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GMM Estimation of Continuous-Time Bilinear Processes

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Abstract This paper examines the moments properties in frequency domain of the class of first order continuous-time bilinear processes ($COBL(1,1)$ for short) driven by a stochastic differential equation defined by a linear drift and certain diffusion processes with time-varying (resp. time-invariant) coefficients. So, we used the associated evolutionary (or time-varying) transfer functions to study the structure of second-order of the process and its powers. In particular, for time-invariant case, an expression of the moments of any order are given as well as some moments properties of special cases. As a consequence, it is observed however, that for this class explicit statistical inference is feasible. So and based on these results we are able to estimate the unknown parameters involved in model via the so-called generalized method of moments (GMM). This method is robust to the misspecification of likelihood functions. However, it suffers from the $ad\ hoc$ choice of moment conditions and must presume the existence of arbitrary population moments, and the $chisquare$ specification test of the overidentifying restrictions is subject to severe overrejection bias. In end, the GMM method is illustrated by a Monte Carlo study and applied to modelling two foreign exchange rates of Algerian Dinar against $U.S.-$Dollar (USD/DZD) and against the single European currency Euro (EUR/DZD).

Keywords Continuous-time bilinear processes, Pearson diffusion process, Evolutionary transfer functions, Spectral representation, GMM estimation.

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1. Introduction

Stochastic differential equation (SDE) models play an interest role in a range of applications areas, including, among others, biology, chemistry, epidemiology, mechanics, economics, and finance. This interest is due to the fact that in many practical situations, the data generating the process, are often observed at discrete-time intervals and irregularly spaced. This phenomenon happens for instance in physics, engineering problems, economy and so on. Therefore, the resort to continuous-time (which can be interpreted as a solution of some SDE models) is unavoidable. Some example we have in mind is in economics, since variables in most models are the result of many large number of microeconomics decision at different points of time, which may be regarded as continuous functions of time. So, during the past years, the theory and applications of stochastic differential equations have been developed very quickly, see e.g. ∅ksendal [18] and it becomes increasingly important in modeling and forecasting financial time series and continues to gain a growing interest of researchers whether in their statistics inference or in their applications.

In this paper we consider the class of continuous-time bilinear processes $(X(t))_{t\in\mathbb{R}_+}$ ($COBL$ for short) generated by the following time-varying SDE

$$\begin{aligned} dX(t) &= (\alpha(t)X(t) + \mu(t))\,dt + (\gamma(t)X(t) + \beta(t))\,dW(t),\ X(0) = X_0, \\ &= \mu_t\,(X(t))\,dt + \sigma_t\,(X(t))\,dW(t), \end{aligned} \tag{1}$$

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where $\mu_t(x) = \alpha(t)x + \mu(t)$ and $\sigma_t(x) = \gamma(t)x + \beta(t)$ represents the drift function and the volatility (or diffusion) term respectively, $(W(t))_{t\geq 0}$ is a standard Brownian motion in $\mathbb{R}$ defined on some basic filtered space $(\Omega, \mathcal{A}, (\mathcal{A}_t)_{t\geq 0}, P)$ with spectral representation $W(t) = \int_{\mathbb{R}} \frac{e^{it\lambda}-1}{i\lambda} dZ(\lambda)$, where $Z(\lambda)$ is an orthogonal complex-valued stochastic measure on $\mathbb{R}$ with zero mean, $E\left\{|dZ(\lambda)|^2\right\} = dF(\lambda) = \frac{d\lambda}{2\pi}$ and uniquely determined by $Z([a,b[) = \frac{1}{2\pi}\int_{\mathbb{R}} \frac{e^{-i\lambda a} - e^{-i\lambda b}}{i\lambda} dW(\lambda)$, for all $-\infty < a < b < +\infty$, the initial state $X(0)$ is a random variable, defined on $(\Omega, \mathcal{A}, P)$ independent of W such that $E\{X(0)\} = m_1(0)$ and $Var\{X(0)\} = R_1(0)$. Special cases of this process are the Brownian motion with drift ($\alpha(t) = 0$ and $\gamma(t) = 0$), the Gaussian Ornstein-Uhlenbeck (GOU) process ($\gamma(t) = 0$) and the volatility of the $COGARCH(1,1)$ process defined by $dX(t) = \sigma(t)\, dW_1(t)$ where $d\sigma^2(t) = \left(\mu(t) - \alpha(t)\sigma^2(t)\right) dt + \gamma(t)\sigma^2(t)\, dW_2(t)$ in which $\mu(t) > 0$, $\alpha(t)$, $\gamma(t) \geq 0$ for all $t \geq 0$ and $W_1(t)$, $W_2(t)$ are independent Bm and independent of $(X(0), \sigma(0))$. The SDE (1) is called time-invariant if there exists some constants α, μ, γ and β such that for all t, $\alpha(t) = \alpha$, $\mu(t) = \mu$, $\gamma(t) = \gamma$ and $\beta(t) = \beta$. Note that in time-invariant case, the correspondent SDE (1) may be rewritten as a Pearson diffusion process, i.e.,

$$dX(t) = -\theta\,(X(t) - u)\, dt + \sqrt{2\theta\left(aX^2(t) + bX(t) + c\right) dW(t)}, \tag{2}$$

where $\theta > 0$ and a, b, c are such that the square root is well defined and reduce to $\sigma(X(t))$. In literature of diffusion processes, there is two functions associated to equation (2) which plays a fundamental role in studying the stationary solution of such SDE (2) called scale and speed densities given respectively by

$$s(x) = \exp\left\{\int_{x_0}^{x} \frac{y-u}{ay^2+by+c} dy\right\} \text{ and } m(x) = \frac{1}{s(x)\,(ax^2+bx+c)},$$

where x_0 is a fixed point such that $ax_0^2 + bx_0 + c > 0$. The main aim here is focused firstly on the conditions ensuring the existence of the processes $(X(t))_{t\in\mathbb{R}_+}$ and its powers $(X^k(t))_{t\in\mathbb{R}_+}$, $k \geq 2$, using the evolutionary transfer functions associated with the model. Secondly, we extend the generalized method of moments (GMM) by Hansen [9] for a discretized time-invariant version of SDE (1) and hence estimates of the parameters involving in the model and hence their asymptotic properties. To ensure the existence and uniqueness of the solution process $(X(t))_{t\geq 0}$ of equation (1) we assume that the parameters $\alpha(t)$, $\mu(t)$, $\gamma(t)$ and $\beta(t)$ are measurable deterministic functions and subject to the following assumption:

Assumption 1
$\alpha(t)$, $\mu(t)$, $\gamma(t)$ and $\beta(t)$ are differentiable functions such that $\forall T > 0$,

$$\int_0^T |\alpha(t)|\, dt < \infty,\ \int_0^T |\mu(t)|\, dt < \infty,\ \int_0^T |\gamma(t)|^2\, dt < \infty \text{ and } \int_0^T |\beta(t)|^2\, dt < \infty.$$

In statistical inference of SDE, estimation methods have usually carried by some discretization schema and hence various techniques are adapted. So, among others, the GMM which may be seen as a semi$-$parametric approach and its implementation is defined by minimizing the weighted distance between the sample moments and the corresponding population moments implied by the model structure, occupies an important place in statistical inference. Indeed, Kallsen and Muhle-Karbe [12], and Haug et al. [8] have proposed an asymptotic inference of moments method (MM) for a discretized continuous $GARCH$ process. Bibi and Merahi [4] have proposed a MM for estimating the parameters of continuous-time bilinear processes. Chan et al. [7] investigated an empirical comparison of GMM for several discretized diffusions processes. Carrasco and Florens [6] have study the GMM in cases where the implementation of this method requires simulations of random numbers. Bollerslev and Zhou [5] have exploit the distributional information contained in high-frequency data in constructing a simple conditional moment estimator for stochastic volatility diffusion. Hlouskova and Sögner [10] have investigates parameter estimation of affine term structure models via GMM. In empirical finance, Zhou [20] perform a Monte Carlo study on MLE, Quasi-MLE $(QMLE)$, GMM, and on efficient method of moments (EMM) for a continuous-time square-root process. He finds the following ranking by decreasing efficiency: MLE, $QMLE$, EMM, and

GMM. All these studies focused on specific models and their conclusions may not carry to another model. For in deep lecture we advised interested readers to see Kessler [14] and the reference therein and to monographs by Rao [19] and Kutoyants [15].

The remainder of the paper is structured as follows. Section 2 outline the Wiener-Itô spectral representation for $SDE\,(1)$, and a recursive evolutionary transfer functions of $SDE\,(1)$ are given so the associated spectral representation of $(X(t))_{t\geq 0}$ and its powers are showed. Section 3, investigated the moments properties of $(X(t))_{t\geq 0}$ and its powers and an explicit formula for time-invariant version are derived. Section 4, is dedicated for the estimate of time invariant $SDE\,(1)$ via GMM estimation, so its consistency and asymptotic normality are studied. Numerical illustrations via Monte Carlo simulation are given in Section 5 followed by an application to model two foreign exchange rates of Algerian Dinar against $U.S.$-Dollar (USD/DZD) and against the single European currency Euro (EUR/DZD).

2. Framework

The existence and uniqueness of the solution process of $SDE\,(1)$ in time domain is ensured by the general results on SDE and under the Assumption 1. Moreover, since the drift and the diffusion functions are Lipschitz with linear growth, i.e., $|\mu_t(x) - \mu_t(y)| \leq \sup_t |\alpha(t)|\,|x-y|$ and $|\sigma_t(x) - \sigma_t(y)| \leq \sup_t |\gamma(t)|\,|x-y|$, then the Itô solution is given by (see Le Breton and Musiela [16] and Bibi and Merahi [3])

$$X(t) = \Phi(t)\left\{X(0) + \int_0^t \Phi^{-1}(s)\left(\mu(s) - \gamma(s)\beta(s)\right)ds + \int_0^t \Phi^{-1}(s)\beta(s)\,dW(s)\right\}, a.e., \tag{3}$$

where the process $(\Phi(t))_{t\geq 0}$ is given by $\Phi(t) = \exp\left\{\int_0^t \left(\alpha(s) - \frac{1}{2}\gamma^2(s)\right)ds + \int_0^t \gamma(s)\,dW(s)\right\}$ its mean function is $\Psi(t) = \exp\left\{\int_0^t \alpha(s)ds\right\}$. In time-invariant case $(\Phi(t))_{t\geq 0}$ reduces to $\Phi(t) = \exp\{-\xi(t)\}$ where $-\xi(t) = \left(\alpha - \frac{1}{2}\gamma^2\right)t + \gamma W(t)$ and thus the solution process (3) reduces to

$$X(t) = e^{-\xi(t)}\left\{X(0) + \int\limits_0^t e^{\xi(s)}d\eta(s)\right\}, t \geq 0, \tag{4}$$

with $\eta(t) = (\mu - \gamma\beta)\,t + \beta W(t)$, that is the solution process of the celebrated generalized Ornstein-Uhlenbeck process. Hence, by Itô formula, we obtain $dX(t) = -\xi(t)X(t)dt + d\eta(t), t \geq 0, X(0) = X_0$. So, the above equation can be interpreted as a random coefficient time-continuous autoregressive version of $SDE\,(1)$. Moreover, the process given by (4) is Markovtan, unique, ergodic and constitute a weak solution to $SDE\,(1)$ with values in the interval $]r, l[$ containing x_0 if and only if $\int_{x_0}^l s(x)dx = +\infty$, $\int_r^{x_0} s(x)dx = +\infty$ and $\int_r^l m(x)dx < +\infty$. Additionally, since

$$\frac{dm(x)}{dx} = \frac{(2a+1)\,x - u + b}{ax^2 + bx + c}m(x),$$

we see that if $SDE\,(1)$ admits a stationary solution, then its invariant distribution belong to the family $\{signU^{-1}\left(\frac{\beta}{\gamma} - \frac{\mu}{\alpha}\right) - \frac{\beta}{\gamma}\}$ where the random variable U has the gamma distribution $\Gamma\left(1 - \frac{2\alpha}{\gamma^2}, \frac{\gamma^2}{2|\alpha|\left|\frac{\beta}{\gamma} - \frac{\mu}{\alpha}\right|}\right)$ (see Lebreton and Musiela [16] for more details).

