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# An Itertive Algorithm with Error Terms for Solving a System of Implicit $n$-Variational Inclusions

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**Abstract** A new system of implicit $n$-variational inclusions is considered. We propose a new algorithm with error terms for computing the approximate solutions of our system. The convergence of the iterative sequences generated by the iterative algorithm is also discussed. Some special cases are also discussed.

**Keywords** Relaxed Operators, Iterative Algorithm, Convergence Results, Resolvent Operators.

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## 1. Historical Perspective and Prelude

Variational inclusions plays an important role in the generalization of classical variational inequalities. So, we have wide range of applications in many of the fields like non-linear programming, economics, optimization, physics etc. Because of its extensive applications various variational inclusions have been established in recent times. Iterative algorithms have been used by different researchers to solve different classes of variational inequalities and variational inclusion problems. For further information one can see [6, 8, 9, 10, 11, 12, 13, 14, 15, 19, 20, 21, 24, 25, 26, 28] and references therein. A new problem of much more interest which is called as system of variational inequalities (inclusions) were introduced and studied in the literature.

In 2007, Xia and Huang [29] studied variational inclusions with a general $H$-monotone operator in Banach spaces, Ahmad et al. [3, 5, 7] considered resolvent operator technique to explain a system of generalized variational-like inclusions in Banach spaces, Verma [27] established and considered some new systems of variational inequalities in Hilbert spaces and generate some iterative algorithms for approximating the solutions of this system. As a generalization of some variational inequalities, Huang [16, 17] introduced Mann and Ishikawa type perturbed iterative algorithms for generalized non-linear implicit quasi-variational inclusions. Then, Agarwal [1] established sensitivity analysis for the new system of generalized non-linear mixed quasi-variational inclusions.

After that, S. Hussain [18] considered an Ishikawa type iterative algorithm for a generalized variational inclusions. In this paper we study and established a system of $n$-variational inclusions in real Hilbert spaces

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called a new system of implicit $n$-variational inclusions. By using resolvent operator technique, we propose a $n$-iterative algorithm with error terms for computing the approximate solutions of a new system of implicit $n$-variational inclusions. We also discussed here criteria of convergence. The mathematical approach of our paper is quite different than the methods discussed above.

Let $X$ be a real Hilbert space whose norm and inner product are denoted by $\|.\|$ and $\langle .,. \rangle$ respectively, $d$ is the metric induced by the norm $\|.\|$, $2^X$ is the family of all non-empty subsets of $X$, $CB(X))$ is the closed and bounded subset of $X$ and $\mathcal{H}(.,.)$ is the Hausdorff metric on $CB(X)$ defined by

$$\mathcal{H}(A,B) = max\left(\sup_{x\in A} d(x,B), \sup_{y\in B} d(A,y)\right)$$

where $d(x,B) = \inf_{y\in B} d(x,y)$ and $d(A,y) = \inf_{x\in A} d(x,y)$.

We require the following definitions and theorems to achieve the main result of this paper.

**Definition 1.1.** A mapping $g : X \to X$ is called

(i) *Lipschitz continuous* if, there exists a constant $\lambda_g > 0$ such that

$$\|g(x_1) - g(x_2)\| \leq \lambda_g \|x_1 - x_2\|, \text{ for all } x_1, x_2 \in X;$$

(ii) *monotone* if,

$$\langle g(x_1) - g(x_2), x_1 - x_2 \rangle \geq 0, \text{ for all } x_1, x_2 \in X;$$

(iii) *strongly monotone* if, there exists a constant $\xi > 0$ such that

$$\langle g(x_1) - g(x_2), x_1 - x_2 \rangle \geq \xi \|x_1 - x_2\|^2, \text{ for all } x_1, x_2 \in X;$$

(iv) *relaxed Lipschitz continuous* if, there exists a constant $r > 0$ such that

$$\langle g(x_1) - g(x_2), x_1 - x_2 \rangle \leq -r\|x_1 - x_2\|^2, \text{ for all } x_1, x_2 \in X.$$

**Definition 1.2.** A mapping $F : X \times X \times X \to X$ is said to be Lipschitz continuous in the first argument if, there exists a constant $\lambda_{F_1}$ such that

$$\|F(x_1, x_2, x_3) - F(y_1, x_2, x_3)\| \leq \lambda_{F_1} \|x_1 - y_1\|, \text{ for all } x_1, y_1, x_2, x_3 \in X.$$

In a similar way, we can define the Lipschitz continuity of $F$ in the rest of the arguments.

