Nonsmooth Vector Optimization Problem Involving Second-Order Semipseudo, Semiquasi Cone-Convex Functions

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Abstract Recently, Suneja et al. [26] introduced new classes of second-order cone-(η, ξ)-convex functions along with their generalizations and used them to prove second-order Karush–Kuhn–Tucker (KKT) type optimality conditions and duality results for the vector optimization problem involving first-order differentiable and second-order directionally differentiable functions. In this paper, we move one step ahead and study a nonsmooth vector optimization problem wherein the functions involved are first and second-order directionally differentiable. We introduce new classes of nonsmooth second-order cone-semipseudoconvex and nonsmooth second-order cone-semiquasiconvex functions in terms of second-order directional derivatives. Second-order KKT type sufficient optimality conditions and duality results for the same problem are proved using these functions.

Keywords Vector optimization, Cones, second-order cone-semipseudoconvexity (semiquasiconvexity), Second-order Optimality, Duality

AMS 2010 subject classifications 90C29, 90C46, 90C25, 90C26

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1. Introduction

Second-order optimality conditions have been widely studied for past many years because they refine first-order by second-order information which is very useful for recognizing efficient solutions. These conditions have important applications in sensitivity analysis and optimal algorithms, for example penalty methods [20, 24]. Various types of second-order (cone) convex functions like second-order (F, ρ) convex [2], second-order (F, α, ρ, d) convex [3], second-order cone-convex [25] and recently many others like second-order univexities, second-order hybrid (Φ, ρ, η, ζ, θ)-invexity [29, 30, 31] along with their weaker notions have been defined for twice differentiable functions and used to study second-order duality results for multiobjective and vector optimization problems. Mangasarian [21] first formulated the second-order dual involving second-order derivatives for nonlinear programming problem and established second-order duality results under certain inclusion conditions. By introducing two additional parameters, Hanson [15] formulated a second-order dual similar to that of Mangasarian [21] and established duality results under the assumption of second-order type I invexity. Mishra [22] deduced second-order duality results involving second-order derivatives for multiobjective programming problem using classes of second order pseudo-type I, second-order quasi-type I and related functions. Recently, Jayswal and Jha [19] and Dubey et al. [8, 9, 10] have studied various second-order symmetric dual programs under the assumptions of second-order F-convexity, G_f-bonvexity/G_f-pseudobonvexity and (G, α_f)-bonvexity/(G, α_f)-pseudobonvexity involving second-order derivatives.

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In the absence of second-order derivatives, Ivanov [17] defined second-order (type I) invexity for first-order differentiable and second-order directionally differentiable functions. He used them to prove necessary and sufficient optimality conditions for nonlinear programming problem. In 2019, using limiting second-order subdifferentials, Feng and Li [12] obtained second-order Fritz-John optimality conditions for (strict) local minimizer of nonlinear programming problem with $C^{1,1}$ functions. Feng and Li [11, 13], Tuyen et al. [28], Ivanov [18], Xiao et al. [32] studied multiobjective/vector optimization problems with inequality constraints as well as ones with both inequality and equality constraints involving $C^{1,1}$ and locally Lipschitz functions. They obtained second-order necessary and sufficient KKT optimality conditions for different kinds of efficiency using various second-order constraint qualifications and regularity conditions in terms of second-order symmetric subdifferential, second-order upper generalized directional derivatives and second-order tangent sets. Using the idea of cones, Suneja et al. [26] extended the functions introduced by Ivanov [17] to second-order cone-$(\eta, \xi)$-convex and its weaker notions and used them to derive second-order KKT type optimality and duality results for vector optimization problem over cones involving first-order differentiable and second-order directionally differentiable vector valued functions.

The present paper is motivated by the works of Ivanov [17] and Suneja et al. [26]. In this paper, we have considered nondifferentiable functions and extended the class of second-order cone-$(\eta, \xi)$-convex functions and their weaker notions [26] for first and second-order directionally differentiable functions. Nonsmooth second-order cone-convex, nonsmooth second-order cone-(strictly) semipseudoconvex and nonsmooth second-order cone-semiquasiconvex functions have been introduced. Interrelations among these functions have been discussed and illustrated by examples. Using these functions, second-order KKT type sufficient optimality conditions for nonsmooth vector optimization problem over cones have been proved. Since first-order differentiable functions are also first-order directionally differentiable, so the results obtained by us can be applied to a wider class of functions as compared to Suneja et al. [26]. Also, second-order Wolfe type and Mond-Weir type duals are formulated and duality results are established. The results are well supported by various examples.

2. Notations and Definitions

Let $K \subseteq \mathbb{R}^m$ be a closed convex pointed $(K \cap (-K) = \{0\})$ cone with non-empty interior $(intK \neq \emptyset)$. We denote $K \setminus \{0\}$ by $K_0$. The positive dual cone $K^+$ and strict positive dual cone $K^{++}$ are defined as follows:

$$K^+ := \{y \in \mathbb{R}^m : z^T y \geq 0 \ \forall \ z \in K\}$$

and

$$K^{++} := \{y \in \mathbb{R}^m : z^T y > 0 \ \forall \ z \in K_0\}.$$ 

Since the cone under consideration is closed and convex, by bipolar theorem $K = (K^+)^+$. In this case,

$$x \in K \iff \lambda^T x \geq 0, \ \forall \ \lambda \in K^+.$$ 

As given by Flores–Baţan et al. [14], we have

$$x \in intK \iff \lambda^T x > 0 \ \forall \ \lambda \in K^+ \setminus \{0\}.$$ 

Let $S \subseteq \mathbb{R}^n$ be a non-empty open subset and $f = (f_1, f_2, \ldots, f_m)^T : S \to \mathbb{R}^m$ be a vector valued function. We recall the definitions of first and second-order directionally differentiable functions which are weaker notions as compared to that of differentiability and twice differentiability respectively.

**Definition 2.1**

The first-order directional derivative of $f_i$ at $x \in S$ in the direction $d \in \mathbb{R}^n$ is defined as an element of $\mathbb{R}$ given by

$$f_i^d(x, d) := \lim_{t \to 0^+} \frac{(f_i(x + td) - f_i(x))}{t}.$$ 

If $f_i^d(x, d)$ exists and is finite, then function $f_i$ is called first-order directionally differentiable at $x$ in the direction $d$. The function $f_i$ is said to be first-order directionally differentiable on $S$ if the derivative $f_i^d(x, d)$ exists finitely for each $x \in S$ and direction $d \in \mathbb{R}^n$. 

Definition 2.2
[7] Suppose $f_i$ is first-order directionally differentiable at $x \in S$ in the direction $d \in \mathbb{R}^n$. The second-order directional derivative of $f_i$ at $x$ in the direction $d$ is defined as an element of $\mathbb{R}$ given by

$$f''_i(x, d) := \lim_{t \to 0^+} \frac{2(f_i(x + td) - f_i(x) - tf'_i(x, d))}{t^2}.$$ 

If $f''_i(x, d)$ exists and is finite, then function $f_i$ is called second-order directionally differentiable at $x$ in the direction $d$. The function $f_i$ is said to be second-order directionally differentiable on $S$ if it is first-order directionally differentiable on $S$ and the derivative $f''_i(x, d)$ exists finitely for each $x \in S$ and direction $d \in \mathbb{R}^n$.