Example 1
In following table, we give the scale and speed densities $s(x)$ and $m(x)$ for some specifications.

Specifications	$s(x)$	$m(x)$
$Brownian\ motion\ with\ drift:$ $dX(t)=\mu dt+dW(t)$	$\exp\{-2\mu x\}$	$2\exp\{2\mu x\}$
$GOU\ \ process:$ $dX(t)=(\alpha X(t)+\mu)\,dt+dW(t)$	$\exp\{-2\mu x+\alpha x^2\}$	$-2\alpha\exp\{2\mu x-\alpha x^2\}$

Table(1) : Scale and speed densities for some specifications.

For the Brownian motion with drift there exists a unique ergodic solution for all $\mu\in\mathbb{R}$. In GOU process, for all $\mu\in\mathbb{R}$, there exists a unique ergodic solution if and only if $\alpha<0$ and the invariant distribution is $\mathcal{N}(\mu,1)$.

2.1. *Wiener-Itô representation*

In the first part of this paper, we shall investigate in frequency domain, some probabilistic and statistical properties of second-order solution process of equation (1) which are regular (or causal), i.e., $X(t)$ is $\sigma\{W(s),s\leq t\}-$measurable. Such solution were given by Iglói and Terdik [11] for time-invariant version of SDE (1). For this purpose, let $\Im=\Im(W):=\sigma(W(t),t\geq 0)$ (resp. $\Im_t:=\sigma(W(s),s\leq t)$) be the σ-algebra generated by $(W(t))_{t\geq 0}$ (resp. by $W(s)$ up to time t) and let $\mathbb{L}_2(\Im)=\mathbb{L}_2(\mathbb{C},\Im,P)$ be the Hilbert space of non-linear $\mathbb{L}_2-$functional of $(W(t))_{t\geq 0}$. It is well-known that any regular second-order process $(X(t))_{t\geq 0}$ (i.e., $X(t)$ is $\Im_t-$measurable) admits the so-called Wiener-Itô orthogonal (or also chaotic) representation (see for instance Major [17]), i.e.,

$$X(t)=g_t(0)+\sum_{r\geq 1}\frac{1}{r!}\int_{\mathbb{R}^r}g_t(\lambda_{(r)})e^{it\underline{\lambda}_{(r)}}dZ(\lambda_{(r)}),\tag{5}$$

wherein $g_t(0)=E\{X(t)\}$, $\lambda_{(r)}=(\lambda_1,...,\lambda_r)\in\mathbb{R}^r$, $\underline{\lambda}_{(r)}=\sum_{i=1}^{r}\lambda_i$ with $\lambda_{(0)}=\underline{\lambda}_{(0)}=0$, and the integrals in (5) are the multiple Wiener-Itô stochastic integrals with respect to the stochastic measure $dZ\left(\lambda_{(r)}\right)=\prod_{i=1}^{r}dZ(\lambda_i)$ and $\left(g_t(\lambda_{(r)})\right)_{r\geq 0}$ are referred as the $r-th$ evolutionary transfer functions (see Bibi [2] for more details) uniquely determined up to symmetrization and $g_t(\lambda_{(r)})\in\mathbb{L}_2(F)=\mathbb{L}_2(\mathbb{C}^n,B_{\mathbb{C}^n},F)$ for all $t\geq 0$, i.e., $\sum_{r\geq 0}\frac{1}{r!}\|g_t\|^2<\infty$ for all t, where $\|g_t\|^2=\int_{\mathbb{R}^r}\left|g_t(\lambda_{(r)})\right|^2dF(\lambda_{(r)})$ with $dF(\lambda_{(r)})=\frac{1}{(2\pi)^r}d\lambda_{(r)}$ and $d\lambda_{(r)}=\prod_{i=1}^{r}d\lambda_i$. As a consequence of the representation (5) is that for any $f_t(\lambda_{(n)})$ and $f_s(\lambda_{(m)})$, we have

$$E\left\{\int_{\mathbb{R}^n}f_t(\lambda_{(n)})dZ(\lambda_{(n)})\overline{\int_{\mathbb{R}^m}f_s(\lambda_{(m)})dZ(\lambda_{(m)})}\right\}=\delta_n^m n!\int_{\mathbb{R}^n}Sym\left\{f_t(\lambda_{(n)})\right\}\overline{Sym\left\{f_s(\lambda_{(n)})\right\}}dF(\lambda_{(n)}),\tag{6}$$

where δ_n^m is the delta function and $Sym\left\{f_t(\lambda_{(n)})\right\}=\frac{1}{n!}\sum_{\pi\in\Pi(n)}f_t\left(\lambda_{\pi(n)}\right)$ where $\Pi(n)$ denotes the group of all permutation of the set $\{1,...,n\}$. Another consequence linked with (5) is the diagram formula which state that

$$\begin{aligned}&\int_{\mathbb{R}}f_t(\lambda)dZ(\lambda)\int_{\mathbb{R}^n}g_s\left(\lambda_{(n)}\right)dZ(\lambda_{(n)})\\&=\int_{\mathbb{R}^{n+1}}g_s\left(\lambda_{(n)}\right)f_t\left(\lambda_{n+1}\right)dZ(\lambda_{(n+1)})+\sum_{k=1}^{n}\int_{\mathbb{R}^{n-1}}\int_{\mathbb{R}}g_s\left(\lambda_{(n)}\right)\overline{f_t\left(\lambda_k\right)}dF\left(\lambda_k\right)dZ(\lambda_{(n\backslash k)}),\end{aligned}\tag{7}$$

where $dZ(\lambda_{(n\backslash k)})=\prod_{i=1,i\neq k}^{n}dZ\left(\lambda_i\right)$. The following theorem due to Bibi and Merahi [3], in which a recursive evolutionary transfer functions associated to the second-order regular solution of SDE (1) is given.

Theorem 1
Assume that everywhere

$$2\alpha\left(t\right)+\gamma^{2}\left(t\right)<0, \tag{8}$$

then the process $(X(t))_{t\geq 0}$ generated by the SDE (1) has a regular second-order solution given by the series (5) where the evolutionary symmetrized transfer functions of this solution are given by the symmetrization of the solution of the following first-order differential equation

$$g_t^{[1](1)}(\lambda_{(r)})=\begin{cases}\alpha(t)g_t^{[1]}(0)+\mu(t), \text{ if } r=0\\ \left(\alpha(t)-i\underline{\lambda}_{(r)}\right)g_t^{[1]}(\lambda_{(r)})+r\left(\gamma(t)g_t^{[1]}(\lambda_{(r-1)})+\delta_{\{r=1\}}\beta(t)\right), \text{if } r\geq 1,\end{cases} \tag{9}$$

where $g_t^{[1]}(0)=E\left\{X(t)\right\}$ and the superscript $^{(j)}$ denotes $j-$fold differentiation with respect to t.

Remark 1
The existence and uniqueness of the evolutionary transfer functions $g_t^{[1]}(\lambda_{(r)}),(t,r)\in\mathbb{R}\times\mathbb{N}$ of (9) are ensured by general results on linear ordinary differential equations, so

$$g_t^{[1]}(\lambda_{(r)})=\begin{cases}\varphi_1\left(t\right)\left(g_0^{[1]}(0)+\int\limits_0^t\varphi_1^{-1}\left(s\right)\mu(s)ds\right) \text{ if } r=0,\\ \varphi_{1,t}\left(\underline{\lambda}_{(r)}\right)\left(g_0^{[1]}(\lambda_{(r)})+r\int\limits_0^t\varphi_{1,s}^{-1}\left(\underline{\lambda}_{(r)}\right)\left(\gamma(s)g_s^{[1]}(\lambda_{(r-1)})+\delta_{\{r=1\}}\beta(s)\right)ds\right) \text{ if } r\geq 1,\end{cases} \tag{10}$$

where $\varphi_{1,t}\left(\underline{\lambda}_{(r)}\right)=\exp\left\{\int\limits_0^t\left(\alpha(s)-i\underline{\lambda}_{(r)}\right)ds\right\}$ and $\varphi_1\left(t\right)=\varphi_{1,t}\left(0\right)$.

In time-invariant case we shall assume through the paper that

$$\alpha,\mu,\gamma,\beta\in\mathbb{R},\ \gamma\neq 0,\alpha\beta\neq\mu\gamma,2\alpha+\gamma^2<0. \tag{11}$$

Remark 2
The condition $\alpha\beta\neq\mu\gamma$ is imposed otherwise the time-invariant version of (1) has only a degenerated solution given by $X(t)=-\dfrac{\beta}{\gamma}=-\dfrac{\mu}{\alpha}$.

Example 2
In time-invariant version and under the condition (11), the transfer functions $g^{[1]}(\lambda_{(r)})$ for all $r\in\mathbb{N}$ are given by

$$g^{[1]}(\lambda_{(r)}))=\begin{cases}-\dfrac{\mu}{\alpha}, r=0,\\ \left(i\underline{\lambda}_{(r)}-\alpha\right)^{-1}\left(r\gamma g^{[1]}(\lambda_{(r-1)})+\delta_{\{r=1\}}\beta\right), r\geq 1,\end{cases}$$

or equivalently $g^{[1]}(\lambda_{(r)})=\gamma^{r-1}r!\left(\beta-\dfrac{\mu}{\alpha}\gamma\right)\prod\limits_{j=1}^{r}\left(i\underline{\lambda}_{(j)}-\alpha\right)^{-1}$ and the symmetrized version can be rewritten as $Sym\left\{g^{[1]}(\lambda_{(r)})\right\}=\left(\mu\gamma-\alpha\beta\right)\gamma^{r-1}\int\limits_0^{+\infty}\exp\left\{\alpha\lambda\right\}\prod\limits_{j=1}^{r}\dfrac{1-\exp\left\{-i\lambda\lambda_j\right\}}{i\lambda_j}d\lambda$, so

$$m_1=-\frac{\mu}{\alpha},\ R_1(\tau)=Cov(X(t),X(t+\tau))=R_1(0)e^{\alpha|\tau|}, \tag{12}$$

where $R_1(0)=\dfrac{\left|\alpha\beta-\mu\gamma\right|^2}{\alpha^2\left|2\alpha+\gamma^2\right|}$. Hence, the second-order properties for time-invariant versions of the nested models can be easily deduced.

2.2. Wiener-Itô representation for $\left(X^k(t)\right)_{t\geq 0}$

In this subsection, we examine the structure of the process $\left(X^k(t)\right)_{t\geq 0}$, $\forall k \geq 2$ in which the condition

$$2\alpha(t) + (2k-1)\gamma^2(t) < 0, \ a.e., \text{ for all } t \geq 0, \tag{13}$$

is imposed. The following lemma give the Wiener-Itô representation of $\left(X^k(t)\right)_{t\geq 0}$.