**Definition 1.3.** A multivalued mapping $A : X \to CB(X)$ is said to be $\mathcal{H}$-Lipschitz continuous if, there exists a constant $\delta_A$ such that

$$\mathcal{H}(A(x_1), A(x_2)) \leq \delta_A \|x_1 - x_2\|, \text{ for all } x_1, x_2 \in X.$$

**Definition 1.4 [2].** Let $I : X \to X$ be an identity mapping and $H : X \to X$ be a mapping. Then for $\lambda > 0$ a multivalued mapping $M : X \to 2^X$ is a said to be $(I - H)$ monotone if, $M$ is monotone, $H$ is relaxed Lipschitz continuous and

$$[(I - H) + \lambda M](X) = X$$

**Definition 1.5 [2].** Let $H : X \to X$ be a relaxed Lipschitz continuous mapping and $I : X \to X$ be an identity mapping. Suppose that $M : X \to 2^X$ is a multivalued, $(I - H)$ - monotone mapping. For $\lambda > 0$, relaxed resolvent operator $R^{I-H}_{\lambda,M} : X \to X$ associated with $I, H$ and $M$ is defined by

$$R^{I-H}_{\lambda,M}(x) = [(I - H) + \lambda M]^{-1}(x), \text{ for all } x \in X, \tag{1.1}$$

The following theorems plays an important role in proving our main results which is due to [2].

**Theorem 1.1 [2].** Let $H : X \to X$ be a relaxed Lipschitz continuous mapping, $I : X \to X$ be an identity mapping and $M : X \to 2^X$ be a mutivalued, $(I - H)$- monotone mapping. Then for $\lambda > 0$, the operator $[(I - H) + \lambda M]^{-1}$ is the single valued.

**Theorem 1.2 [2].** Let $I : X \to X$ be an identity mapping, $H : X \to X$ be a $r$-relaxed Lipschitz continuous mapping and $M : X \to 2^X$ be a multivalued, $(I - H)$- monotone mapping. Then the relaxed resolvent operator $R^{I-H}_{\lambda,M} : X \to X$ is $\frac{1}{1+r}$ Lipschitz continuous. i.e.,

$$\|R^{I-H}_{\lambda,M}(x_1) - R^{I-H}_{\lambda,M}(x_2)\| \le \frac{1}{1+r}\|x_1 - x_2\|, \ \forall \ x_1, x_2 \in X.$$

## 2. Formulation of the Problem

We introduce a new system of implicit $n$-variational inclusions in Hilber spaces and develop an iterative algorithm with error terms for solving this system. For each $i \in \{1, 2, 3, ...n\}$, let $X_i$ be a real Hilbert space, let $H_i, g_i : X_i \to X_i, F_i, P_i : X_1 \times X_2 ... \times X_n \to X_i$ be the single valued mappings and $A_{i1}, A_{i2}, ....., A_{in} : X_i \to CB(X_i)$ be the multivalued mappings. Let $I_i : X_i \to X_i$ be the identity mappings and $M_i : X_i \times X_i \to 2^{X_i}$ be the multivalued, $(I_i - H_i)$- monotone mappings. We consider the following system of implicit $n$ variational inclusions (in short, SIVI):

$$\text{(SIVI)} \quad \begin{cases} \text{Find } (x_1, x_2, ..., x_n, u_{11}, u_{12}, u_{13}, ..., u_{1n}, ..., u_{n1}, u_{n2}, u_{n3}, ...u_{nn}) \\ (x_1, x_2, ..., x_n) \in X_1 \times X_2 \times .... \times X_n, \\ u_{i1} \in A_{i1}(x_1), u_{i2} \in A_{i2}(x_2), .....u_{in} \in A_{in}(x_n) \text{ such that} \\ 0 \in F_1(x_1, x_2...., x_n) + P_1(u_{11}, u_{12}, ...., u_{1n}) + M_1(g_1(x_1), x_1) \\ 0 \in F_2(x_1, x_2, ...., x_n) + P_2(u_{21}, u_{22}, ...., u_{2n}) + M_2(g_2(x_2), x_2) \\ . \\ . \\ . \\ 0 \in F_n(x_1, x_2, ...., x_n) + P_n(u_{n1}, u_{n2}, ...., u_{nn}) + M_n(g_n(x_n), x_n). \end{cases}$$