Remark 2.1
$f$ is said to be first-order directionally differentiable at $x \in S$ in the direction $d \in \mathbb{R}^n$ if each $f_i$ is first-order directionally differentiable at $x$ in the direction $d$. The first-order directional derivative of $f$ at $x$ in the direction $d$ is defined to be the vector:

$$(f'_1(x, d), f'_2(x, d), \ldots, f'_m(x, d))^T.$$ 

Remark 2.2
Suppose $f$ is first-order directionally differentiable at $x \in S$ in the direction $d \in \mathbb{R}^n$, $f$ is said to be second-order directionally differentiable at $x$ in the direction $d$ if each $f_i$ is second-order directionally differentiable at $x$ in the direction $d$. The second-order directional derivative of $f$ at $x$ in the direction $d$ is defined to be the vector:

$$(f''_1(x, d), f''_2(x, d), \ldots, f''_m(x, d))^T.$$ 

Next, we introduce new classes of nonsmooth second-order cone-convex, nonsmooth second-order cone-semipseudoconvex and nonsmooth second-order cone-semi/quasiconvex functions that will be used to study second-order KKT type optimality conditions and duality results for nonsmooth vector optimization problem. Let $\bar{x} \in S$ where $S$ is a non-empty open subset of $\mathbb{R}^n$, $K \subset \mathbb{R}^m$ be a closed convex pointed cone with $intK \neq \emptyset$ and $f : S \to \mathbb{R}^m$ be first and second-order directionally differentiable vector valued function.

Definition 2.3
$f$ is said to be nonsmooth second-order $K$-convex at $\bar{x}$, if there exists a real valued function $\omega : S \times S \to [0, \infty)$ such that for all $x \in S$

$$f(x) - f(\bar{x}) - f'(\bar{x}, x - \bar{x}) - \omega(x, \bar{x})f''(\bar{x}, x - \bar{x}) \in K.$$ 

Remark 2.3
Suppose $f$ is first-order differentiable at $\bar{x}$. Then, $f'(\bar{x}, x - \bar{x}) = \nabla f(\bar{x})(x - \bar{x})$ where \( \nabla f(\bar{x}) = \begin{bmatrix} \nabla f_1(\bar{x}), \nabla f_2(\bar{x}), \ldots, \nabla f_m(\bar{x}) \end{bmatrix}^T \) is the $m \times n$ Jacobian matrix of $f$ at $\bar{x}$ and for each $i = 1, 2, \ldots, m$, $\nabla f_i(\bar{x}) = \begin{bmatrix} \frac{\partial f_i}{\partial x_1}(\bar{x}), \frac{\partial f_i}{\partial x_2}(\bar{x}), \ldots, \frac{\partial f_i}{\partial x_n}(\bar{x}) \end{bmatrix}^T$ is the $n \times 1$ Gradient vector of $f_i$ at $\bar{x}$. If $\omega(., .) = 1$, then nonsmooth second-order $K$-convex becomes second-order $K$-$(\eta, \xi)$-convex with $\eta(x, \bar{x}) \equiv \xi(x, \bar{x}) \equiv x - \bar{x}$ defined by Suneja et al. [26]. Further, if $m = 1$, $K = \mathbb{R}_+$, then nonsmooth second-order $K$-convex becomes second-order invex defined by Ivanov [17].

Definition 2.4
$f$ is said to be nonsmooth second-order $K$-semipseudoconvex at $\bar{x}$, if there exists a real valued function $\omega : S \times S \to [0, \infty)$ such that for all $x \in S$

$$-[f'(\bar{x}, x - \bar{x}) + \omega(x, \bar{x})f''(\bar{x}, x - \bar{x})] \notin intK \iff -[f(x) - f(\bar{x})] \notin intK.$$ 

Remark 2.4
Clearly, every nonsmooth second-order $K$-convex function with respect to $\omega(., .)$ is nonsmooth second-order $K$-semipseudoconvex with respect to same $\omega(., .)$ but the converse is not true as can be seen from the following example.
Remark 2.5

Let $\bar{x} = 0$, then

$$f'(0, x) = \begin{cases} (-x, x), & x \geq 0 \\ (x, 0), & x < 0 \end{cases} \quad \text{and} \quad f''(0, x) = \begin{cases} (2x^2, 0), & x \geq 0 \\ (2x^2, 2x^2), & x < 0 \end{cases}.$$ 

Let $K = \{(x_1, x_2)^T \in \mathbb{R}^2 : x_2 \leq 0, x_1 \leq -x_2 \}$ and $\omega : S \times S \to [0, \infty)$ be defined as

$$\omega(x, \bar{x}) = \frac{1 - x}{4(1 + x)(1 + x^2)} + \bar{x}^2.$$ 

Now, $f$ is nonsmooth second-order $K$-semipseudoconvex at $\bar{x} = 0$ with respect to $\omega(\cdot, \cdot)$ as

$$\text{int}K \ni -[f(x) - f(0)] = \begin{cases} \left( \frac{x}{x + 1}, \frac{-x}{x^2 + 1}, \frac{-x}{1 - x}, \frac{-x}{x^2} \right), & x \geq 0 \\ \left( \frac{x}{x + 1}, \frac{-x}{x^2 + 1}, \frac{-x}{1 - x}, \frac{-x}{x^2} \right), & x < 0. \end{cases}$$ 

This shows that $x \in \{x : 0 < x < 1\} \cup \{x : -1 < x < \frac{1 - \sqrt{5}}{2}\}$

$$\implies \text{int}K \ni -[f'(0, x) + \omega(x, 0)f''(0, x)] = \begin{cases} \left( x - \frac{(1-x)x^2}{2(1+x)(1+x^2)}, -x \right), & x \geq 0 \\ \left( -x - \frac{(1-x)x^2}{2(1+x)(1+x^2)}, \frac{-x}{2(1+x)(1+x^2)} \right), & x < 0. \end{cases}$$

However, $f$ is not nonsmooth second-order $K$-convex at $\bar{x}$ with respect to $\omega(\cdot, \cdot)$ as for $x = \frac{1}{2}$

$$f(x) - f(0) - f'(0, x) - \omega(x, 0)f''(0, x) = \left( \frac{2}{15}, \frac{-1}{10} \right) \notin K.$$ 

Definition 2.5

$f$ is said to be nonsmooth second-order $K$-semiquasiconvex at $\bar{x}$, if there exists a real valued function $\omega : S \times S \to (0, \infty)$ such that for all $x \in S$

$$[f(x) - f(\bar{x})] \notin \text{int}K \implies -[f'(x, x - \bar{x}) + \omega(x, \bar{x})f''(x, x - \bar{x})] \in K.$$ 

Definition 2.6

$f$ is said to be nonsmooth second-order $K$-strictly semipseudoconvex at $\bar{x}$, if there exists a real valued function $\omega : S \times S \to [0, \infty)$ such that for all $x \in S$

$$-[f(x) - f(\bar{x})] \in \text{K}_0 \implies -[f'(x, x - \bar{x}) + \omega(x, \bar{x})f''(x, x - \bar{x})] \in \text{int}K.$$ 

Remark 2.5

We glance at few important reductions of the new classes defined above.