Lemma 1
Suppose that the solution process of SDE (1) is regular. Then under the condition (13), the process $\left(X^k(t)\right)_{t\geq 0}$ is regular and has a Wiener-Itô representation, i.e.,

$$X^k(t) = g_t^{[k]}(0) + \sum_{r\geq 1} \frac{1}{r!} \int_{\mathbb{R}^r} e^{it\underline{\lambda}_{(r)}} g_t^{[k]}(\lambda_{(r)}) dZ(\lambda_{(r)}),$$

where the evolutionary transfer functions $g_t^{[k]}(\lambda_{(r)})$, $r \geq 0$ satisfying the following first-order differential equation

$$g_t^{[k](1)}(\lambda_{(r)}) \tag{14}$$
$$= \begin{cases} k\left(\alpha(t) + \frac{1}{2}\gamma^2(t)(k-1)\right) g_t^{[k]}(0) + k\left(\gamma(t)\beta(t)(k-1) + \mu(t)\right) g_t^{[k-1]}(0) + \frac{1}{2}\beta^2(t)k(k-1)g_t^{[k-2]}(0) & \text{if } r = 0, \\ \left(k\left(\alpha(t) + \frac{1}{2}\gamma^2(t)(k-1)\right) - i\underline{\lambda}_{(r)}\right) g_t^{[k]}(\lambda_{(r)}) + k\left(\gamma(t)\beta(t)(k-1) + \mu(t)\right) g_t^{[k-1]}(\lambda_{(r)})) \\ + \frac{1}{2}\beta^2(t)k(k-1)g_t^{[k-2]}(\lambda_{(r)}) + kr\left(\gamma(t)g_t^{[k]}(\lambda_{(r-1)}) + \beta(t)g_t^{[k-1]}(\lambda_{(r-1)})\right) & \text{if } r \geq 1 \end{cases}$$

Proof
The proof follows upon the observation that by applying the Itô's formulae for $f(x) = x^k$ for any integer $k \geq 2$, then the process $\left(X^k(t)\right)_{t\geq 0}$ satisfying the following stochastic differential equation

$$dX^k(t) = \left(k\left(\alpha(t) + \frac{1}{2}\gamma^2(t)(k-1)\right)X^k(t) + k\left(\mu(t) + \gamma(t)\beta(t)(k-1)\right)X^{k-1}(t) + \frac{1}{2}\beta^2(t)k(k-1)X^{k-2}(t)\right)dt$$
$$+ k\left(\gamma(t)X^k(t) + \beta(t)X^{k-1}(t)\right)dW(t), \ a.e.,$$

so, using the diagram formula (7) the result follows. □

Remark 3
The existence and uniqueness of the evolutionary symmetrized transfer functions $g_t^{[2]}(\lambda_{(r)}), (t,r) \in \mathbb{R}\times\mathbb{N}$ given by (14) is ensured by general results on linear ordinary differential equations (see, e.g., [13], chap. 1) so, the evolutionary transfer functions $g_t^{[k]}$ are given recursively by

$$g_t^{[k]}(\lambda_{(r)}) = \begin{cases} \varphi_{k,t}(0)\left(g_0^{[k]}(0) + \int_0^t \varphi_{k,s}^{-1}(0)\,\mu_s^{[k]}(0)ds\right) & \text{if } r = 0, \\ \varphi_{k,t}\left(\underline{\lambda}_{(r)}\right)\left(g_0^{[k]}(\lambda_{(r)}) + \int_0^t \varphi_{k,s}^{-1}\left(\underline{\lambda}_{(r)}\right)\mu_s^{[k]}(\lambda_{(r)})ds\right) & \text{if } r \geq 1, \end{cases} \tag{15}$$

in which $\varphi_{k,t}\left(\underline{\lambda}_{(r)}\right) = \exp\left\{\int_0^t \left(k\left(\alpha(s) + \frac{1}{2}\gamma^2(s)(k-1)\right) - i\underline{\lambda}_{(r)}\right)ds\right\}$, $g_t^{[k]}(0) = m_k(t) = E\left\{X^k(t)\right\}, t \geq 0,$ and

$$\mu_t^{[k]}(\lambda_{(r)}) = \begin{cases} 2\left(\gamma(t)\beta(t) + \mu(t)\right) g_t^{[1]}\left(\lambda_{(r)}\right) + \beta^2(t)\delta_{\{r=0\}} + 2r\left(\gamma(t)g_t^{[2]}(\lambda_{(r-1)}) + \beta(t)g_t^{[1]}(\lambda_{(r-1)})\right), k = 2, \\ k\left((k-1)\gamma(t)\beta(t) + \mu(t)\right) g_t^{[k-1]}\left(\lambda_{(r)}\right) + \frac{1}{2}k(k-1)\beta^2(t)g_t^{[k-2]}\left(\lambda_{(r)}\right) \\ +kr\left(\gamma(t)g_t^{[k]}(\lambda_{(r-1)}) + \beta(t)g_t^{[k-1]}(\lambda_{(r-1)})\right), k \geq 3, \end{cases}$$

which reduces in time-invariant case to an elegant non recursive form and the moments of the process $\left(X^k(t)\right)_{t\geq 0}$ may be evaluated in function of their transfer functions.

3. Moments properties of $\left(X^{k}\left(t\right)\right)_{t\geq 0}$

Since (1) is non linear with deterministic coefficients, the solution process (5) is non Gaussian in general, its first and second moment however are insufficient for its identification and hence the resort to higher order moments for the identifiability purpose is therefore necessary. In this section, we examine the moments properties of the process $\left(X^{k}(t)\right)_{t\geq 0}, \forall k\geq 2$.

Theorem 2
Let $(X(t))_{t\geq 0}$ be the solution process of SDE (1), then under the condition (13), the mean $m_k(t)$, variance $R_k(t)$ and covariance functions $R_k(t,s)$ of $\left(X^{k}(t)\right)_{t\geq 0}$, $k\geq 2$ are given respectively for all $t\geq s\geq 0$ by

$$m_k(t) \tag{16}$$
$$= \varphi_k(t)\varphi_k^{-1}(s)\{m_k(s)+k\int_s^t \varphi_k(s)\varphi_k^{-1}(u)\left(\left(\mu(u)+(k-1)\gamma(u)\beta(u)\right)m_{k-1}(u)+\frac{1}{2}\beta^2(u)(k-1)m_{k-2}(u)\right)du\},$$
$$R_k(t)$$
$$= \phi_k(t)\phi_k^{-1}(s)R_k(s)+\int_s^t \phi_k(t)\phi_k^{-1}(u)[\gamma^2(u)m_k^2(u)+2k\left(\mu(u)+(2k-1)\gamma(u)\beta(u)\right)m_{2k-1}(u) \tag{17}$$
$$-2k\left(\mu(u)+(k-1)\gamma(u)\beta(u)\right)m_k(u)m_{k-1}(u)+k(2k-1)\beta^2(u)m_{2k-2}(u)-k(k-1)\beta^2(u)m_k(u)m_{k-2}(u)]du,$$
$$R_k(t,s)$$
$$= \varphi_k(t)\varphi_k^{-1}(s)\{R_k(s)+k\int_s^t \varphi_k(s)\varphi_k^{-1}(u)((\mu(u)+\gamma(u)\beta(u)(k-1))\,Cov\left(X^{k-1}(u),X^k(s)\right) \tag{18}$$
$$+\frac{1}{2}\beta^2(u)(k-1)Cov\left((X^{k-2}(u),X^k(s))\right)du\},$$

where $\varphi_k(t)=\exp\left\{k\int_0^t\left(\alpha(u)+\frac{1}{2}\gamma^2(u)(k-1)\right)du\right\}$ and $\phi_k(t)=\exp\left\{k\int_0^t\left(2\alpha(u)+(2k-1)\gamma^2(u)\right)du\right\}$.

Proof
The fact that $g_t^{[k]}(0)=m_k(t)$, then from (14) we can obtain the following ordinary differential equation

$$m_k^{(1)}(t)=k\left(\alpha\left(t\right)+\frac{1}{2}\gamma^2(t)(k-1)\right)m_k(t)+k\left(\gamma(t)\beta(t)(k-1)+\mu(t)\right)m_{k-1}(t)+\frac{1}{2}\beta^2(t)k(k-1)m_{k-2}(t), \tag{19}$$

and the expression (16) is obtained by solving the above differential equation. To prove (17) we have $Var\left\{X^k(t)\right\}=R_k(t)=E\left\{X^{2k}(t)\right\}-\left(E\left\{X^k(t)\right\}\right)^2$ with $E\left\{X^k(t)\right\}=m_k(t)=g_t^{[k]}(0)$ and $E\left\{X^{2k}(t)\right\}=m_{2k}(t)=g_t^{[2k]}(0), \forall t\geq 0$ which implies that $R_k(t)=g_t^{[2k]}(0)-\left(g_t^{[k]}(0)\right)^2$. By differentiating with respect to t and from the formula (14) in which we substitute respectively $\frac{dg_t^{[k]}(0)}{dt}$, $\frac{dg_t^{[2k]}(0)}{dt}$ we get

$$\begin{aligned}\frac{dR_k(t)}{dt} &= \frac{dg_t^{[2k]}(0)}{dt}-2g_t^{[k]}(0)\frac{dg_t^{[k]}(0)}{dt}\\
&= k\left(2\alpha(t)+\gamma^2(t)(2k-1)\right)\left(g_t^{[2k]}(0)-\left(g_t^{[k]}(0)\right)^2\right)+2k\left(\gamma(t)\beta(t)(2k-1)+\mu(t)\right)g_t^{[2k-1]}(0)\\
&\quad -2k\left(\gamma(t)\beta(t)(k-1)+\mu(t)\right)g_t^{[k]}(0)g_t^{[k-1]}(0)+\beta^2(t)k(2k-1)g_t^{[2k-2]}(0)-\beta^2(t)k(k-1)g_t^{[k]}(0)g_t^{[k-2]}(0)\\
&= k\left(2\alpha(t)+\gamma^2(t)(2k-1)\right)R_k(t)+2k\left(\gamma(t)\beta(t)(2k-1)+\mu(t)\right)m_{2k-1}(t)\\
&\quad -2k\left(\gamma(t)\beta(t)(k-1)+\mu(t)\right)m_k(t)m_{k-1}(t)+\beta^2(t)k(2k-1)m_{2k-2}(t)-\beta^2(t)k(k-1)m_k(t)m_{k-2}(t).\end{aligned}$$

Therefore, the expression (17) is ensured by applying the general results on linear ordinary differential equations. It remains to prove (18), then we have for all $t \geq s$

$$R_k(t,s) = Cov(X^k(t), X^k(s)) = \sum_{r\geq 1} \frac{1}{(r!)^2} E\left\{ \int_{\mathbb{R}^r} g_t^{[k]}\left(\lambda_{(r)}\right) e^{it\underline{\lambda}_{(r)}} dZ(\lambda_{(r)}) \overline{\int_{\mathbb{R}^r} g_s^{[k]}\left(\lambda_{(r)}\right) e^{is\underline{\lambda}_{(r)}} dZ(\lambda_{(r)})} \right\}.$$