Equivalently

$$0 \in F_i(x_1, x_2, ..., x_n) + P_i(u_{i1}, u_{i2}, ....u_{in}) + M_i(g_i(x_i), x_i).$$

**Special Cases**:

(i) If $F_1(x_1, x_2, ..., x_n) \equiv F_1(x_1, x_2, x_3)$, $F_2(x_1, x_2, ..., x_n) \equiv F_2(x_1, x_2, x_3)$, $F_3(x_1, x_2, ..., x_n) \equiv F_3(x_1, x_2, x_3), F_4, F_5, ..., F_n = 0$, $P_1(u_{11}, u_{22}, ..., u_{1n}) \equiv P_1(u_{11}, u_{22}, u_{33})$, $P_2(u_{21}, u_{22}, ..., u_{2n}) \equiv P_2(u_{21}, u_{22}, u_{23})$, $P_3(u_{31}, u_{32}, ..., u_{3n}) \equiv P_3(u_{31}, u_{32}, u_{33}), P_4, P_5, ..., P_n \equiv 0$. Then the problem (SIVI) reduces to find $(x_1, x_2, x_3, u_{11}, u_{12}, u_{13}, u_{21}, u_{22}, u_{23}, u_{31}, u_{32}, u_{33})$ such that for each $i \in \{1, 2, 3\}$, $x_1, x_2, x_3 \in X_1 \times X_2 \times X_3$, $u_{i1} \in A_{i1}(x_1), u_{i2} \in A_{i2}(x_2), u_{i3} \in A_{i3}(_3)$ such that

$$\text{(SGIVI)} \quad \begin{cases} 0 \in F_1(x_1, x_2, x_3) + P_1(u_{11}, u_{12}, u_{13}) + M_1(g_1(x_1), x_1) \\ 0 \in F_2(x_1, x_2, x_3) + P_2(u_{21}, u_{22}, u_{23}) + M_2(g_2(x_2), x_2) \\ 0 \in F_3(x_1, x_2, x_3) + P_3(u_{31}, u_{32}, u_{33}) + M_3(g_3(x_3), x_3). \end{cases}$$

System of generalized implicit variational inclusion (SGIVI) introduced and studied by Ahmad et al [4].

(ii) If $F_1(x_1, x_2, x_3) \equiv F(x_1, x_2)$, $F_2(x_1, x_2, x_3) \equiv G(x_1, x_2)$, $F_3 \equiv 0, P_1(., ., .) \equiv P(., .)$, $P_2(., ., .) \equiv Q(., .)$, $P_3 \equiv 0$. $M_1(g_1(x_1), x_1) \equiv M_1(g_1(x_1)), M_2(g_2(x_2), x_2) \equiv M_2(g_2(x_2))$, $M_3 \equiv 0$, then the problem (SGIVI) reduces to the problem of finding $(x_1, x_2) \in X_1 \times X_2$ such that

$$\text{(SGMQI)} \quad \begin{cases} 0 \in F(x_1, x_2) + P(u, v) + M_1(g_1(x_1)) \\ 0 \in G(x_1, x_2) + Q(u, v) + M_2(g_2(x_2)), \end{cases}$$

which is called the system of generalized mixed quasivariational inclusions with $(H,\eta)$-monotone operators (SGMQI) introduced and studied by Peng and Zhu [23].