1. If $\omega(\cdot, \cdot) \equiv 0$, then nonsmooth second-order $K$-(strictly) semipseudoconvex with respect to $K$ and nonsmooth second-order $K$-semiquasiconvex function becomes quasiconvex with respect to $K$ defined by Aggarwal [1].
2. Suppose $f$ is first-order differentiable and $\omega(.,.) \equiv 1$. Then, nonsmooth second-order $K$-(strictly) semipseudoconvex becomes second-order $K$-(strictly) pseudoconvex function and nonsmooth second-order $K$-semiquasiconvex becomes second-order $K$-(strictly) quasiconvex function with $\eta(x, \bar{x}) \equiv \xi(x, \bar{x}) \equiv x - \bar{x}$ defined by Suneja et al. [26].

**Remark 2.6**

Every nonsmooth second-order $K$-strictly semipseudoconvex function with respect to $\omega(.,.)$ is nonsmooth second-order $K$-semipseudoconvex function with respect to same $\omega(.,.)$. However, the converse is not true as illustrated by the following example.

**Example 2.2**

Let $S = (-8, 8) \subseteq \mathbb{R}$. Define $f = (f_1, f_2)^T : S \rightarrow \mathbb{R}^2$ as

$$f_1(x) = \begin{cases} 0, & x \geq 0 \\ x^2, & x < 0 \end{cases}$$

and $f_2(x) = x^2$.

Let $\bar{x} = 0$. Then,

$$f'(0, x) = (0, 0)^T$$

and

$$f''(0, x) = \begin{cases} (0, 2x^2)^T, & x \geq 0 \\ (2x^2, 2x^2)^T, & x < 0 \end{cases}.$$

Let $K = \{(x_1, x_2)^T \in \mathbb{R}^2 : x_2 \leq 0, x_2 \leq x_1\}$ and $\omega : S \times S \rightarrow [0, \infty)$ be a constant real valued function with $\omega(.,.) \equiv 1$. Now, $f$ is nonsmooth second-order $K$-semipseudoconvex at $\bar{x} = 0$ with respect to $\omega(.,.)$ as

$$\text{int}K \ni -[f(x) - f(0)] = \begin{cases} (0, -x^2)^T, & x \geq 0 \\ (-x^2, -x^2)^T, & x < 0 \end{cases}.$$ 

This shows that

$$x > 0 \implies \text{int}K \ni -[f'(0, x) + \omega(x, 0)f''(0, x)] = \begin{cases} (0, -2x^2)^T, & x \geq 0 \\ (-2x^2, -2x^2)^T, & x < 0 \end{cases}.$$ 

However, $f$ is not nonsmooth second-order $K$-strictly semipseudoconvex at $\bar{x} = 0$ with respect to $\omega(.,.)$ as for $x < 0$,

$$K_0 \ni -[f(x) - f(0)] = \begin{cases} (0, -x^2)^T, & x \geq 0 \\ (-x^2, -x^2)^T, & x < 0 \end{cases}$$

but

$$-[f'(0, x) + \omega(x, 0)f''(0, x)] = \begin{cases} (0, -2x^2)^T, & x \geq 0 \\ (-2x^2, -2x^2)^T, & x < 0 \notin \text{int}K.$$ 

3. **Second-Order Optimality Conditions**

We consider the following nonsmooth vector optimization problem:

$$K\text{-Minimize} \quad f(x) \tag{VOP}$$

subject to $-g(x) \in Q$, 

where $f = (f_1, f_2, \ldots, f_m)^T : S \rightarrow \mathbb{R}^m$, $g = (g_1, g_2, \ldots, g_p)^T : S \rightarrow \mathbb{R}^p$ are first and second-order directionally differentiable on $S$. $S$ is non-empty open subset of $\mathbb{R}^n$, $K$ and $Q$ are closed convex pointed cones with non-empty interiors in $\mathbb{R}^m$ and $\mathbb{R}^p$ respectively. $S_0 = \{x \in S : -g(x) \in Q\}$ denotes the set of all feasible solutions of (VOP).

**Definition 3.1**

Let $\bar{x} \in S_0$. Then, $\bar{x}$ is called a

(i) weak minimum of (VOP) if for all $x \in S_0$, $f(\bar{x}) - f(x) \notin \text{int} K$;

(ii) minimum of (VOP) if for all $x \in S_0$, $f(\bar{x}) - f(x) \notin K_0$;

(iii) strong minimum of (VOP) if for all $x \in S_0$, $f(\bar{x}) - f(x) \in K$.

Next, we prove second-order KKT type sufficient optimality conditions for (VOP) using second-order cone-convexity.

**Theorem 1**

Let $f$ be nonsmooth second-order $K$-convex and $g$ be nonsmooth second-order $Q$-convex at $\bar{x} \in S_0$ with respect to same $\omega : S \times S \rightarrow [0, \infty)$. Suppose there exist $\lambda \in K^+ \setminus \{0\}, \mu \in Q^+$ such that for all $x \in S_0$,

$$\lambda^T[f'(\bar{x}, x - \bar{x}) + \omega(x, \bar{x})f''(\bar{x}, x - \bar{x})] + \mu^T[g'(\bar{x}, x - \bar{x}) + \omega(x, \bar{x})g''(\bar{x}, x - \bar{x})] \geq 0,$$

$$\mu^T g(\bar{x}) \geq 0. \quad (1)$$

Then, $\bar{x}$ is a weak minimum of (VOP).

**Proof**

Let if possible $\bar{x}$ be not a weak minimum of (VOP). Then, there exists $\hat{x} \in S_0$ such that

$$f(\hat{x}) - f(\bar{x}) \in \text{int} K.$$

Using $\lambda \in K^+ \setminus \{0\}$, we get

$$\lambda^T[f(\bar{x}) - f(\hat{x})] > 0. \quad (3)$$

As $f$ is nonsmooth second-order $K$-convex at $\bar{x}$ with respect to $\omega(\cdot, \cdot)$ and $\lambda \in K^+ \setminus \{0\}$, we get

$$\lambda^T[f(\hat{x}) - f(\bar{x}) - f'(\bar{x}, \hat{x} - \bar{x}) - \omega(\bar{x}, \hat{x})f''(\bar{x}, \hat{x} - \bar{x})] \geq 0. \quad (4)$$

Adding (3) and (4), we get

$$-\lambda^T[f'(\bar{x}, \hat{x} - \bar{x}) + \omega(\hat{x}, \bar{x})f''(\bar{x}, \hat{x} - \bar{x})] > 0.$$

Using (1), we obtain

$$\mu^T[g'(\bar{x}, \hat{x} - \bar{x}) + \omega(\hat{x}, \bar{x})g''(\bar{x}, \hat{x} - \bar{x})] > 0. \quad (5)$$

Since $g$ is nonsmooth second-order $Q$-convex at $\bar{x}$ with respect to $\omega(\cdot, \cdot)$ and $\mu \in Q^+$, therefore

$$\mu^T[g(\hat{x}) - g(\bar{x}) - g'(\bar{x}, \hat{x} - \bar{x}) - \omega(\bar{x}, \hat{x})g''(\bar{x}, \hat{x} - \bar{x})] \geq 0. \quad (6)$$

Adding (5) and (6), we get $\mu^T[g(\hat{x}) - g(\bar{x})] > 0$. Using (2), we get $\mu^T g(\hat{x}) > 0$ which is a contradiction to $\hat{x} \in S_0$. Thus, $\bar{x}$ is a weak minimum of (VOP).