By differentiating with respect to t and the use of the formula (14) we obtain

$$\begin{aligned}
&\frac{dR_k(t,s)}{dt} \\
&= \sum_{r\geq 1} \frac{1}{(r!)^2} E\left\{ \int_{\mathbb{R}^r} \frac{d\left(g_t^{[k]}\left(\lambda_{(r)}\right) e^{it\underline{\lambda}_{(r)}}\right)}{dt} dZ(\lambda_{(r)}) \overline{\int_{\mathbb{R}^r} g_s^{[k]}\left(\lambda_{(r)}\right) e^{is\underline{\lambda}_{(r)}} dZ(\lambda_{(r)})} \right\} \\
&= \sum_{r\geq 1} \frac{1}{(r!)^2} E\left\{ \int_{\mathbb{R}^r} \left(k\left(\alpha(t) + \frac{1}{2}(k-1)\gamma^2(t)\right) g_t^{[k]}\left(\lambda_{(r)}\right) + \mu_t^{[k]}\left(\lambda_{(r)}\right)\right) e^{it\underline{\lambda}_{(r)}} dZ(\lambda_{(r)}) \overline{\int_{\mathbb{R}^r} g_s^{[k]}\left(\lambda_{(r)}\right) e^{is\underline{\lambda}_{(r)}} dZ(\lambda_{(r)})} \right\},
\end{aligned}$$

Now apply the property of orthogonality (6) to get

$$\begin{aligned}
&\frac{dR_k(t,s)}{dt} = \\
&= k\left(\alpha(t) + \frac{1}{2}(k-1)\gamma^2(t)\right) \sum_{r\geq 1} \frac{1}{(r!)^2} E\left\{ \int_{\mathbb{R}^r} \left(g_t^{[k]}\left(\lambda_{(r)}\right)\right) e^{it\underline{\lambda}_{(r)}} dZ(\lambda_{(r)}) \overline{\int_{\mathbb{R}^r} g_s^{[k]}\left(\lambda_{(r)}\right) e^{is\underline{\lambda}_{(r)}} dZ(\lambda_{(r)})} \right\} \\
&+ k\left((k-1)\gamma(t)\beta(t) + \mu(t)\right) \sum_{r\geq 1} \frac{1}{(r!)^2} E\left\{ \int_{\mathbb{R}^r} \left(g_t^{[k-1]}\left(\lambda_{(r)}\right)\right) e^{it\underline{\lambda}_{(r)}} dZ(\lambda_{(r)}) \overline{\int_{\mathbb{R}^r} g_s^{[k]}\left(\lambda_{(r)}\right) e^{is\underline{\lambda}_{(r)}} dZ(\lambda_{(r)})} \right\} \\
&+ \frac{1}{2}k(k-1)\beta^2(t) \sum_{r\geq 1} \frac{1}{(r!)^2} E\left\{ \int_{\mathbb{R}^r} \left(g_t^{[k-2]}\left(\lambda_{(r)}\right)\right) e^{it\underline{\lambda}_{(r)}} dZ(\lambda_{(r)}) \overline{\int_{\mathbb{R}^r} g_s^{[k]}\left(\lambda_{(r)}\right) e^{is\underline{\lambda}_{(r)}} dZ(\lambda_{(r)})} \right\},
\end{aligned}$$

which implies

$$\begin{aligned}
&\frac{dR_k(t,s)}{dt} \\
&= k\left(\alpha(t) + \frac{1}{2}(k-1)\gamma^2(t)\right) R_k(t,s) + k\left((k-1)\gamma(t)\beta(t) + \mu(t)\right) Cov(X^{k-1}(t), X^k(s)) \\
&+ \frac{1}{2}k(k-1)\beta^2(t) Cov(X^{k-2}(t), X^k(s)), t \geq s,
\end{aligned}$$

and the expression (17) is now obtained by solving the above ordinary differential equation. □

In time-invariant case, we have

Theorem 3
Consider the time-invariant of SDE (1), then under the condition (13) the moments up to the order k of the process solution are given by the following expressions

1. If $\beta \neq 0$, we have

$$\begin{aligned}
m_1 &= -\frac{\mu}{\alpha},\ m_2 = \frac{2(\gamma\beta + \mu)\mu - \alpha\beta^2}{\alpha(2\alpha + \gamma^2)}, \\
m_3 &= \frac{-2(\gamma\beta + \mu)\mu^2 + \beta^2\mu(2\alpha + \gamma^2)}{\alpha(\alpha + \gamma^2)(2\alpha + \gamma^2)}, \\
m_4 &= -\frac{2(3\gamma\beta + \mu)}{(2\alpha + 3\gamma^2)} m_3 - \frac{3\beta^2}{(2\alpha + 3\gamma^2)} m_2.
\end{aligned}$$

2. If $\beta = 0$, then $m_k = (-1)^k \prod_{j=1}^{k} \dfrac{\mu}{\alpha + \frac{1}{2}(j-1)\gamma^2}$ for all $k \geq 0$.

Proof

1. If $\beta \neq 0$, then since $g_t^{[k]}(0) = m_k(t)$, for $k \geq 1$, thus from (9) we obtain $m_1^{(1)}(t) = \alpha(t)m_1(t) + \mu(t)$. In time-invariant case and under the condition (13), the process $X(t)$ is second order stationary, its moments are independent of t, so $m_1(t) = m_1$ which implies $\alpha m_1 + \mu = 0$ and $m_1 = -\dfrac{\mu}{\alpha}$. For the same raison and from the expression (14) we can obtain a difference equation for all $k \geq 2$ as follows
$$\left(\alpha + \frac{1}{2}(k-1)\gamma^2\right) m_k + \left((k-1)\gamma\beta + \mu\right) m_{k-1} + \frac{1}{2}(k-1)\beta^2 m_{k-2} = 0, \tag{20}$$
hence, $m_2 = -\dfrac{2(\mu+\gamma\beta)}{2\alpha+\gamma^2} m_1 - \dfrac{\beta^2}{2\alpha+\gamma^2}$. The expressions for m_3 and m_4 maybe obtained from (19).
2. If $\beta = 0$, then in time-invariant case we obtain the difference equation (20) becomes as $\left(\alpha + \frac{1}{2}(k-1)\gamma^2\right) m_k + \mu m_{k-1} = 0$ which implies $m_k = -\dfrac{\mu}{\left(\alpha + \frac{1}{2}(k-1)\gamma^2\right)} m_{k-1}, \forall k \geq 1$ with $m_0 = 1$ and hence
$$m_k = (-1)^k \prod_{j=1}^{k} \frac{\mu}{\alpha + \frac{1}{2}(j-1)\gamma^2}, \forall k \geq 0,$$
and the proof of the theorem is complete.

□

Example 3
Table(2) illustrated some finite-order moments for the GOU process defined by $dX(t) = (\mu - \alpha X(t))\,dt + \beta dW(t)$ with $\alpha > 0$ and $\beta \neq 0$.

m_1	m_2	m_3	m_4	Kurtosis	Skewness
$\dfrac{\mu}{\alpha}$	$\dfrac{2\mu^2+\alpha\beta^2}{2\alpha^2}$	$\dfrac{\mu\left(2\mu^2+3\alpha\beta^2\right)}{2\alpha^3}$	$\dfrac{4\mu^4+10\alpha\beta^2\mu^2+3\alpha^2\beta^4}{4\alpha^4}$	$-\dfrac{12}{\alpha}\left(\dfrac{\mu}{\beta}\right)^2$	$-\left(\dfrac{2}{\alpha}\right)^{\frac{3}{2}}\left(\dfrac{\mu}{\beta}\right)^3$

Table(2): First finite-order moment of GOU process.

Remark 4
By equations (12) and Table (2) the parameters μ, α, β and γ can be expressed as function of the finite moment of the process. Indeed,

1. For GOU process, we have $\alpha = -\log\left(\left|\dfrac{R_1(1)}{R_1(0)}\right|\right)$, $\mu = m_1\alpha$ and $\beta^2 = -\dfrac{12\alpha m_1^2}{Kurtosis\,(X)}$.

2. For $COBL(1,1)$ and when $\beta = 0$ we obtain $\alpha = \log\left(\left|\dfrac{R_1(1)}{R_1(0)}\right|\right)$, $\mu = -m_1\alpha$ and $\gamma^2 = \dfrac{-2\alpha Var(X)}{Var(X)+m_1^2}$.

These relationships can be used for estimating the processes by the moment method (MM).

4. GMM estimation

In what follows, we focus on estimating of the unknown parameters of time-invariant version. For this purpose we shall assume that $\beta = 0$ in SDE (1) i.e.,
$$dX(t) = (\alpha X(t) + \mu)\,dt + \gamma X(t)\,dW(t). \tag{21}$$

this assumption can be fulfilled by the transformation $Y(t) = \frac{\mu}{(\gamma\mu - \alpha\beta)}(\beta + \gamma X(t))$. So, the parameters of interest are gathered in the vector $\underline{\theta} = (\mu, \alpha, \gamma)' \in \mathbb{R}^3$, its true values denoted by $\underline{\theta}_0 = (\mu_0, \alpha_0, \gamma_0)'$ belonging to an Euclidean compact permissible parameter subspace Θ of $\mathbb{R}^3$. In statistical literature of continuous-time models, several techniques of estimation were proposed (interested readers are advised to see the monographs by Bergstrom [1], Rao [19] and Kutoyants [15].and the references therein). Since, in practice, data are observed at discrete interval and the likelihood of $X(t+1)$ conditional on $X(t)$ does not have a simple expression. However, in recent years, a number of diffusion processes which have a similar second-order properties as a continuous-time autoregressive moving average ($CARMA$) processes have been estimated via some discretization schema and hence adaptive methods related to discrete-time linear models are however applied. So, for the SDE (21), the Euler-Maruyama scheme yields

$$X(t+\Delta) = X(t) + \int_t^{t+\Delta} (\alpha X(s) + \mu)\, ds + \gamma \int_t^{t+\Delta} X(s)\, dW(s),$$

where Δ is some small enough constant sampling interval, hence an approximation of discrete-time version of SDE (21) is given by

$$X(t+1) = X(t) + (\alpha X(t) + \mu)\,\Delta + \eta\,(t+1)\,, \tag{22}$$

in which $(\eta\,(t+1))_{t\geq 0}$ is a some white noise with $E\{\eta\,(t+1)\,|I_t\} = 0$ and $Var\{\eta\,(t+1)\,|I_t\} = \gamma^2 X^2(t)\Delta$ and I_t denotes the information available up a time t, and hence (22) can be viewed as an $AR(1)$ model with heteroskedasticity. This finding leads us to estimate the vector $\underline{\theta}_0$ of the process in discrete-time using GMM. For this purpose, we use the orthogonality conditions given by the vector

$$\underline{g}_t\,(\underline{\theta}) = \begin{pmatrix} \eta\,(t+1) \\ \eta^2\,(t+1) - \gamma^2 X^2(t)\Delta \\ \left(\eta^2\,(t+1) - \gamma^2 X^2(t)\Delta\right) X(t) \end{pmatrix},$$

with $E_{\underline{\theta}_0}\left\{\underline{g}_t\,(\underline{\theta}_0)\right\} = \underline{O}$. A GMM estimator of $\underline{\theta}_0$ is defined as any measurable solution $\widehat{\underline{\theta}}_n$ of $\widehat{\underline{\theta}}_n = Arg \min_{\underline{\theta}\in\Theta} \left\{\widehat{Q}_n = \widehat{\underline{g}}'_n\,(\underline{\theta})\, W_n \widehat{\underline{g}}_n\,(\underline{\theta})\right\}$ where $\widehat{g}_n\,(\underline{\theta}) = \frac{1}{n}\sum_{t=1}^{n} \underline{g}_t\,(\underline{\theta})$ and W_n is a sequence of positive definite weighting matrices. Under the condition (13) for each $\underline{\theta} \in \Theta$, the process $\left(\underline{g}_t\,(\underline{\theta})\right)_{t\in\mathbb{Z}}$ is stationary, ergodic and fulfilled $\left\|E_{\underline{\theta}_0}\left\{\underline{g}_t\,(\underline{\theta})\right\}\right\| < +\infty$ for any $\underline{\theta} \in \Theta$ and hence, almost surely $\widehat{g}_n\,(\underline{\theta}) \to E_{\underline{\theta}_0}\left\{\underline{g}_0\,(\underline{\theta})\right\}$ as $n \to +\infty$. To analyze the large sample properties of the proposed estimator, it is necessary to impose the following regularity conditions on the process $(X(t))_{t\in\mathbb{Z}}$, on the matrix W_n and on the parameter space Θ.