(iii) If $P = Q \equiv 0$, $g_1 = I_1$ (the identity map on $X_1$), $g_2 \equiv I_2$(the identity map on $X_2$), $M_1(g_1(x_1)) = M_1(x_1)$, $M_2(g_2(x_2)) = M_2(x_2)$ then (SGMQI) reduces to the system of variational inclusion with $(H,\eta)$-monotone operators (SVI) which is to find $(x,y) \in X_1 \times X_2$ such that

$$\text{(SVI)} \quad \begin{cases} 0 \in F(x_1,x_2) + M_1(x_1) \\ 0 \in G(x_1,x_2) + M_2(x_2). \end{cases}$$

Problem (SVI) was introduced and studied by Fang et al [14].

**Lemma 2.1.** For each $i \in \{1,2,...,n\}$ let $X_i$ be a real Hilbert space, $H_i, g_i : X_i \to X_i$, $F_i, P_i : X_1 \times X_2 \times ...... \times X_n \to X_i$ be single-valued mappings and $A_{i1}, A_{i2}, ...., A_{in} : X_i \to CB(X_i)$ be the multivalued mappings. Let $I_i : X_i \to X_i$ be the identity mappings and $M_i : X_i \times X_i \to 2^{X_i}$ be the multivalued,, $(I_i - H_i)$-monotone mappings. Then, $(x_1, x_2, .., x_n, u_{11}, u_{12}, .....u_{1n}, u_{21}, u_{22}, ......u_{2n}, .., u_{n1}, u_{n2}, .., u_{nn})$ with $(x_1, x_2, ...., x_n) \in X_1 \times X_2 \times .... \times X_n$, $u_{i1} \in A_{i1}(x_1), u_{i2} \in A_{i2}(x_2), ...., u_{in} \in A_{in}(x_n)$ is a solution of problem (SIVI), if following equations are satisfied:

$$g_i(x_i) = R^{I_i - H_i}_{\lambda_i, M_i(.,x_i)}[(I_i - H_i)(g_i(x_i)) - \lambda_i F_i(x_1, x_2, ..., x_n) - \lambda_i P_i(u_{i1}, u_{i2}, ..., u_{in})],$$

where, $R^{I_i - H_i}_{\lambda_i, M_i(.,x_i)} = [(I_i - H_i) + \lambda_i M_i(., x_i)]^{-1}$ are the relaxed resolvent operators and $\lambda_i > 0$ are constants.

**Proof.** The proof is a direct consequence of the definition of the relaxed resolvent operator (1.1).

On the basis of the above observations, we propose the following iterative algorithm with error terms for computing the approximate solution of (SIVI).

**Algorithm 2.1.** For each $i \in \{1,2,...,n\}$, given $x_i^o \in X_i$, take $u_{i1}^o \in A_{i1}(x_1^o), u_{i2}^o \in A_{i2}(x_2^o), ...., u_{in}^o \in A_{in}(x_n^o)$ and let

$$\begin{aligned} x_i^1 \quad = \quad & (1-\mu_i)x_i^o + \mu_i[x_i - g_i(x_i^o) + R^{I_i - H_i}_{\lambda_i, M_i(.,x_i^o)}((I_i - H_i)(g_i(x_i) - \lambda_i F_i(x_1^o, x_2^o, ..., x_n^o) \\ & -\lambda_i P_i(u_{i1}^o, u_{i2}^o ..., u_{in}^o)] + \mu_i e_i^o. \end{aligned}$$

Since, $u_{i1}^o \in A_{i1}(x_1^o), u_{i2}^o \in A_{i2}(x_2^o), ......u_{in}^o \in A_{in}(x_n^o)$, by Nadlers theorem, there exist $u_{i1}^1 \in A_{i1}(x_1^1), u_{i2}^1 \in A_{i2}(x_2^1), ..., u_{in} \in A_{in}(x_n^1)$, such that

$$\|u_{i1}^1 - u_{i1}^o\| \le (1+1)\mathcal{H}_1(A_{i1}(x_1^1), A_{i1}(x_1^o))$$

$$\|u_{i2}^1 - u_{i2}^o\| \le (1+1)\mathcal{H}_2(A_{i2}(x_2^1), A_{i2}(x_2^o))$$

$$\vdots$$

$$\|u_{in}^1 - u_{in}^o\| \le (1+1)\mathcal{H}_n(A_{in}(x_n^1), A_{in}(x_n^o)).$$