Following second-order KKT type sufficient optimality conditions for minimum and strong minimum of (VOP) can be proved on the similar lines.

**Theorem 2**

Let $f$ be nonsmooth second-order $K$-convex and $g$ be nonsmooth second-order $Q$-convex at $\bar{x} \in S_0$ with respect to same $\omega(\cdot, \cdot) : S \times S \rightarrow [0, \infty)$. Suppose there exist $\lambda \in K^+; \mu \in Q^+$ such that for all $x \in S_0$, (1) and (2) hold. Then, $\bar{x}$ is a minimum of (VOP).
Theorem 3
Let \( f \) be nonsmooth second-order \( K \)-convex and \( g \) be nonsmooth second-order \( Q \)-convex at \( \bar{x} \in S_0 \) with respect to same \( \omega(\cdot,\cdot) : S \times S \rightarrow [0, \infty) \). Suppose there exists \( \mu \in Q^+ \) such that for all \( x \in S_0 \), (1) and (2) hold and (1) holds for all \( \lambda \in K^+ \). Then, \( \bar{x} \) is a strong minimum of (VOP).

In the next theorem, we obtain second-order KKT type sufficient optimality conditions under the weaker assumption of nonsmooth second-order cone-semipseudoconvexity and nonsmooth second-order cone-semiquasiconvexity.

Theorem 4
Let \( f \) be nonsmooth second-order \( K \)-semipseudoconvex and \( g \) be nonsmooth second-order \( Q \)-semiquasiconvex at \( \bar{x} \in S_0 \) with respect to same \( \omega(\cdot,\cdot) : S \times S \rightarrow [0, \infty) \). If there exist \( \lambda \in K^+ \setminus \{0\}, \mu \in Q^+ \) such that for all \( x \in S_0 \), (1) and (2) hold, then \( \bar{x} \) is a weak minimum of (VOP).

Proof
For all \( x \in S_0 \), \( \mu^T g(x) \leq 0 \). Using (2), we can write
\[
\mu^T g(x) - \mu^T g(\bar{x}) \leq 0, \quad \forall x \in S_0.
\]
If \( \mu \neq 0 \), then
\[
g(x) - g(\bar{x}) \notin \text{int} Q, \quad \forall x \in S_0.
\]
Since \( g \) is nonsmooth second-order \( Q \)-semiquasiconvex at \( \bar{x} \) with respect to \( \omega(\cdot,\cdot) \) and \( \mu \in Q^+ \setminus \{0\} \), therefore
\[
-\mu^T g'(\bar{x}, x - \bar{x}) - \omega(x, \bar{x}) \mu^T g''(\bar{x}, x - \bar{x}) \geq 0, \quad \forall x \in S_0.
\]

Above inequality also holds for \( \mu = 0 \). From (1), we get
\[
\lambda^T f'(\bar{x}, x - \bar{x}) + \omega(x, \bar{x}) \lambda^T f''(\bar{x}, x - \bar{x}) \geq 0, \quad \forall x \in S_0.
\]
This implies for all \( x \in S_0 \),
\[
-[f'(\bar{x}, x - \bar{x}) + \omega(x, \bar{x}) f''(\bar{x}, x - \bar{x})] \notin \text{int} K.
\]
As \( f \) is nonsmooth second-order \( K \)-semipseudoconvex at \( \bar{x} \) with respect to \( \omega(\cdot,\cdot) \), we have
\[
-(f(x) - f(\bar{x})) \notin \text{int} K \quad \forall x \in S_0.
\]
Thus, \( \bar{x} \) is a weak minimum of (VOP).

We give an example to illustrate Theorem 4.

Example 3.1
Let \( S = (-1, 2) \subseteq \mathbb{R}, K = \{(x_1, x_2)^T \in \mathbb{R}^2 : x_2 \geq 0, x_2 \geq x_1\} \) and \( Q = \{(x_1, x_2)^T \in \mathbb{R}^2 : x_2 \geq 0, x_1 \geq x_2\} \). Define \( f = (f_1, f_2)^T : S \rightarrow \mathbb{R}^2 \) and \( g = (g_1, g_2)^T : S \rightarrow \mathbb{R}^2 \) as
\[
f_1(x) = \begin{cases} x \quad & x \geq 0 \\ x^3 \quad & x < 0 \end{cases}, \quad f_2(x) = \sin |x| + x^2, \quad g_1(x) = -|x| - x^2 - 1 \quad \text{and} \quad g_2(x) = -|x|.
\]

The feasible set of corresponding problem (VOP) is \( S_0 = (-1, 2) \). Let \( \bar{x} = 0 \).

Then,
\[
f'(0, x) = \begin{cases} (x, x) \quad & x \geq 0 \\ (0, -x) \quad & x < 0 \end{cases}\quad \text{and} \quad f''(0, x) = (0, 2x^2).
\]
\[
g'(0, x) = \begin{cases} (-x, -x) \quad & x \geq 0 \\ (x, x) \quad & x < 0 \end{cases}\quad \text{and} \quad g''(0, x) = (-2x^2, 0).
\]
Let $\omega : S \times S \to [0, \infty)$ be defined as
\[
\omega(x, \bar{x}) = \begin{cases} 
\frac{1}{4|x|} + \bar{x}^2, & x \neq 0 \\
\frac{1}{\bar{x}^2 + 1}, & x = 0.
\end{cases}
\]

Now, $f$ is nonsmooth second-order $K$-semipseudoconvex at $\bar{x} = 0$ with respect to $\omega(\cdot, \cdot)$ as
\[
-[f'(0, x) + \omega(x, 0)f''(0, x)] = \begin{cases} 
(-x, \frac{3x^2}{2}), & x \geq 0 \\
(0, \frac{3x}{2}), & x < 0 \notin \text{int}K
\end{cases}
\]

$\Rightarrow x \in (-1, 2)$ and for all such $x$,
\[
-[f(x) - f(\bar{x})] = \begin{cases} 
\left(\frac{-x}{x^2 + 1}, -\sin x - x^2\right), & x \geq 0 \\
(-x^3, \sin x - x^2), & x < 0 \notin \text{int}K.
\end{cases}
\]

(see Fig. 1, Fig. 2). Also, $g$ is nonsmooth second-order $Q$-semiquasiconvex at $\bar{x} = 0$ with respect to $\omega(\cdot, \cdot)$ as

\[
[g(x) - g(0)] = \begin{cases} 
(-x - x^2, -x), & x \geq 0 \\
(x - x^2, x), & x < 0 \notin \text{int}Q
\end{cases}
\]

$\Rightarrow x \in (-1, 2)$

$\Rightarrow -[g'(0, x) + \omega(x, 0)g''(0, x)] = \begin{cases} 
(x + 2\omega(x, 0)x^2, x), & x \geq 0 \\
(-x + 2\omega(x, 0)x^2, -x), & x < 0 \in Q.
\end{cases}
\]

Here,
\[
K^+ = \{(x_1, x_2) : x_2 \geq 0, x_1 \geq -x_2\} \text{ and } Q^+ = \{(x_1, x_2) : x_1 \geq 0, x_1 \geq -x_2\}.
\]

For $\lambda = (-1, 1) \in K^+ \setminus \{0\}$ and $\mu = (0, \frac{1}{4}) \in Q^+$, following conditions hold for all $x \in S_0$:
\[
\lambda^T[f'(0, x) + \omega(x, 0)f''(0, x)] + \mu^T[g'(0, x) + \omega(x, 0)g''(0, x)] = \begin{cases} 
\frac{7}{4}, & x \geq 0 \\
-\frac{5x}{4}, & x < 0 \geq 0,
\end{cases}
\]

$\mu^T g(\bar{x}) = 0 \geq 0$.