A1. The sequence of matrices (W_n) converges in probability to a non-random positive definite matrix W.

A2. The matrix $E_{\underline{\theta}_0}\left\{\frac{\partial \underline{g}'_t\,(\underline{\theta}_0)}{\partial \underline{\theta}}\right\} W E_{\underline{\theta}_0}\left\{\frac{\partial \underline{g}_t\,(\underline{\theta}_0)}{\partial \underline{\theta}}\right\}$ is a finite non-singular matrix of constants.

A3. The parameter $\underline{\theta}_0$ is in the interior of Θ.

We are now in a position to state the following results.

Theorem 4
Beside the assumption (13), under the conditions $\mathbf{A}1-\mathbf{A}3$, $\widetilde{\underline{\theta}}_n$ converges in probability to $\underline{\theta}_0$.

Proof
From the first-order conditions (organized as column vector) for the minimization of $\widehat{Q}_n(\underline{\theta})$ we have

$$\frac{\partial \widehat{\underline{g}}'_n(\widehat{\underline{\theta}}_n)}{\partial \underline{\theta}} W_n \widehat{\underline{g}}_n(\widehat{\underline{\theta}}_n) = \underline{O}. \tag{23}$$

Taking the first-order Taylor-series expansion of the score vector $\widehat{g}_n(\underline{\theta})$ around $\underline{\theta}_0$, we obtain $\widehat{g}_n\left(\widehat{\underline{\theta}}_n\right) = \widehat{\underline{g}}_n(\underline{\theta}_0) - \frac{\partial \widehat{\underline{g}}_n(\underline{\theta}_*)}{\partial \underline{\theta}}\left(\widehat{\underline{\theta}}_n - \underline{\theta}_0\right)$ where $\underline{\theta}_*$ is an intermediate point on the line segment joining $\widehat{\underline{\theta}}_n$ and $\underline{\theta}_0$. Substituting for $\widehat{\underline{g}}_n\left(\widehat{\underline{\theta}}_n\right)$ into (23) yields $\frac{\partial \widehat{\underline{g}}_n'(\widehat{\underline{\theta}}_n)}{\partial \underline{\theta}} W_n \left\{\widehat{\underline{g}}_n(\underline{\theta}_0) - \frac{\partial \widehat{\underline{g}}_n(\underline{\theta}_*)}{\partial \underline{\theta}}\left(\widehat{\underline{\theta}}_n - \underline{\theta}_0\right)\right\} = \underline{O}$. Rearranging the above expression gives almost surely

$$\widetilde{\underline{\theta}}_n - \underline{\theta}_0 = \left\{\frac{\partial \widehat{\underline{g}}_n'(\widetilde{\underline{\theta}}_n)}{\partial \underline{\theta}} W_n \frac{\partial \widehat{\underline{g}}_n(\underline{\theta}_*)}{\partial \underline{\theta}}\right\}^{-1} \frac{\partial \widehat{\underline{g}}_n'(\underline{\theta}_n)}{\partial \underline{\theta}} W_n \widehat{\underline{g}}_n(\underline{\theta}_0).$$

Since the process $(X(t))_{t\geq 0}$, is an ergodic process, then under the conditions $A1. - A3.$, we have

$$p \lim_{n\to\infty} \frac{\partial \widehat{\underline{g}}_n(\widehat{\underline{\theta}}_n)}{\partial \underline{\theta}} W_n = B = E_{\underline{\theta}_0}\left\{\frac{\partial \underline{g}(\underline{\theta}_0)}{\partial \underline{\theta}}\right\} W,$$

$$p \lim_{n\to\infty} \frac{\partial \widehat{\underline{g}}_n'(\widehat{\underline{\theta}}_n)}{\partial \underline{\theta}} W_n \frac{\partial \widehat{\underline{g}}_n(\underline{\theta}_*)}{\partial \underline{\theta}} = A = E_{\underline{\theta}_0}\left\{\frac{\partial \underline{g}_t'(\underline{\theta}_0)}{\partial \underline{\theta}}\right\} W E_{\underline{\theta}_0}\left\{\frac{\partial \underline{g}_t(\underline{\theta}_0)}{\partial \underline{\theta}}\right\}.$$

Hence, from Slutsky's and the dominated convergence theorem, it follows that

$$p \lim_{n\to\infty} \left\{\frac{\partial \widehat{\underline{g}}_n'(\widehat{\underline{\theta}}_n)}{\partial \underline{\theta}} W_n \frac{\partial \widehat{\underline{g}}_n(\underline{\theta}_*)}{\partial \underline{\theta}}\right\}^{-1} \frac{\partial \widehat{\underline{g}}_n'(\widehat{\underline{\theta}}_n)}{\partial \underline{\theta}} W_n = A^{-1}B',$$

is finite, and since $p \lim_{n\to\infty} \widehat{\underline{g}}_n(\underline{\theta}_0) = \underline{O}$, the weak consistency of $\widetilde{\underline{\theta}}_n$ follows. □

Theorem 5
Under the conditions of theorem 4, we have $\sqrt{n}\left(\widehat{\underline{\theta}}_n - \underline{\theta}_0\right) \rightsquigarrow \mathcal{N}(\underline{O}, \Sigma(\underline{\theta}_0))$ where

$$\Sigma(\underline{\theta}_0) = A^{-1} E_{\underline{\theta}_0}\left\{\frac{\partial \underline{g}'(\underline{\theta}_0)}{\partial \underline{\theta}}\right\} W \Sigma_{as} W E_{\underline{\theta}_0}\left\{\frac{\partial \underline{g}(\underline{\theta}_0)}{\partial \underline{\theta}}\right\} A'^{-1},$$

with $\Sigma_{as} = \lim_{n\to+\infty} Var\left\{\sqrt{n}\widehat{\underline{g}}_n(\underline{\theta})\right\}$.

Proof
The proof rests classically on a Taylor-series expansion of the score vector $\widehat{g}_n(\underline{\theta})$ around $\underline{\theta}_0$. Thus, by the same argument used in Theorem 4, we have

$$\left\{\frac{\partial \widehat{\underline{g}}'(\widehat{\underline{\theta}}_n)}{\partial \underline{\theta}} W_n \frac{\partial \widehat{\underline{g}}(\widehat{\underline{\theta}}_*)}{\partial \underline{\theta}}\right\}\left(\widehat{\underline{\theta}}_n - \underline{\theta}_0\right) = \frac{\partial \widehat{\underline{g}}'(\widehat{\underline{\theta}}_n)}{\partial \underline{\theta}} W_n \widehat{\underline{g}}_n(\underline{\theta}_0). \tag{24}$$

On the other hand, for any $\underline{\lambda} \in \mathbb{R}^3$, the sequence $\left\{\underline{\lambda}'\widehat{\underline{g}}_n(\underline{\theta}), I_t\right\}_t$ is a square-integrable martingale difference. The central limit theorem and the Wold-Cramer device show that $\widehat{\underline{g}}_n(\underline{\theta}) \rightsquigarrow N(\underline{O}, \Sigma_{as})$. Moreover, by (24) and by setting $A = p \lim_{n\to\infty} \frac{\partial \widehat{\underline{g}}'(\widehat{\underline{\theta}}_n)}{\partial \underline{\theta}} W_n \frac{\partial \widehat{\underline{g}}(\widehat{\underline{\theta}}_*)}{\partial \underline{\theta}}$ and $B = p \lim_{n\to\infty} \frac{\partial \widehat{\underline{g}}'(\widehat{\underline{\theta}}_n)}{\partial \underline{\theta}} W_n$, the result follows from Slutsky's theorem. □

4.1. Discussion

4.1.1. The major difference between GMM and MM methods

The GMM constitute a flexible alternative to the common quasi-maximum likelihood (QML) and semi-parametric estimators. GMM has the advantage of being simple to compute compared with semi-parametric estimator. As a competitor to QML estimator, GMM is locally robust, asymptotically efficient with the coefficient of skewness and excess kurtosis being important for the goodness of fit of a statistical model. Moreover, as suggested by a referee, we list the main difference between MM and GMM.

MM This method use sample moments to estimate the parameters of interest.

GMM GMM extends the method MM in two important ways. The first treat the problem of conditions ensuring the existence of two or more moments which have informations about unknown parameters. The second one is that the quantities other than sample moments can be used to estimate the parameters.

MM The obtained estimators have explicit expressions of moments

GMM The GMM is obtained by minimizing a certain quadratic form based on the averages of some functions calculated on the available data.

MM The MM method cannot incorporate more moments than parameters, i.e., we have the same number of sample moment conditions as the number of parameters.

GMM The GMM combines the observed data with the information in population moment conditions to produce estimates of the unknown parameters, i.e., we may have more sample moment conditions than the number of parameters.

MM The MM estimator is consistent, and we use a demanding condition on the moments of order greater than 8 to prove their asymptotic normality.

GMM The GMM estimators are known to be consistent, asymptotically normal and asymptotically efficient in the class of all estimators that don't use any extra information aside from the data contained in the moment conditions.

Beside the above differences, in this paper, the GMM estimator is obtained from $AR(1)$ model after a discretization of SDE (21), and hence the reference by Bibi and Merahi [4] is entirely different from the present.

4.1.2. Optimal choice of weight matrix

We now discuss the optimal choice of weight matrix W which matters for asymptotic efficiency. Although the choice does not affect the asymptotic properties of GMM, very little is known about the impacts in finite samples. Indeed, it is clear that the asymptotic variance of $\widehat{\underline{\theta}}_n$ depends on W_n via W. Then, with an appropriate choice of W, it is possible to minimize the asymptotic variance of $\widehat{\underline{\theta}}_n$. Indeed, the minimum variance that can be achieved is when $W = \Sigma_{as}^{-1}$. In this particular case, the asymptotic variance of $\widehat{\underline{\theta}}_n$ is $\left\{ E_{\underline{\theta}_0} \left\{ \frac{\partial \underline{g}'_{\underline{\theta}_0}}{\partial \underline{\theta}} \right\} \Sigma_{as}^{-1} E_{\underline{\theta}_0} \left\{ \frac{\partial \underline{g}_{\underline{\theta}_0}}{\partial \underline{\theta}} \right\} \right\}^{-1}$ and $n\widehat{Q}_n$ has an asymptotic chi-square distribution with an appropriate degrees of freedom. One can note that this choice is only sufficient for efficiency. Hence, estimating the matrix Σ_{as} by a consistent estimator $\widehat{\Sigma}_{as}$ is crucial since: i) it is the optimal weighting matrix of GMM; ii) it is a part of the construction of $\widehat{\underline{\theta}}_n$ and its asymptotic variance (needed to construct confidence intervals and to make statistical tests available based on $\widehat{\underline{\theta}}_n$). In practice, the Newey-West estimator can be used $\widehat{V}_n = \widehat{\Omega}_n(0) + 2\sum_{j=1}^{q} K\left(\frac{j}{q}\right) \widehat{\Omega}_n(j)$ where $\widehat{\Omega}_n(j) = n^{-1}\sum_{t=j+1}^{n} \underline{G}(t)\underline{G}(t-j)$ with $\underline{G}(n) = \frac{1}{n}\sum_{t=1}^{n} \underline{g}_t\left(\widehat{\underline{\theta}}_n\right)$. The truncated lag q needs to go to infinity at some appropriate rate with respect to the sample, and the kernel $K(.)$ is assumed to belong to $\{k : R \to [-1,1] \mid k(0) = 1,\ k(x) = k(-x), \forall x \in R,\ \int |k(x)|dx < \infty$, and k is continuous but at some countable points$\}$. Examples of such kernel weights are given in Table(3) below.