Again, let

$$\begin{aligned} x_i^2 \quad = \quad & (1-\mu_i)x_i^1 + \mu_i[x_i^1 - g_i(x_i^1) + R^{I_i - H_i}_{\lambda_i, M_i(.,x_i^1)}((I_i - H_i)(g_i(x_i^1)) - \lambda_i F_i(x_1^1, x_2^1 ..., x_n^1) \\ & -\lambda_i P_i(u_{i1}^1, u_{i2}^1, ..., u_{in}^1))] + \mu_i e_i^1. \end{aligned}$$

By Nadler's theorem [22], there exists $u_{i1}^2 \in A_{i1}(x_1^2), u_{i2}^2 \in A_{i2}(x_2^2), ..., u_{in}^2 \in A_{in}(x_n^2)$ such that,

$$\|u_{i1}^2 - u_{i1}^1\| \le \left(1 + \frac{1}{2}\right)\mathcal{H}_1(A_{i1}(x_1^2), A_{i1}(x_1^1))$$

$$\|u_{i2}^2 - u_{i2}^1\| \le \left(1 + \frac{1}{2}\right)\mathcal{H}_2(A_{i2}(x_2^2), A_{i2}(x_2^1))$$

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$$\|u_{in}^2 - u_{in}^1\| \le \left(1 + \frac{1}{2}\right)\mathcal{H}_n(A_{in}(x_n^2), A_{in}(x_n^1)).$$

By induction, we obtain the sequences, $\{x_i^n\}, \{u_{i2}^n\}, ...., \{u_{in}^n\}$ satisfying

$$x_i^{n+1} = (1-\mu_i)x_i^n + \mu_i[x_i^n - g_i(x_i^n) + R^{I_i-H_i}_{\lambda_i, M_i(.,x_i^n)}((I_i - H_i)(g_i(x_i^n))) - \lambda_i F_i(x_1^n, x_2^n..., x_n^n)$$
$$-\lambda_i P_i(u_{i1}^n, u_{i2}^n, ..., u_{in}^n)] + \mu_i e_i^n \quad (2.1)$$

$$\|u_{i1}^{n+1} - u_{i1}^n\| \le \left(1 + \frac{1}{n+1}\right)\mathcal{H}_1(A_{i1}(x_1^{n+1}), A_{i1}(x_1^n)) \quad (2.2)$$

$$\|u_{i2}^{n+1} - u_{i2}^n\| \le \left(1 + \frac{1}{n+1}\right)\mathcal{H}_2(A_{i2}(x_2^{n+1}), A_{i2}(x_2^n)) \quad (2.3)$$

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$$\|u_{in}^{n+1} - u_{in}^n\| \le \left(1 + \frac{1}{n+1}\right)\mathcal{H}_n(A_{in}(x_n^{n+1}, A_{in}(x_n^n))), \quad (2.4)$$

where $n = 0, 1, 2, ....$ for $i = \{1, 2, , ..., n\}$, $\mu_i > 0, \lambda_i > 0$ are constants, $e_i^n \in X_i$ for $n \ge 0$, are errors to take into account a possible inexact computation of the resolvent operator point and $\mathcal{H}_i(.,.)$ are the Hausdroff metrics on $CB(X_i)$.

## 3. Existence and Convergence Analysis

In this section, we consider those conditions under which the solution of the problem (SIVI) exists and the sequences of the approximate solutions obtained by Algorithm 2.1, converge strongly to the exact solution of the problem (SIVI).