Thus, by Theorem 4, $\bar{x} = 0$ is a weak minimum of (VOP).

**Theorem 5**

Let $f$ be nonsmooth second-order $K$-strictly semipseudoconvex and $g$ be nonsmooth second-order $Q$-semiquasiconvex at $\bar{x} \in S_0$ with respect to same $\omega(\cdot, \cdot) : S \times S \to [0, \infty)$. If there exist $\lambda \in K^+ \setminus \{0\}, \mu \in Q^+$ such that for all $x \in S_0$, (1) and (2) hold, then $\bar{x}$ is a minimum of (VOP).
Let if possible \( \overline{x} \) be not a minimum of (VOP), then there exists \( \hat{x} \in S_0 \) such that

\[
f(\overline{x}) - f(\hat{x}) \in K_0.
\]

Since \( f \) is nonsmooth second-order \( K \)-strictly semipseudoconvex at \( \overline{x} \) with respect to \( \omega(,.) \), therefore

\[-[f'(\overline{x}, \hat{x} - \overline{x}) + \omega(\hat{x}, \overline{x})f''(\overline{x}, \hat{x} - \overline{x})] \in \text{int} K.
\]

As \( \lambda \in K^+ \setminus \{0\} \), we get

\[\lambda^T[f'(\overline{x}, \hat{x} - \overline{x}) + \omega(\hat{x}, \overline{x})f''(\overline{x}, \hat{x} - \overline{x})] < 0.\]  

Using (2) and the fact that \( \hat{x} \in S_0 \), we get

\[\mu^T[g(\hat{x}) - g(\overline{x})] \leq 0.
\]

If \( \mu \neq 0 \), then

\[g(\hat{x}) - g(\overline{x}) \notin \text{int} Q.
\]

Again \( g \) is nonsmooth second-order \( Q \)-semiquasiconvex at \( \overline{x} \) with respect to \( \omega(,.) \) and \( \mu \in Q^+ \), we get

\[\mu^T[g'(\overline{x}, \hat{x} - \overline{x}) + \omega(\hat{x}, \overline{x})g''(\overline{x}, \hat{x} - \overline{x})] \leq 0.\]  

If \( \mu = 0 \), still above inequality holds. Adding (7) and (8), we get

\[\lambda^Tf'(\overline{x}, \hat{x} - \overline{x}) + \mu^Ti'(\overline{x}, \hat{x} - \overline{x}) + \omega(\hat{x}, \overline{x})[\lambda^Tf''(\overline{x}, \hat{x} - \overline{x}) + \mu^Tg''(\overline{x}, \hat{x} - \overline{x})] < 0
\]

which is contradiction to (1). Thus, \( \overline{x} \) is a minimum of (VOP).

\[\square\]

4. Second-Order Duality

Aggarwal [1] associated a first-order dual in terms of first-order directional derivatives with (VOP) and proved duality results under the assumption of pseudonconvexity and quasiconvexity with respect to cone. Suneja et al. [26] formulated a second-order dual involving first-order derivatives and second-order directional derivatives for (VOP) and established duality results using second-order \( (\eta, \xi) \)-cone-convexity and its weaker notions.

In this section, we formulate second-order Wolfe type and Mond-Weir type duals for (VOP) in terms of first and second-order directional derivatives and prove duality results using nonsmooth second-order cone-convexity and its weaker notions. We begin with following second-order Wolfe type dual (WD).

Let \( k \in \text{int} K \) be any arbitrary fixed vector.

\[
\begin{align*}
K\text{-Maximize} & \quad f(u) + \mu^Tg(u)k \\
\text{subject to} & \quad \lambda^Tf'(u, x - u) + \mu^Tg'(u, x - u) \\
& \quad + \xi[\lambda^Tf''(u, x - u) + \mu^Tg''(u, x - u)] \geq 0 \quad \forall \ x \in S_0, \\
& \quad \lambda^Tk = 1,
\end{align*}
\]

\(\lambda \in K^+ \setminus \{0\}, \mu \in Q^+ \), \( u \in S, \xi \in \mathbb{R}_+ \). In general, \( \xi \) can be regarded as a function.

Let \( D_0 \) be the feasible set of (WD).

**Definition 4.1**

A point \((\bar{u}, \lambda, \bar{\mu}, \xi) \in D_0 \) is called weakly efficient solution (weak maximum) of (WD) if for all \((u, \lambda, \mu, \xi) \in D_0 \),

\[f(u) + \mu^Tg(u)k - f(\bar{u}) - \bar{\mu}^Tg(\bar{u})k \notin \text{int} K.
\]

**Theorem 6** (Weak Duality)

Let \( \overline{x} \in S_0 \) and \((u, \lambda, \mu, \xi) \in D_0 \). Assume that \( f \) is nonsmooth second-order \( K \)-convex and \( g \) is nonsmooth second-order \( Q \)-convex at \( u \) with respect to \( \xi(,.) \). Then,

\[f(u) + \mu^Tg(u)k - f(\overline{x}) \notin \text{int} K.
\]
Proof
Let if possible \( f(u) + \mu^T g(u)k - f(\bar{x}) \in intK \). Then,
\[
\lambda^T [f(u) - f(\bar{x})] + \mu^T g(u) > 0.
\] (11)
As \( f \) is nonsmooth second-order \( K \)-convex at \( u \) with respect to \( \xi(\cdot, \cdot) \) and \( \lambda \in K^+ \setminus \{ 0 \} \), we get
\[
\lambda^T [f(\bar{x}) - f(u) - f'(u, \bar{x} - u) - \xi(\bar{x}, u)f''(u, \bar{x} - u)] \geq 0.
\] (12)
Adding (11) and (12), we get
\[
\mu^T g(u) - \lambda^T [f'(u, \bar{x} - u) + \xi(\bar{x}, u)f''(u, \bar{x} - u)] > 0.
\]
Using (9), we get
\[
\mu^T g(u) + g'(u, \bar{x} - u) + \xi(\bar{x}, u)g''(u, \bar{x} - u) > 0.
\] (13)
Again \( g \) is nonsmooth second-order \( Q \)-convex at \( u \) with respect to \( \xi(\cdot, \cdot) \) and \( \mu \in Q^+ \), therefore
\[
\mu^T [g(\bar{x}) - g(u) - g'(u, \bar{x} - u) - \xi(\bar{x}, u)g''(u, \bar{x} - u)] \geq 0.
\] (14)
Adding (13) and (14), we get \( \mu^T g(\bar{x}) > 0 \) which is a contradiction to \( \bar{x} \in S_0 \). Hence \( f(u) + \mu^T g(u)k - f(\bar{x}) \notin intK \).

