X(t) = X^T I_s \otimes \Psi(t) = (I_s \otimes \Psi(t))X,$$

$$D^\alpha X(t) = I^{1-\alpha} X(t) = I^{1-\alpha} (X^T (I_s \otimes \Psi(t)))' = X^T I^{1-\alpha} (I_s \otimes (D \Psi(t))) = X^T I_s \otimes (I^{1-\alpha} D \Psi(t))$$

$$= X^T I_s \otimes (D I^{1-\alpha} \Psi(t)) = X^T I_s \otimes (D \Psi_{m \times \tilde{m}} Q^{1-\alpha} \Phi_{m \times \tilde{m}}^{-1} \Psi(t)),$$

using dynamic system of (19) and (23), we set

$$R(t) = X^T I_s \otimes (D \Psi_{m \times \tilde{m}} Q^{1-\alpha} \Phi_{m \times \tilde{m}}^{-1}) \Psi(t) - AX^T I_s \otimes \Psi(t) - BU^T I_q \otimes \Psi(t),$$

$$R(t) = [X^T I_s \otimes (D \Psi_{m \times \tilde{m}} Q^{1-\alpha} \Phi_{m \times \tilde{m}}^{-1}) - AX^T I_s \otimes I_{2^k(2l+1)} - BU^T I_q \otimes I_{2^k(2l+1)}] \Psi(t).$$
As in a typical tau method \[11\] we generate \(2^k(2l + 1) - 1\) linear equations by applying
\[
\langle R(t), \psi_{n,m}(t) \rangle = \int_0^1 R(t)\psi_{n,m}(t)dt = 0, \tag{24}
\]
also, for boundary value we have
\[
X(0) = X^T \Psi(0) = X_0. \tag{25}
\]
Eqs. (24) and (25) generate \(2^k(2l + 1)\) set of linear equations. These linear equations can be solved for unknown coefficients of the vectors \(X^T\) and \(U^T\). Consequently, \(X(t)\) and \(U(t)\) can be calculated.

7. Illustrative example

We applied the method presented in this paper and solved the underlying examples.

**Example 1**
Consider the following time invariant FOCP \[25\]
\[
\begin{align*}
\min J &= \frac{1}{2} \int_0^1 (x^2(t) + u^2(t))dt, \\
D^\alpha x(t) &= -x(t) + u(t), \\
x(0) &= 1.
\end{align*} \tag{26}
\]
We want to find a control variable \(u(t)\) which minimizes the quadratic performance index \(J\). This problem is solved by proposed method, the numerical value obtained for \(J\) is 0.1932, which is close to the exact solution in the case \(\alpha = 1\) (0.1929).

**Example 2**
Consider the following functional
\[
\begin{align*}
\min J(t) &= \int_0^1 [(D^\alpha x(t))^2 + t D^\alpha x(t)]dt, \\
\text{and the boundary conditions} & \quad x(0) = 0 \text{ and } x(1) \text{ is unspecified.}
\end{align*} \tag{27}
\]
For solving the above problem we use the underlying relation
\[
D^\alpha x(t) = C^T \Psi(t) \Rightarrow x(t) = C^T P^\alpha \Psi(t) + x(0), \tag{28}
\]
we expand \(t\) in terms of considered wavelet as \(t = d^T \Psi(t)\) where \(d\) is as follows
\[
d = 2^k \left[ \begin{array}{cccccc}
1, & 0, & \cdots, & 0, & -\frac{\sqrt{2}}{\pi}, & \cdots, & -\frac{\sqrt{2}}{l\pi}, \\
0, & \cdots, & 0, & -\frac{\sqrt{2}}{\pi}, & \cdots, & -\frac{\sqrt{2}}{l\pi}, \\
& \cdots, & 0, & \cdots, & 0, & -\frac{\sqrt{2}}{\pi}, & \cdots, & -\frac{\sqrt{2}}{l\pi} \end{array} \right] T, \tag{29}
\]
because \(x(1)\) is unspecified we have
\[
2D^\alpha x(t) + t|_{t=1} = 0 \Rightarrow C^T \Psi(1) = -\frac{1}{2}, \tag{30}
\]
by substituting Eqs.(28) and (29) in (27) we get
\[
J(t) = \int_0^1 [C^T \Psi(t)\Psi(t)^TC + d^T \Psi(t)\Psi(t)^TC]dt = C^TC + d^TC, \tag{31}
\]
now we have to minimize quadratic function (31) subject to constraint (30). The exact value for \(J\) is -0.0833 \[22\] and the value obtained via above method is -0.08328635, which is acceptable.
8. Conclusion

In this paper, we use a direct numerical method for fractional problems based on two new operational matrices of fractional integration and differentiation. The procedure of constructing these matrices is summarized. This matrices are utilized along with tau method in order to reduce the fractional differential equations into the algebraic equations which can be efficiently solved. The proposed approach is computationally simple. Two examples are given to show the efficiency of method. The result obtained in this paper is more acceptable in comparison with [24], where the fractional operational matrices of the Sine-Cosine wavelet are obtained using block-pulse functions. The obtained matrices can also be used to solve fractional optimal control with delay or multi-delay.

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