**Theorem 3.1.** For each $i \in \{1, 2, ..., n\}$, consider $X_i$ is a Hilbert space, $I_i : X_i \to X_i$ be the identity mappings and $H_i, g_i : X_i \to X_i$ be the single-valued mappings such that $g_i$ is $\xi_i$-strongly monotone, $\lambda_{g_i}$-Lipschitz continuous and $H_i$ is $\lambda_{H_i}$-Lipschitz continuous, $r_i$-relaxed Lipschitz continuous. Suppose that $A_{i1}, A_{i2}, ...., A_{in} : X_i \to CB(X_i)$ are the multivalued mappings such that $A_{i1}$ is $\delta_{Ai_1} - D_1$-Lipschitz continuous and $A_{i2}$ is $\delta_{Ai_2} - D_2$-Lipschitz continuous....$\delta_{Ai_n} - D_n$-Lipschitz continuous, respectively. Let $F_i, P_i : X_1 \times X_2 \times ..... \times X_n \to X_i$ be the single-valued mappings such that $F_i$'s are Lipschitz continuous in all $n$-arguments with onstants $\lambda_{P_{i1}} > 0, \lambda_{P_{i2}} > 0, ...., \lambda_{P_{in}} > 0$, respectively. Suppose that $M_i : X_i \times X_i \to 2^{X_i}$ are the multivalued $(I_i - H_i)$-monotone mappings. For $\lambda_i > 0$ and $h_i > 0$ assume

$$\|R^{I_i-H_i}_{\lambda_i, M_i(.,x)}(z) - R^{I_i-H_i}_{\lambda_i, M_i(.,y)}(z)\| \le h_i\|x - y\|, \ \forall \ x, y, z \in X_i, \quad (3.1)$$

and

$$\begin{cases} k_i = 1 - \mu_i + \mu_i h_i + \mu_i\sqrt{1 - 2\xi_i + \lambda_{g_i}^2} + \frac{\mu_i\lambda_{g_i} + \mu_i\lambda_{H_i}\lambda_{g_i}}{1+r_i} + \sum_{i=1}^n \frac{\mu_i\lambda_j\lambda_{F_{ji}}}{1+r_j} < 1 \\ v_i = \mu_i\left(\sum_{j=1}^{i=n} \frac{\mu_j\lambda_j\lambda_{P_{ji}}\delta_{A_{ji}}}{1+r_j}\right) < 1, \\ k_i + v_i < 1, \text{ and } 2\xi_i < 1 + \lambda_{g_i}^2, \text{ for each } i \in \{1, 2, ..., n\} \\ \sum_{q=1}^\infty \|e_1^q - e_1^{q-1}\|k^{-q} < \infty, \sum_{q=1}^\infty \|e_2^q - e_2^{q-1}\|k^{-q} < \infty, ..., \sum_{q=1}^\infty \|e_n^q - e_n^{q-1}\| < \infty, \\ lim_{n\to\infty} e_1^n = lim_{n\to\infty} e_2^n .... = lim_{n\to\infty} e_n^n = 0, \text{ for each } k \in (0, 1). \end{cases} \quad (3.2)$$

Then the problem (SIVI) admits a solution $(x_1, x_2, ...x_n, u_{11}, u_{12}, ....u_{1n}, u_{21}, u_{22}, ....u_{2n}, u_{31}, u_{32}, ....u_{3n}, u_{n1}, u_{n2}, ..., u_{nn})$ and iterative sequences $\{x_i^n\}, \{u_{i1}^n\}, \{u_{i2}^n\}, ....\{u_{in}^n\}$ generated by iterative Algorithm 2.1 strongly converge to $x_i, u_{i1}, u_{i2}, ....u_{in}$, respectively, for each $i \in \{1, 2, 3, ..., n\}$.

**Proof.** For each $i \in \{1, 2, ..., n\}$, let $d_i^n = [(I_i - H_i)(g_i(x_i^n)) - \lambda_i F_i(x_1^n, x_2^n ..., x_n^n) - \lambda_i P_i(u_{i_1}^n, u_{i_2}^n, ..., u_{i_n}^n)]$.