Names	Expressions	Names	Expressions
Truncated	$k_T(x) = \begin{cases} 1 & if\ \lvert x\rvert \leq 1, \\ 0 & otherwise, \end{cases}$	Parzen	$k_P(x) = \begin{cases} 1 - 6x^2 + 6\lvert x\rvert^3 & if\ \lvert x\rvert \leq 1/2, \\ 2(1-\lvert x\rvert)^3 & if\ 1/2 < \lvert x\rvert \leq 1, \\ 0 & otherwise \end{cases}$
Bartlett	$k_B(x) = \begin{cases} 1 - \lvert x\rvert & if\ \lvert x\rvert \leq 1, \\ 0 & otherwise, \end{cases}$	Tukey-Hanning	$k_H(x) = \begin{cases} (1 + \cos \pi x)/2 & if\ \lvert x\rvert \leq 1, \\ 0 & otherwise, \end{cases}$

Table(3): Example of kernel weights.

It can be shown that Bartlett and Parzen kernels all product positive semi-definite estimates of V while this is not necessarily with truncated and Tukey-Hanning kernels.

4.1.3. Computation the matrices W in Newey-West estimator

In practice, the construction of the weight matrix W is obtained by the following algorithm

Algorithm Matlab

1. Given kernel $K(.)$, q : lags, N: number of observations
2. Input: Data $\{X(1), ...X(N)\}$,

First step; $(W = I)$
for $i = 0 : N$
$GU = G(1 + i : end, :)$
$GL = G(1 : end - i, :)$
$V(:, :, i + 1) = GU' * GL/N$
end
Second step: (HAC)
$S = V(:, :, 1)$
for $i = 1 : q$
$Snext = 2 * k(v) * V(:, :, i + 1)$
$S = S + Snext$
end
$W = inv(S)$

5. Empirical evidence

5.1. Simulation study

This subsection, discusses a simulation study in order to clarify the theory investigated in the previous chapters by confirming the consistency and asymptotic normality of the GMM estimator and by illustrating its performance compared with respect to moment method using the equations in Table(2). In order to do so, we simulated $n = 500$ independent trajectories via some specifications of $COBL(1, 1)$ model with length $N \in \{1000, 2000\}$ driven by a standard Bm distribution and vector of parameters $\underline{\theta}$ described in the bottom of each table below. The vector $\underline{\theta}$ is chosen to satisfied the second order stationarity and the existence of moments up to fourth order. For the purpose of illustration, we consider the following models

$$\text{Model}\,(1) : dX(t) = (\alpha X(t) + \mu)\, dt + \beta dW(t),$$
$$\text{Model}\,(2) : dX(t) = (\alpha X(t) + \mu)\, dt + \beta X(t) dW(t),$$

their vector of parameters $\underline{\theta} = (\mu, \alpha, \beta)'$ is estimated by the GMM algorithm noted $\widehat{\underline{\theta}}_g$ and as a parameter of configuration we estimate $\underline{\theta}$ by the moment method noted $\widehat{\underline{\theta}}_m$. Both methods have been executed under the $MATLAB/8$ using "$fminsearch.m$" as a minimizer function. In Tables below, the column "Mean" correspond to the average of the parameters estimates over the $n = 500$ simulations. In order to show the performance of $(G)\,MM$, we have reported in each table the root mean squared error $(RMSE)$ (results between brackets). First, from theorem 3, Table(2) and under the condition (13), the fourth-order moments for Model(1) and Model(2) are given by Table(4).

	m_1	m_2	m_3	m_4
Model (1)	$-\dfrac{\mu}{\alpha}$	$\dfrac{2\mu^2 - \alpha\beta^2}{2\alpha^2}$	$\dfrac{\mu\left(3\alpha\beta^2 - 2\mu^2\right)}{2\alpha^3}$	$\dfrac{4\mu^4 - 10\alpha\beta^2\mu^2 + 3\alpha^2\beta^4}{4\alpha^4}$
Model (2)	$-\dfrac{\mu}{\alpha}$	$\dfrac{-2\mu}{2\alpha + \beta^2} m_1$	$\dfrac{\mu}{\alpha + \beta^2} m_2$	$\dfrac{2\mu}{2\alpha + 3\mu^2} m_3$

Table(4). Fourth-order moment of Model(i), $i = 1, 2$.

The vectors $\underline{g}_t^{(i)}(\underline{\theta})$, $i = 1, 2$, of orthogonality conditions associated to Model(i), $i = 1, 2$ are

$$\underline{g}_t^{(1)}(\underline{\theta}) = \begin{pmatrix} X(t+1) - a^* X(t) - \mu^* \\ (X(t+1) - a^* X(t) - \mu^*)^2 - \gamma^{*2} \\ \left((X(t+1) - a^* X(t) - \mu^*)^2 - \gamma^{*2}\right) X(t) \end{pmatrix},$$

$$\underline{g}_t^{(2)}(\underline{\theta}) = \begin{pmatrix} X(t+1) - a^*X(t) - \mu^* \\ (X(t+1) - a^*X(t) - \mu^*)^2 - \gamma^{*2}X^2(t) \\ \left((X(t+1) - a^*X(t) - \mu^*)^2 - \gamma^{*2}X^2(t)\right)X(t) \end{pmatrix},$$

where $\mu^* = \mu\Delta, a^* = 1 + a\Delta$ and $\gamma^* = \gamma\sqrt{\Delta}$. Based on relationships given in theorem 3, Table(4) and the vectors $\underline{g}_{t,}^{(i)}(\underline{\theta})$, $i = 1, 2$, we obtain

5.1.1. Model (1)

The results of estimating the Model(1) are summarized in Table(5).

$\widehat{\underline{\theta}}$	$N=1000$ Mean GMM	$N=1000$ Mean MM	$N=2000$ Mean GMM	$N=2000$ Mean MM	$N=1000$ Mean GMM	$N=1000$ Mean MM	$N=2000$ Mean GMM	$N=2000$ Mean MM
$\widehat{\mu}$	0.2556 (0.0352)	0.2538 (0.0402)	0.2516 (0.0183)	0.2569 (0.0271)	2.0475 (0.0374)	2.0042 (0.0271)	2.0031 (0.0344)	2.0200 (0.0331)
$\widehat{\alpha}$	-1.5338 (0.0287)	-1.5391 (0.0302)	-1.5218 (0.0325)	-1.5310 (0.0210)	-0.5304 (0.0862)	-0.5151 (0.0672)	-0.5003 (0.0802)	-0.5031 (0.0770)
$\widehat{\beta}$	0.7462 (0.0101)	0.7493 (0.0213)	0.7450 (0.1022)	0.7492 (0.0151)	-1.4903 (0.0441)	-1.5359 (0.0451)	-1.4947 (0.0345)	-1.4947 (0.0345)
	Design(1): $\underline{\theta} = (0.25, -1.5, 0.75)'$				Design(2): $\underline{\theta} = (2.0, -0.5, -1.5)'$			

Table(5): $(G)\,MM$ estimation of Model(1).

The plots of asymptotic density of each component of $\widehat{\underline{\theta}}$ according to two methods are summarized in the Figure 1.

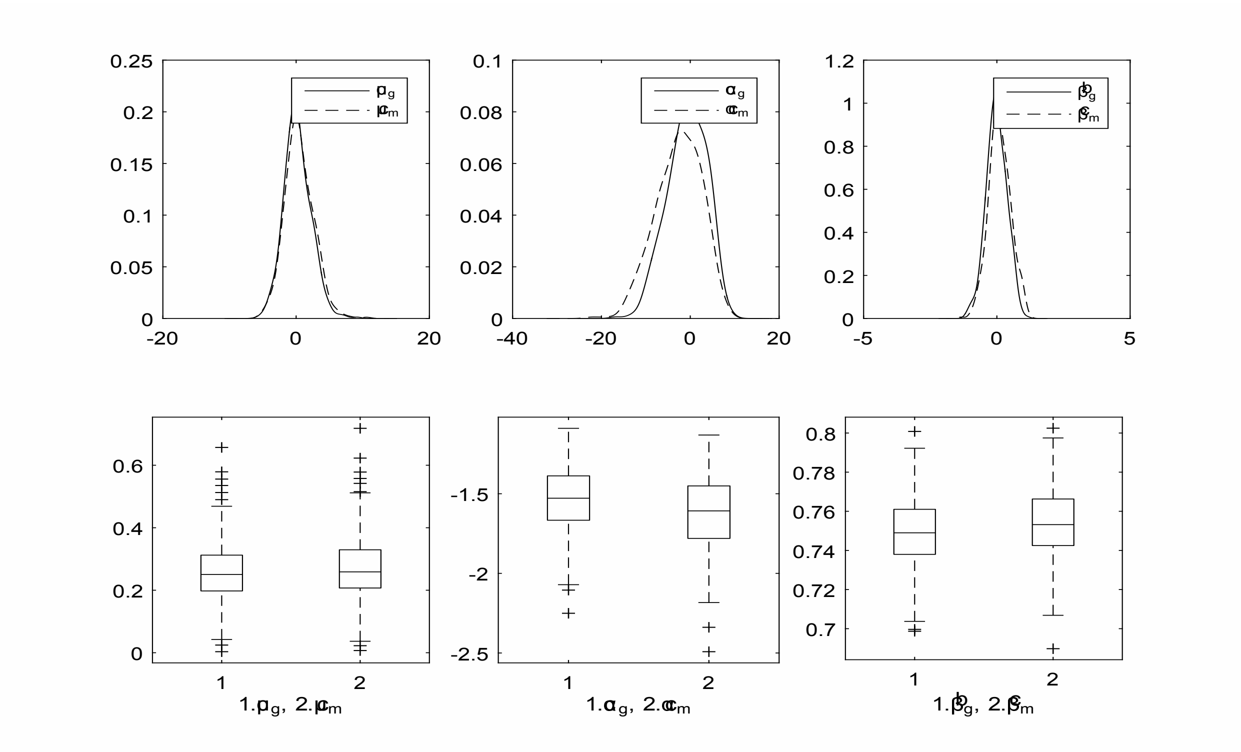

Figure 1. Top panels: the overlay of asymptotic distribution of $\sqrt{n}(\widehat{\theta}_g(i) - \theta(i))$ (resp. $\sqrt{n}(\widehat{\theta}_m(i) - \theta(i))$). Bottom panels: Box plot summary of $\widehat{\theta}_g(i)$ (resp, $\widehat{\theta}_m(i)$) $i = 1, ..., 3$, according to Design(1) of Table(5).

5.1.2. Model (2)

For the second model, we report the results of its estimation in the Table(6).