To prove Strong Duality result, we use the KKT type necessary optimality conditions derived by Aggarwal [1] under the following regularity condition.

Definition 4.2
The function \( g \) is said to satisfy the regularity condition at \( \bar{x} \in S \) if
\[
g'(\bar{x}; S - \bar{x}) + \{ \alpha g(\bar{x}) \mid \alpha \geq 0 \} + Q = \mathbb{R}^p.
\] (15)

Theorem 7
[1] Let \( \bar{x} \) be a weak minimum of (VOP). If \( f'(\bar{x}, x - \bar{x}) \) is \( K \)-subconvexlike, \( g'(\bar{x}, x - \bar{x}) \) is \( Q \)-subconvexlike on \( S \) and the regularity condition (15) holds at \( \bar{x} \), then there exist \( \lambda \in K^+ \setminus \{ 0 \}, \mu \in Q^+ \) such that
\[
\lambda^T f'(\bar{x}, x - \bar{x}) + \mu^T g'(\bar{x}, x - \bar{x}) \geq 0 \quad \forall \; x \in S,
\] (16)
\[
\mu^T g(\bar{x}) = 0.
\] (17)

Theorem 8 (Strong Duality)
Let \( \bar{x} \) be a weak minimum of (VOP). Assume that \( f'(\bar{x}, x - \bar{x}) \) is \( K \)-subconvexlike, \( g'(\bar{x}, x - \bar{x}) \) is \( Q \)-subconvexlike on \( S \) and the regularity condition (15) holds at \( \bar{x} \). Then, there exist \( \lambda \in K^+ \setminus \{ 0 \}, \bar{\mu} \in Q^+ \) such that \( (\bar{x}, \lambda, \bar{\mu}, \xi = 0) \) is feasible for the dual problem (WD) and the objective function values of (VOP) and (WD) are equal. Moreover, if the conditions of Weak Duality Theorem 6 hold for all \((u, \lambda, \mu, \xi) \in D_0\), then \((\bar{x}, \lambda, \bar{\mu}, \xi = 0)\) is a weak maximum of (WD).

Proof
Since \( \bar{x} \) is a weak minimum of (VOP), by Theorem 7 there exist \( \lambda \in K^+ \setminus \{ 0 \}, \mu \in Q^+ \) such that (16) and (17) are satisfied. Since \( \lambda \in K^+ \setminus \{ 0 \} \) and \( k \in intK \), therefore \( \lambda^T k > 0 \). Set \( \lambda = \frac{\lambda}{\sqrt{\lambda^T k}} \in K^+ \setminus \{ 0 \}, \bar{\mu} = \frac{\mu}{\sqrt{\lambda^T k}} \in Q^+ \), then \((\bar{x}, \lambda, \bar{\mu}, \xi = 0)\) is feasible for the dual problem (WD) and objective function values of (VOP) and (WD) are equal. Let if possible \((\bar{x}, \lambda, \bar{\mu}, \xi = 0)\) be not a weak maximum of (WD), then there exists \((u, \lambda, \mu, \xi) \in D_0\) such that \( f(u) + \mu^T g(u)k - f(\bar{x}) \in intK \) which is a contradiction to Weak Duality Theorem 6. Hence \((\bar{x}, \lambda, \bar{\mu}, \xi = 0)\) is a weak maximum of (WD).

Strong Duality result in literature has mainly been proved by taking the parameter \( \xi \) (usually denoted by \( p \)) associated with the second-order derivative as zero (for instance [2, 3, 4, 16, 23, 26, 27, 33]). However, we shall next prove the Strong Duality result in which the variable \( \xi \) may not be equal to zero and hence we will be having the Strong Duality result for the non-trivial case.
Theorem 9 (Non-trivial Strong Duality)
Let \( \bar{x} \) be a weak minimum of (VOP). Assume that \( f'(\bar{x}, x - \bar{x}) \) is \( K \)-subconvexlike, \( g'(\bar{x}, x - \bar{x}) \) is \( Q \)-subconvexlike on \( S \) and the regularity condition (15) holds at \( \bar{x} \). If \( f''(\bar{x}, x - \bar{x}) \in K \) and \( g''(\bar{x}, x - \bar{x}) \in Q \) for all \( x \in S_0 \), then there exist \( \lambda \in K^+ \setminus \{0\}, \mu \in Q^+ \) such that \( (\bar{x}, \lambda, \mu, \xi) \) is feasible for the dual problem (WD) for all \( \xi \in \mathbb{R}_+ \) and the objective function values of (VOP) and (WD) are equal. Moreover, if the conditions of Weak Duality Theorem 6 hold for all \( (u, \lambda, \mu, \xi) \in D_0 \), then \( (\bar{x}, \lambda, \mu, \xi) \) is a weak maximum of (WD).

Proof
Since \( \bar{x} \) is a weak minimum of (VOP), by Theorem 7 there exist \( \lambda \in K^+ \setminus \{0\}, \mu \in Q^+ \) such that (16) and (17) are satisfied. Using \( f''(\bar{x}, x - \bar{x}) \in K \) and \( g''(\bar{x}, x - \bar{x}) \in Q \), we get

\[
\lambda^T f'(\bar{x}, x - \bar{x}) + \mu^T g'(\bar{x}, x - \bar{x}) + \xi [\lambda^T f''(\bar{x}, x - \bar{x}) + \mu^T g''(\bar{x}, x - \bar{x})] \geq 0 \quad \forall x \in S_0, \xi \in \mathbb{R}_+.
\]

Since \( \lambda \in K^+ \setminus \{0\}, k \in int K \), therefore \( \lambda^T k \geq 0 \). Set \( \lambda = \frac{\lambda^T}{k_\lambda} \in K^+ \setminus \{0\}, \mu = \frac{\mu^T}{k_\mu} \in Q^+ \), then \( (\bar{x}, \lambda, \mu, \xi) \) is feasible for the dual problem (WD) for all \( \xi \in \mathbb{R}_+ \) and objective function values of (VOP) and (WD) are equal. Let if possible \( (\bar{x}, \lambda, \mu, \xi) \) be not a weak maximum of (WD), then there exists \( (u, \lambda, \mu, \xi) \in D_0 \) such that \( f(u) + \mu^T g(u)k - f(\bar{x}) \in int K \) which is a contradiction to Weak Duality Theorem 6. Hence \( (\bar{x}, \lambda, \mu, \xi) \) is a weak maximum of (WD).

Following is an example to illustrate Theorem 6.