Using Algorithm 2.1, condition (3.1) and Theorem 2.2, we have

$$\begin{aligned}
\|x_1^{n+1} - x_1^n\| &= \|(1-\mu_i)x_1^n + \mu_1[x_1^n - g_1(x_1^n) + R^{I_1-H_1}_{\lambda_1, M_1(.,x_1^n)}(d_1^n)] + \mu_1 e_1^n - (1-\mu_1)x_1^{n-1} \\
&\quad -\mu_1[x_1^{n-1} - g_1(x_1^{n-1}) + R^{I_1-H_1}_{\lambda_1, M_1(.,x_1^{n-1})}(d_1^{n-1})] - \mu_1 e_1^{n-1}\| \\
&\leq (1-\mu_1)\|x_1^n - x_1^{n-1}\| + \mu_1\|x_1^n - x_1^{n-1} - (g_1(x_1^n) - g_1(x_1^{n-1})\| \\
&\quad +\mu_1\|R^{I_1-H_1}_{\lambda_1, M_1(.,x_1^n)}(d_1^n) - R^{I_1-H_1}_{\lambda_1, M_1(.,x_1^n)}(d_1^{n-1})\| + \mu_1\|R^{I_1-H_1}_{\lambda_1, M_1(.,x_1^n)}(d_1^{n-1}) \\
&\quad -R^{I_1-H_1}_{\lambda_1, M_1(.,x_1^n)}(d_1^{n-1}) + \mu_1\|e_1^n - e_1^{n-1}\| \\
&\leq (1-\mu_1)\|x_1^n - x_1^{n-1}\| + \mu_1\|x_1^n - x_1^{n-1} - (g_1(x_1^n) - g_1(x_1^{n-1}))\| \\
&\quad +\mu_1\|R^{I_1-H_1}_{\lambda_1, M_1(.,x_1^n)}(d_1^n) - R_{\lambda_1, M_1(.,x_1^n)}(d_1^{n-1})\| \\
&\quad +\mu_1\|R^{I_1-H_1}_{\lambda_1, M_1(.,x_1^n)}(d_1^{n-1}) - R_{\lambda_1, M_1(.,x_1^{n-1})(d_1^{n-1})}\| + \mu_1\|e_1^n - e_1^{n-1}\| \\
&\leq (1-\mu_1)\|x_1 - x_{n-1}\| + \mu_1\|x_1^n - x_1^{n-1} - (g_1(x_1^n) - g_1(x_1^{n-1}))\| \\
&\quad +\frac{\mu_1}{1+r_1}\|d_1^n - d_1^{n-1}\| + \mu_1 h_1\|x_1^n - x_1^{n-1}\| + \mu_1\|e_1^n - e_1^{n-1}\| \\
&\leq (1-\mu_1+\mu_1 h_1)\|x_1^n - x^{n-1}\| + \mu_1\|x_1^n - x_1^{n-1} - (g_1(x_1^n - g_1^{n-1})\| \\
&\quad +\frac{\mu_1}{1+r_1}\|d_1^n - d_1^{n-1}\| + \mu_1\|e_1^n - e_1^{n-1}\|,
\end{aligned} \tag{3.3}$$

and since $g_1$ is $\lambda_{g_1}$-Lipschitz continuous and $\xi_1$- strongly monotone, we obtain

$$\begin{aligned}
\|x_1^n - x_1^{n-1} - (g1(x_1^n) - g_1(x_1^{n-1}))\|^2 &= \|x_1^n - x_1^{n-1}\|^2 - 2\langle x_1^n - x_1^{n-1}, g_1(x_1^n) - g_1(x_1^{n-1})\rangle \\
&\quad +\|g_1(x_1^n) - g_1(x_1^{n-1})\|^2 \\
&\leq (1 - 2\xi_1 + \lambda_{g_1}^2)\|x_1^n - x_1^{n-1}\|^2.
\end{aligned} \tag{3.4}$$

As $g_1$ is $\lambda_{g_1}$-Lipschitz continuous, $F_i$ is Lipschitz continuous in all $n$-arguments with constants $\lambda_{F_{11}}, \lambda_{F_{12}}, ...., \lambda_{F_{1n}}$, respectively, $P_1$ is Lipschitz continuous in all the $n$-arguments with constants $\lambda_{P_{11}}, \lambda_{P_{12}}, \lambda_{P_{13}}, ..., \lambda_{P_{1n}}$ respectively, $A_{11}$ is $\delta_{A_{11}}$-$D_1$- Lipschitz continuous, $A_{12}$ is $\delta_{A_{12}}$-$D_2$- Lipschitz continuous, ......$A_{1n}$ is $\delta_{A_{1n}}$-$D_n$- Lipschitz continuous, respectively, we get