	$N=1000$		$N=2000$		$N=1000$		$N=2000$	
$\underline{\widehat{\theta}}$	$\underline{Mean}$		$\underline{Mean}$		$\underline{Mean}$		$\underline{Mean}$	
	GMM	MM	GMM	MM	GMM	MM	GMM	MM
$\widehat{\mu}$	0.2520 (0.0192)	0.2532 (0.0187)	0.2501 (0.0172)	0.2506 (0.0162)	0.5026 (0.0171)	0.5060 (0.0191)	0.5038 (0.0143)	0.5021 (0.0201)
$\widehat{\alpha}$	−1.4879 (0.0307)	−1.5352 (0.0452)	−1.5080 (0.0217)	−1.5064 (0.0251)	−1.5724 (0.0161)	−1.5307 (0.0142)	−1.5058 (0.0201)	−1.5072 (0.0162)
$\widehat{\beta}$	0.7537 (0.0121)	0.7449 (0.0157)	0.7449 (0.0157)	0.7512 (0.0609)	−0.50139 (0.0201)	−0.4946 (0.0211)	−0.4982 (0.0125)	−0.4920 (0.0302)
	Design(1): $\underline{\theta} = (0.25, -1.5, 0.75)'$				Design(2): $\underline{\theta} = (0.5, -1.5, -0.5)'$			

Table(6): $(G)\,MM$ estimation of Model(2) .

The plots of asymptotic density of each parameter in $\underline{\widehat{\theta}}$ according to two methods are summarized in Figure 2.

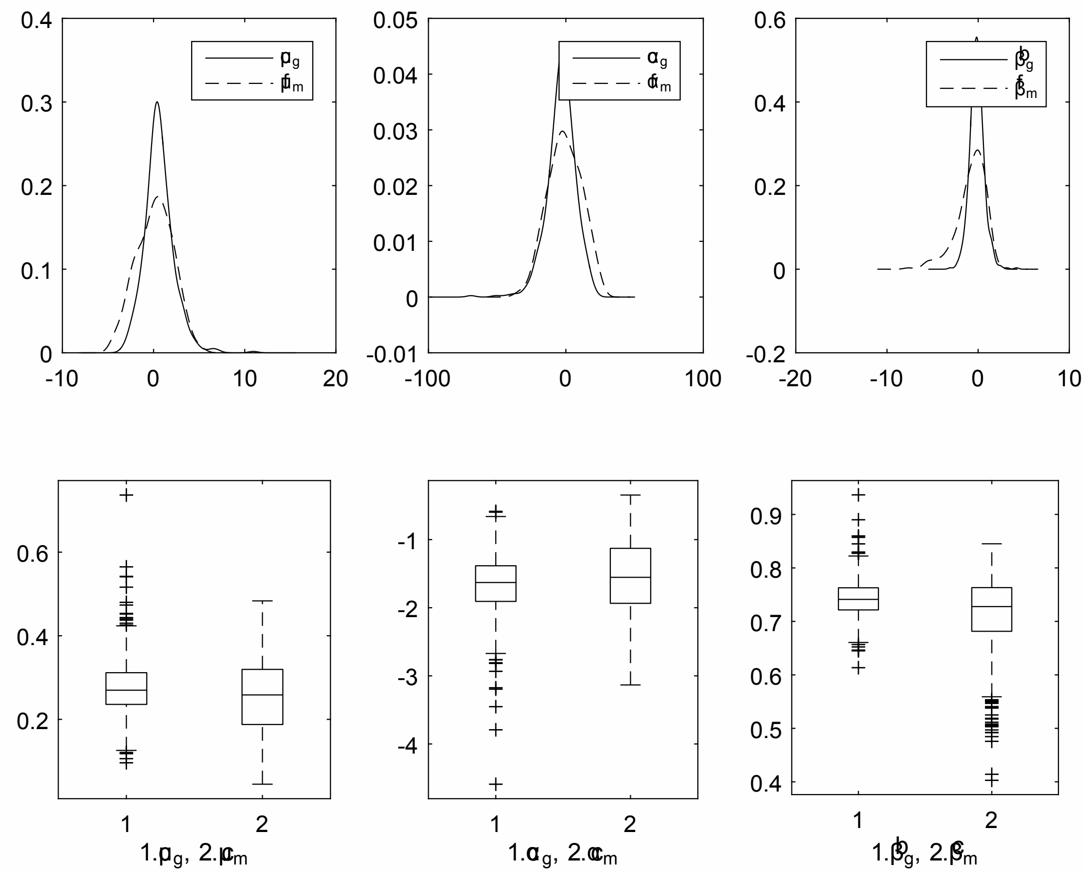

Figure 2. Top panels: the overlay of asymptotic distribution of $\sqrt{n}(\widehat{\theta}_g(i) - \theta(i))$ (resp. $\sqrt{n}(\widehat{\theta}_m(i) - \theta(i))$). Bottom panels: Box plot summary of $\widehat{\theta}_g(i)$ (resp, $\widehat{\theta}_m(i)$) $i = 1, 2, 3$, according to Design(1) of Table(6).

Now, a few comments are in order. By inspecting Table(5), it is clear that the results of GMM and of MM methods are reasonably close on each other and also for their $RMSE$ with non significant deviation. These observations maybe seen by regarding the plots of asymptotic distributions of their kernels estimates and their elementary statistics summarized in box plots which represents a strong similarities. This finding is however violated in Table(6). Indeed, it is obvious that the estimators $\underline{\widehat{\theta}}_m$ and $\underline{\widehat{\theta}}_g$ of the true values of unknown parameters, shows certain differences regarding the plots presented in Figure 2. in particular, the asymptotic variances of $\underline{\widehat{\theta}}_g\,(i)$ are smaller than of $\underline{\widehat{\theta}}_g(i)$, $i = 1, 2, 3$ Moreover, it can be seen from their box plots, that the elementary statistics of two methods represents a significant dissimilarities, in particular the GMM displays more outliers than MM, this is not surprising due to robustness properties of GMM (if and only if the function defining the orthogonality restrictions imposed on the underlying model is bounded) and hence its capability to detect the outliers in large data.

5.2. Real data analysis

In this subsection, the proposed method is now investigated to real financial time series. So we apply the method to two foreign exchange rates of Algerian Dinar against $U.S.$-Dollar (USD/DZD) and against the single European currency Euro (EUR/DZD), noted respectively $\left(y^d\,(t)\right)$ and $(y^e(t))$ collected from January $3, 2000$ to September $29, 2011$. After removing the days when the market was closed (weekends, holidays,...), we provide 3055 observations of each series Table(7) below provides descriptive statistics of such series.

Series	means	Std.Dev	Median	Skewness	Kurtosis	J. Bera
$y^e(t)$	88.61	11.57	91.09	-0.51	2.13	232.46
$y^d(t)$	73.45	4.24	73.12	-0.60	3.76	258.00

Table(7): Descriptive statistics of the series $(y^e(t))_{t\geq 1}$, $\left(y^d(t)\right)_{t\geq 1}$.

As a first finding, it is seen from the Jarque-Bera ($J.Bera$) normality test that the series $y^d\,(t)$ and $y^e(t)$ are not normally distributed, this excludes its modelling by a GOU model. Figure 3 displays a plot of the series $\left(y^d\,(t)\right)$ and $(y^e(t))$.

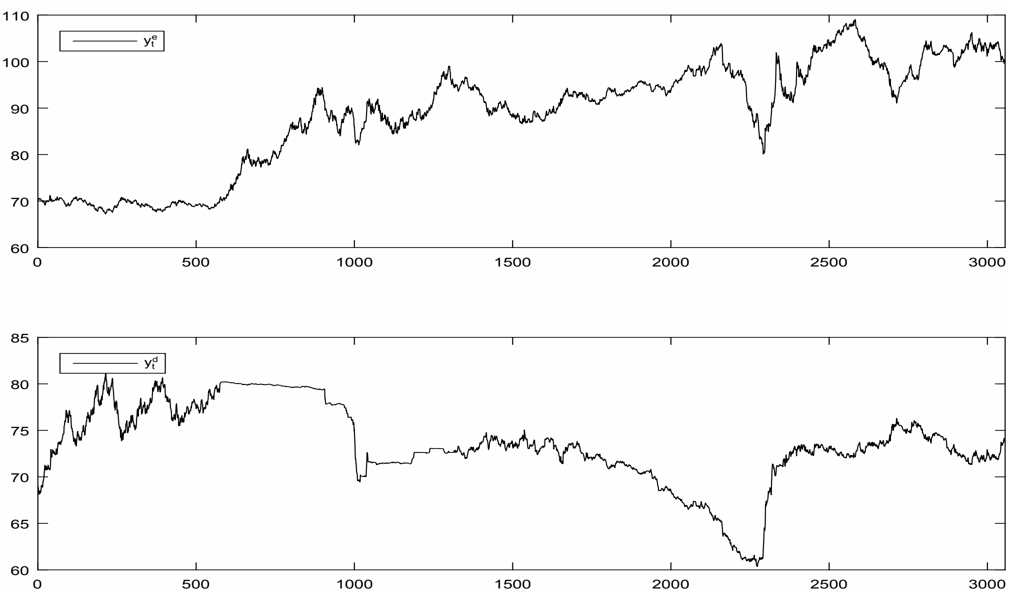

Figure 3. Top panel: trajectory of price of USD/DZD. Bottom panel: trajectory of price of EUR/DZD.

A rapid examination shows that the series $\left(y^d(t)\right)$ and $(y^e(t))$ exhibit nonlinear behavior. Moreover, the sample partial autocorrelation function $(PACF)$ of prices series $\left(y^d(t)\right)$ and $(y^e(t))$ are plotted in Figure 4.

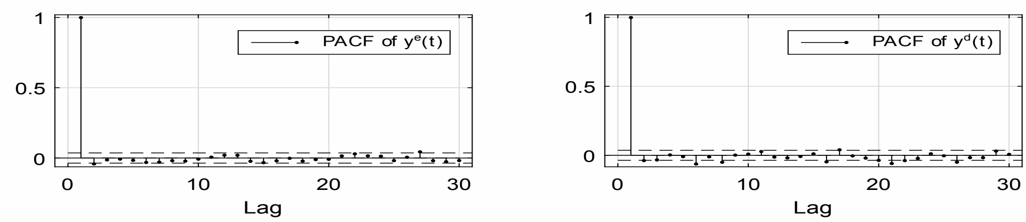

Figure 4. The $PACF$ of the prices series $\left(y^d(t)\right)$ and $(y^e(t))$.

It can be observed that the decreasing towards zero of the empirical $PACF$ computed for 30 lags with 95% confidence-limits for two series is insignificant. More precisely, the $PACF$ in Figure 4 appears to die out slowly, showing the possibility of the processes $\left(y^d(t)\right)$ and $(y^e(t))$ are mostly close to being non-stationary. Part of this non-stationarity is caused by some outliers due to fall of the prices values of the series between March 19, 2010 and September 21, 2010, The $PACF$ of prices reveals also some mild serial correlation and illustrates significant degree of persistence in variance, which implies that we need a $AR(1)$ with first and second conditional moment restriction of residual subjected to $E\{\eta(t)\,|I_{t-1}\} = 0$ and $Var\{\eta(t)\,|I_{t-1}\} = \gamma^2 X^2(t-1)\Delta$. The results of $(G)\,MM$ estimates followed by their $RMSE$ (results between bracket) of the series of prices $(y^e(t))_{t\geq 1}$ and $\left(y^d(t)\right)_{t\geq 1}$ noted hereafter $(\widehat{y}^e(t))_{t\geq 1}$ and $\left(\widehat{y}^d(t)\right)_{t\geq 1}$ via model(2) are given in Table(8).