Example 4.1
Let \( S = (-0.5, 2) \subseteq \mathbb{R}, K = \{(x_1, x_2)^T \in \mathbb{R}^2: x_1 \geq 0, x_1 \geq x_2\} \) and \( Q = \{(x_1, x_2)^T \in \mathbb{R}^2: x_2 \leq 0, x_2 \leq x_1\} \). Define \( f = (f_1, f_2)^T: S \rightarrow \mathbb{R}^2 \) and \( g = (g_1, g_2)^T: S \rightarrow \mathbb{R}^2 \) as

\[
f_1(x) = \sin \left| x \right| + x^2 \quad \text{and} \quad f_2(x) = \begin{cases} \frac{x}{x + 1}, & x \geq 0 \\ \frac{x^2}{x^2 + \frac{2}{3}}, & x < 0 \end{cases}.
\]

\[
g_1(x) = - \left| x \right| \quad \text{and} \quad g_2(x) = \begin{cases} \sin x, & x \geq 0 \\ \cos x - 1, & x < 0 \end{cases}.
\]

The feasible set of corresponding problem (VOP) is \( S_0 = [0, 2) \) and let \( u = 0 \). Now,

\[
f'(0, x) = \begin{cases} (x, x)^T, & x \geq 0 \\ (-x, 0)^T, & x < 0 \end{cases} \quad \text{and} \quad f''(0, x) = \begin{cases} (2x^2, -2x^2)^T, & x \geq 0 \\ (2x^2, 2x^2)^T, & x < 0 \end{cases},
\]

\[
g'(0, x) = \begin{cases} (-x, x)^T, & x \geq 0 \\ (0, 0)^T, & x < 0 \end{cases} \quad \text{and} \quad g''(0, x) = \begin{cases} (0, 0)^T, & x \geq 0 \\ (0, -x^2)^T, & x < 0 \end{cases}.
\]

Let \( \xi: S \times S \rightarrow [0, \infty) \) be defined as \( \xi(x, u) = \frac{1}{2} + u^2 x^2 \). Then, \( f \) is nonsmooth second-order \( K \)-convex at \( u = 0 \) with respect to \( \xi(\cdot, \cdot) \) as for all \( x \in S \)

\[
K \ni f(x) - f(0) - f'(0, x) - \xi(x, 0)f''(0, x) = \begin{cases} \left( \sin x + \frac{x^2}{2} - x, \frac{x^2}{x+1} - x + \frac{x^2}{2} \right)^T, & x \geq 0 \\ \left( -\sin x + x + \frac{x^2}{2}, \frac{x^2}{2} + \frac{x^2}{3} \right)^T, & x < 0 \end{cases}
\]

[see Figure 3, Figure 4, Figure 5].

Also, $g$ is nonsmooth second-order $Q$-convex at $u = 0$ with respect to $\xi(\cdot, \cdot)$ as for all $x \in S$

\[ Q \ni g(x) - g(0) - g'(0, x) - \xi(x, 0)g''(0, x) = \begin{cases} (0, \sin x - x)^T, & x \geq 0 \\ (0, \cos x - 1 + \frac{x^2}{4})^T, & x < 0 \end{cases} \]

[see Figure 6, Figure 7]. Here,

\[ K^+ = \{(x_1, x_2)^T \in \mathbb{R}^2 : x_1 \geq -x_2 \geq 0 \} \quad \text{and} \quad Q^+ = \{(x_1, x_2)^T \in \mathbb{R}^2 : 0 \leq x_1 \leq -x_2 \}. \]

For $\lambda = (1, 0)^T \in K^+ \setminus \{0\}, \mu = (0, -1)^T \in Q^+, k = (1, \frac{1}{2})^T \in int K$ and for all $x \in S_0$, following conditions hold:

(i) $\lambda^T f'(0, x) + \mu^T g'(0, x) + \xi(x, 0)[\lambda^T f''(0, x) + \mu^T g''(0, x)] = \begin{cases} \frac{x^2}{4}, & x \geq 0 \\ -x + \frac{3x^2}{4}, & x < 0 \geq 0; \end{cases}$

(ii) $\lambda^T k = 1.$
Thus, \((u = 0, \lambda = (1, 0)^T, \mu = (0, -1)^T, \xi = \frac{1}{4})\) is a dual feasible point. Moreover, for all \(\bar{x} \in S_0\)

\[
f(u) + \mu^T g(u)k - f(\bar{x}) = \left( -\sin \bar{x} - \bar{x}^2, \frac{-\bar{x}}{\bar{x} + 1} \right)^T \notin \text{int}K.
\]

Hence Weak Duality Theorem 6 holds for all feasible point \(\bar{x}\) of (VOP) and the dual feasible point \((u = 0, \lambda = (1, 0)^T, \mu = (0, -1)^T, \xi = \frac{1}{4})\).

Next, we associate following second-order Mond-Weir type dual with (VOP) and establish duality results using nonsmooth second-order cone-semipseudoconvexity and nonsmooth second-order cone-semiquasiconvexity.

\[
K\text{-Maximize } f(u) \quad \text{(MD)}
\]

subject to

\[
\lambda^T f'(u, x - u) + \mu^T g'(u, x - u) + \xi [\lambda^T f''(u, x - u) + \mu^T g''(u, x - u)] \geq 0, \quad \forall x \in S_0
\]

\[
\mu^T g(u) \geq 0,
\]

\[
\lambda \in K^+ \setminus \{0\}, \mu \in Q^+, u \in S, \xi \in \mathbb{R}_+. \text{ In general, } \xi \text{ can be regarded as a function.}
\]

Let \(D_1\) be the feasible set of (MD).

**Definition 4.3**

A point \((\bar{u}, \bar{\lambda}, \bar{\mu}, \bar{\xi}) \in D_1\) is called weakly efficient solution (weak maximum) of (MD) if for all \((u, \lambda, \mu, \xi) \in D_1, f(u) - f(\bar{u}) \notin \text{int}K.

**Theorem 10** (Weak Duality)

Let \(\bar{x} \in S_0\) and \((u, \lambda, \mu, \xi) \in D_1\). Assume \(f\) is nonsmooth second-order \(K\)-semipseudoconvex and \(g\) is nonsmooth second-order \(Q\)-semiquasiconvex at \(u\) with respect to \(\xi(\cdot, \cdot)\). Then \(f(u) - f(\bar{x}) \notin \text{int}K.

**Proof**

The proof follows on the lines of Theorem 4.

**Theorem 11** (Strong Duality)

Let \(\bar{x}\) be a weak minimum of (VOP). Assume \(f'(\bar{x}, x - \bar{x})\) is \(K\)-subconvexlike, \(g'(\bar{x}, x - \bar{x})\) is \(Q\)-subconvexlike on \(S\) and the regularity condition (15) holds at \(\bar{x}\). Then, there exist \(\bar{\lambda} \in K^+ \setminus \{0\}, \bar{\mu} \in Q^+\) such that \((\bar{x}, \bar{\lambda}, \bar{\mu}, \bar{\xi} = 0)\) is feasible for the dual problem (MD) and the objective function values of (VOP) and (MD) are equal. Moreover, if the conditions of Weak Duality Theorem 10 hold for all \((u, \lambda, \mu, \xi) \in D_1\), then \((\bar{x}, \bar{\lambda}, \bar{\mu}, \bar{\xi} = 0)\) is a weak maximum of (MD).