	MM			GMM		
$\underline{\theta}$	μ	α	β	μ	α	β
$\widehat{y}^e(t)$	17.2911 (0.0307)	-0.1952 (0.0101)	0.0807 (0.0812)	17.4418 (0.0725)	-0.2125 (0.0613)	0.0771 (0.0621)
$\widehat{y}^d(t)$	25.3216 (0.0501)	-0.3447 (0.0817)	0.0479 (0.1002)	24.1221 (0.1320)	-1.0447 (0.1522)	0.0378 (0.0204)

Table(8): The $(G)\,MM$ estimates of $(y^e(t))_{t\geq 1}$, $\left(y^d(t)\right)_{t\geq 1}$.

The graphics of original series $y^e(t)$ and $y^d(t)$ stacked on their estimates series $\widehat{y}^e(t)$ and $\widehat{y}^d(t)$ by a MM method are shown in Figure 5 and Figure 6 below.

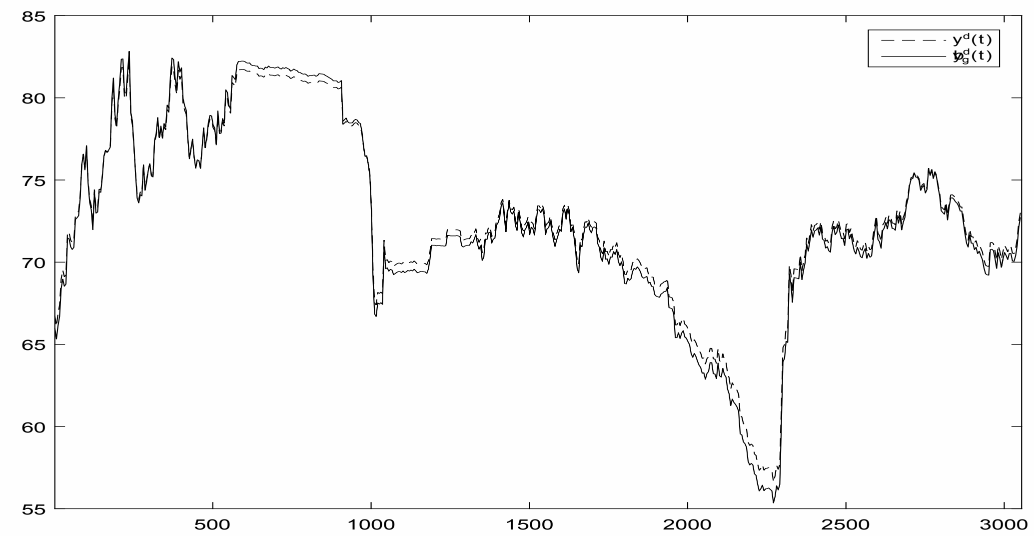

Figure 5. The MM fit of $y^e(t)$.

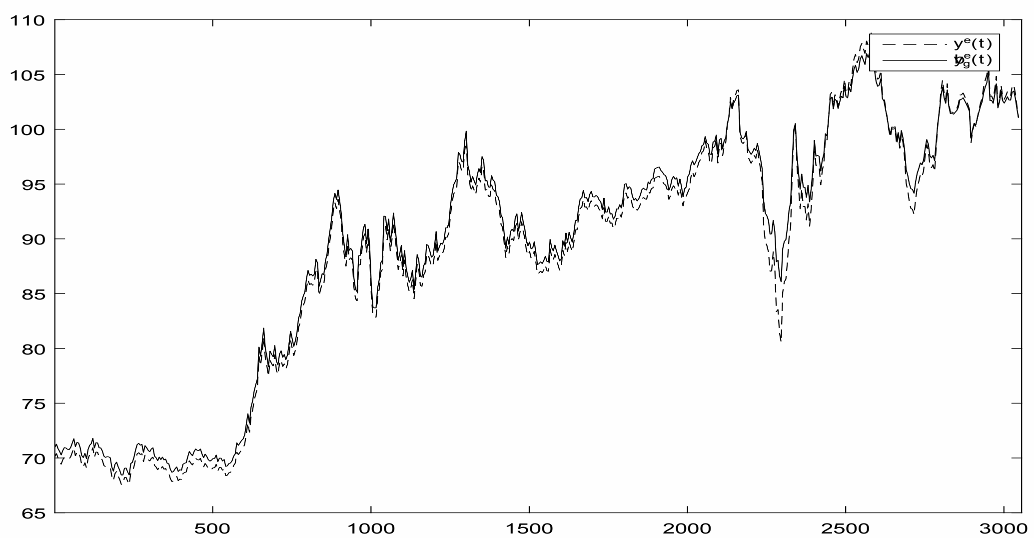

Figure 6. The MM fit of $y^d(t)$.

It is clear, regarding each of the Figure 5 and Figure 6, that the plots of actual series $y^e(t)$ and $y^d(t)$ (dashed line) display a very similar pattern with respect to their estimates $\widehat{y}^e(t)$ and $\widehat{y}^d(t)$ (continuous line) via MM method, so the resulting models provide good estimates for the data, The second tentative is to fit the series $y^e(t)$ and $y^d(t)$ by a GMM method. The following overlay in Figure 7 and Figure 8 show the trajectories of the original series compared with their estimates via a GMM method.

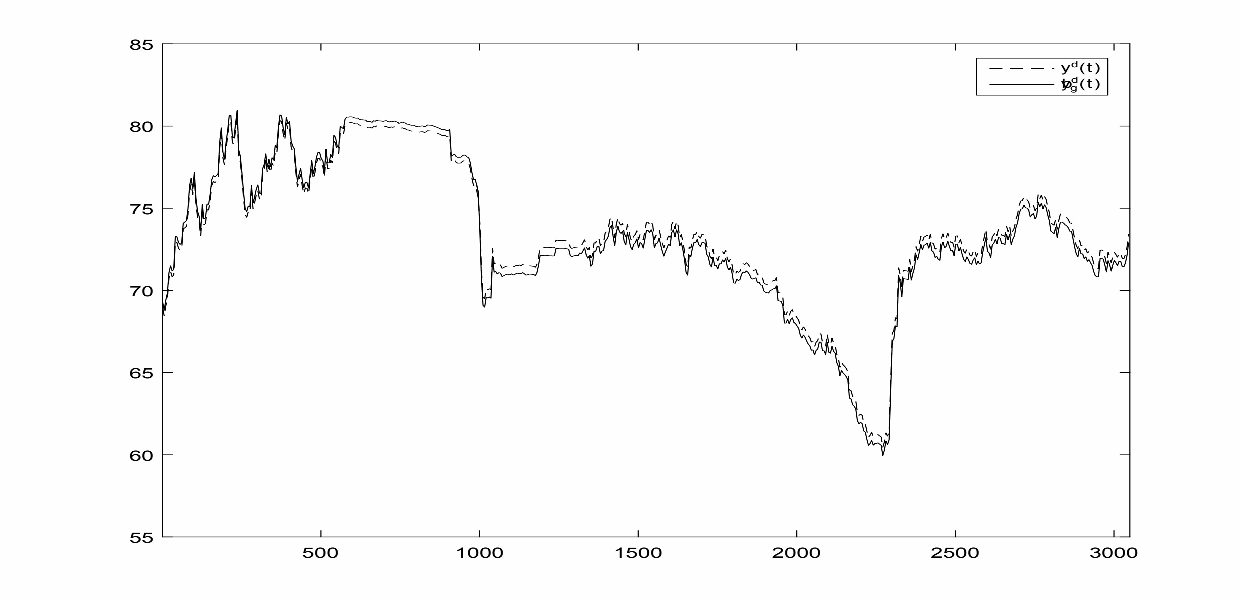

Figure 7. The GMM fit of $y^e(t)$.

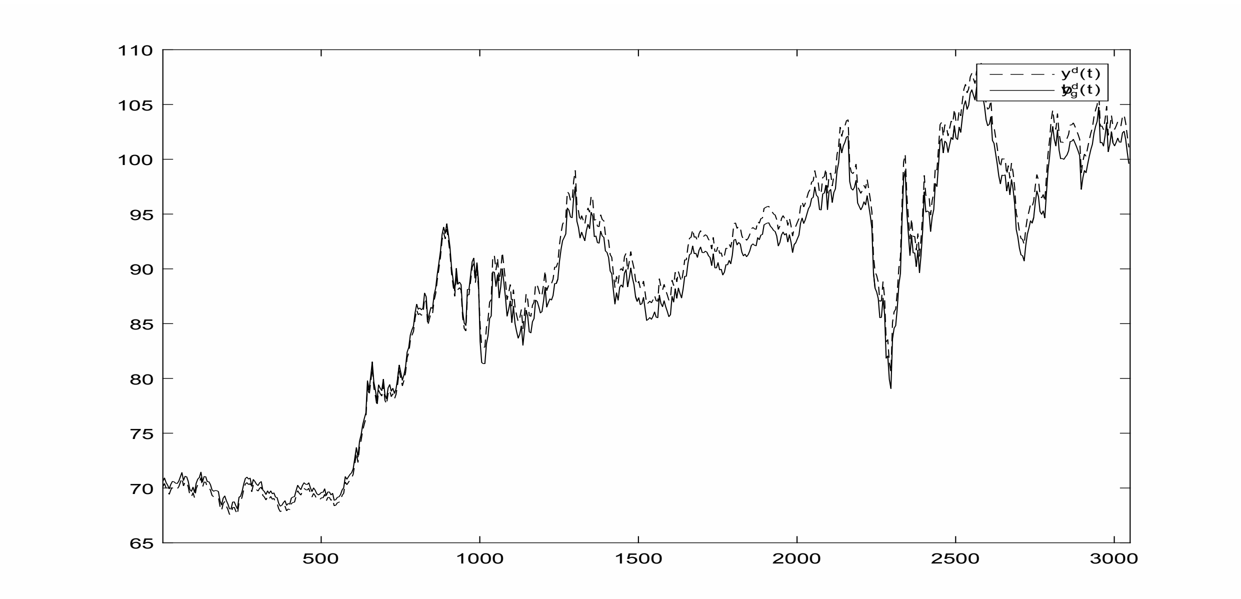

Figure 8. The GMM fit of $y^d(t)$.

As seen in the overlay plots for each series, there is no significant difference between the actual and the estimated values, and hence, the goodness fit of the series $y^e(t)$ and $y^d(t)$ by GMM method is proved.

6. Summary

In this paper, we have provided a probabilistic and statistical analysis followed by empirical evidence of time-continuous bilinear models characterized by certain stochastic differential equation with time varying coefficients driven by a Bm. The main aim of considering this class of models is twofold, the first is to give a solution in frequency domain based on associated evolutionary transfer functions system which is pertinent to the continuation of the development of non-linear SDE identification procedures. We have observed the second-order properties of solutions process as $\mathbb{L}_2-$functional in Wiener space. Moreover, Wiener-Itô representation is given under the assumption of regularity, providing a natural generalization of the stochastic spectral representation in time-discrete Gaussian case.

The second aim, we propose an empirical methodology motivated by the estimation of the discretized time-invariant model via a GMM method. So, the consistency and the asymptotic normality are proved. This method is illustrated by a Monte-Carlo study and in order to make our results comparable with the existing literature, we have preferred to compare it with a MM method. The motivation of the investigation in this work is the application of diffusion models to model two foreign exchange rates of Algerian Dinar against U.S.-Dollar (USD/DZ) and against the single European currency Euro (EUR/DZ). The results of simulation and/or of the application shows the interest of the proposal methods whether their asymptotic properties or in modelling the real data. Note at the end, that the results of such nature have never appeared in the literature of diffusion models, although the area has been considered for a long time.

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