**Proof**

Since \(\bar{x}\) is a weak minimum of (VOP), by Theorem 7 there exist \(\bar{\lambda} \in K^+ \setminus \{0\}, \bar{\mu} \in Q^+\) such that (16) and (17) are satisfied. Then, \((\bar{x}, \bar{\lambda}, \bar{\mu}, \bar{\xi} = 0)\) is feasible for the dual problem (MD) and objective function values of (VOP) and (MD) are equal. Let if possible \((\bar{x}, \bar{\lambda}, \bar{\mu}, \bar{\xi} = 0)\) be not a weak maximum of (MD), then there exists \((u, \lambda, \mu, \xi) \in D_1\) such that \(f(u) - f(\bar{x}) \notin \text{int}K\) which is a contradiction to Weak Duality Theorem 10. Hence \((\bar{x}, \bar{\lambda}, \bar{\mu}, \bar{\xi} = 0)\) is a weak maximum of (MD).

Next, we have the Strong Duality result in which the variable \(\xi\) may not be equal to zero.

**Theorem 12** (Non-trivial Strong Duality)

Let \(\bar{x}\) be a weak minimum of (VOP). Assume \(f'(\bar{x}, x - \bar{x})\) is \(K\)-subconvexlike, \(g'(\bar{x}, x - \bar{x})\) is \(Q\)-subconvexlike on \(S\) and the regularity condition (15) holds at \(\bar{x}\). If \(f''(\bar{x}, x - \bar{x}) \in K\) and \(g''(\bar{x}, x - \bar{x}) \in Q\) for all \(x \in S_0\), then there exist \(\bar{\lambda} \in K^+ \setminus \{0\}, \bar{\mu} \in Q^+\) such that \((\bar{x}, \bar{\lambda}, \bar{\mu}, \bar{\xi})\) is feasible for the dual problem (MD) for all \(\xi \in \mathbb{R}_+\) and the objective function values of (VOP) and (MD) are equal. Moreover, if the conditions of Weak Duality Theorem 10 hold for all \((u, \lambda, \mu, \xi) \in D_1\), then \((\bar{x}, \bar{\lambda}, \bar{\mu}, \bar{\xi})\) is a weak minimum of (MD).

Proof
Since \( \bar{x} \) is a weak minimum of (VOP), by Theorem 7 there exist \( \bar{\lambda} \in K^+ \setminus \{0\}, \bar{\mu} \in Q^+ \) such that (16) and (17) are satisfied. Using \( f''(\bar{x}, x - \bar{x}) \in K \) and \( g''(\bar{x}, x - \bar{x}) \in Q \), we get
\[
\bar{\lambda}^T f'(\bar{x}, x - \bar{x}) + \bar{\mu}^T g'(\bar{x}, x - \bar{x}) + \xi \left[ \bar{\lambda}^T f''(\bar{x}, x - \bar{x}) + \bar{\mu}^T g''(\bar{x}, x - \bar{x}) \right] \geq 0 \quad \forall x \in S_0, \xi \in \mathbb{R}_+.
\]
Then, \( (\bar{x}, \bar{\lambda}, \bar{\mu}, \bar{\xi}) \) is feasible for the dual problem (MD) for all \( \bar{\xi} \in \mathbb{R}_+ \) and objective function values of (VOP) and (MD) are equal. Let if possible \( (\bar{x}, \bar{\lambda}, \bar{\mu}, \bar{\xi}) \) be not a weak maximum of (MD), then there exists \( (u, \lambda, \mu, \xi) \in D_1 \) such that \( f(u) - f(\bar{x}) \in intK \) which is a contradiction to Weak Duality Theorem 10. Hence \( (\bar{x}, \bar{\lambda}, \bar{\mu}, \bar{\xi}) \) is a weak maximum of (MD).

We conclude this section with an example in which we find a feasible solution of (MD) given a weak minimum of (VOP) using Theorem 12.

Example 4.2
Let \( S = (-4, 4) \subseteq \mathbb{R}, K = \{(x_1, x_2)^T \in \mathbb{R}^2 : x_1 \leq 0, x_2 \geq 0\} \) and \( Q = \{x_1 \in \mathbb{R} : x_1 \geq 0\} \). Define \( f = (f_1, f_2)^T : S \rightarrow \mathbb{R}^2 \) and \( g : S \rightarrow \mathbb{R} \) as
\[
f_1(x) = \begin{cases} \frac{x}{x^2 + 1}, & x \geq 0 \\ \frac{x}{x^3}, & x < 0 \end{cases}, \quad f_2(x) = \sin x | x | + x^2 \quad \text{and} \quad g(x) = \begin{cases} -1, & x \geq 0 \\ -2x - 1, & x < 0 \end{cases}.
\]
The feasible set of corresponding problem (VOP) is \( S_0 = [-\frac{3}{2}, 4) \) and let \( u = 0 \). Clearly, \( u \) is a weak minimum of (VOP) as
\[
f(u) - f(x) = \begin{cases} \left( -\frac{x}{x^2 + 1}, -\sin x - x^2 \right)^T, & x \geq 0 \\ \left( -\frac{x}{x^3}, \sin x - x^2 \right)^T, & x < 0 \end{cases} \notin intK \quad \forall x \in S_0 \quad \text{[see Figure 8, Figure 9]}.\]

Figure 8. Graph of \( -\sin x - x^2 \)

Figure 9. Graph of \( \sin x - x^2 \)

\[
f'(0, x) = \begin{cases} (x, x)^T, & x \geq 0 \\ (0, -x)^T, & x < 0 \end{cases} \quad \text{and} \quad f''(0, x) = (0, 2x^2).
\]
\[
g'(0, x) = \begin{cases} 0, & x \geq 0 \\ -2x, & x < 0 \end{cases} \quad \text{and} \quad g''(0, x) = 0.
\]

Since \( f'(0; S) + intK = \{(a, b) \in \mathbb{R}^2 : a < 4, b > -4, a < b\} \) and \( g'(0; S) + intQ = \{c \in \mathbb{R} : c > 0\} \) are convex sets, therefore by Proposition 6.4 [5], \( f'(0, x) \) and \( g'(0, x) \) are \( K \)-subconvexlike and \( Q \)-subconvexlike respectively on \( S \). Also, \( g'(0; S) + \{a\xi(0) : a \geq 0\} + Q = \mathbb{R} \) implies that regularity condition (15) holds at \( u = 0 \). Since \( f''(0, x) \in K \) and \( g''(0, x) \in Q \) for all \( x \in S_0 \), therefore \( (u, \lambda, \mu, \xi) = (0, (-1, 1), 0, \xi) \) is a feasible solution of associated second-order Mond-Weir type dual (MD), for every \( \xi \in \mathbb{R}_+ \).

Conclusion

In this article, we have studied nonsmooth vector optimization problem (VOP) wherein the functions are first and second-order directionally differentiable. New classes of second-order cone-semi(pseudoconvex)quasiconvex functions have been introduced in terms of second-order directional derivative. These functions generalize the ones studied by Suneja et al. [26]. Further, these functions are used to establish second-order KKT type sufficient optimality conditions for (VOP). Second-order Mond-Weir type and Wolfe type duals are formulated and duality results are proved. It may be explored that whether some conditions in the Strong Duality Theorem (Non-trivial Strong Duality Theorem) can be relaxed.

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