$$\begin{aligned}
\|d_1^n - d_1^{n-1}\| &= \|(I_1 - H_1)(g_1(x_1^n)) - \lambda_1 F_1(x_1^n, x_2^n, ..., x_n^n) - \lambda_1 P_1(u_{11}^n, u_{12}^n, ..., u_{1n}^n) \\
&\quad -(I_1 - H_1)(g_1(x_1^{n-1}) + \lambda_1 F_1(x_1^{n-1}, x_2^{n-1}, ..., x_n^{n-1}) \\
&\quad +\lambda_1 P_1(u_{11}^{n-1}, u_{12}^{n-1}, ..., u_{1n}^{n-1})\| \\
&\leq \|g_1(x_1^n) - g_1(x_1^{n-1}\| + \|H_i(g_1(x_1^n)) - H_1(g_1(x_1^{n-1}))\| \\
&\quad +\lambda_1\|F_1(x_1^n, x_2^n, ..., x_n^n) - F_1(x_1^{n-1}, x_2^{n-1}, ..., x_n^{n-1})\| \\
&\quad +\lambda_1\|P_1(u_{11}^n, u_{12}^n, ..., u_{1n}^n) - P_1(u_{11}^{n-1}, u_{12}^{n-1}, ..., u_{1n}^{n-1})\| \\
&\leq \|g_1(x_1^n) - g_1(x_1^{n-1}\| + \|H_1(g_1(x_1^n)) - H_1(g_1(x_1^{n-1}))\| \\
&\quad +\lambda_1\|F_1(x_1^n, x_2^n, ..., x_n^n) - F_1(x_1^{n-1}, x_2^{n-1}, ..., x_n^n)\| \\
&\quad +\lambda_1\|F_1(x_1^{n-1}, x_2^n, x_3^n, ..., x_n^n) - F_1(x_1^{n-1}, x_2^{n-1}, ..., x_n^n)\|
\end{aligned}$$

$$
\begin{aligned}
&+\lambda_1\|F_1(x_1^{n-1},x_2^{n-1},....,x_n)-F_1(x_1^{n-1},x_2^{n-1},x_3^{n-1},...,x_n^{n-1})\|\\
&+\lambda_1\|P_1(u_{11}^n,u_{12}^n,...,u_{1n}^n)-P_1(u_{11}^{n-1},u_{12}^n,u_{13}^n,...,u_{1n}^n)\|\\
&+\lambda_1\|P_1(u_{11}^{n-1},u_{12}^n,...,u_{1n}^n)-P_1(u_{11}^{n-1},u_{12}^{n-1},...,u_{1n}^n)\|\\
&+\lambda_1\|P_1(u_{11}^{n-1},u_{12}^{n-1},...,u_{1n}^n)-P_1(u_{11}^{n-1},u_{12}^{n-1},...,u_{1n}^{n-1})\|\\
\leq\;& \lambda_{g_1}\|x_1^n-x_1^{n-1}\|+\lambda_{H_1}\lambda_{g_1}\|x_1^n-x_1^{n-1}\|+\lambda_1\lambda_{F_{11}}\|x_1^n-x_1^{n-1}\|\\
&+\lambda_1F_{12}\|x_2^n-x_2^{n-1}\|+\lambda_1\lambda_{F_{13}}\|x_3^n-x_3^{n-1}\|+....+\lambda_1\lambda_{F_{1n}}\|x_n^n-x_n^{n-1}\|\\
&+\lambda_1\lambda_{P_{11}}\|u_{11}^n-u_{11}^{n-1}\|+\lambda_1\lambda_{P_{12}}\|u_{12}^n-u_{12}^{n-1}\|\\
&+\lambda_1\lambda_{P_{13}}\|u_{13}^n-u_{13}^{n-1}\|+....+\lambda_1P_{1n}\|u_{1n}^n-u_{1n}^{n-1}\|\\
\leq\;& \lambda_{g_1}\|x_1^n-x_1^{n-1}\|+\lambda_{H_1}\lambda_{g_1}\|x_1^n-x_1^{n-1}\|+\lambda_1\lambda_{F_{11}}\|x_1^n-x_1^{n-1}\|\\
&+\lambda_1\lambda_{F_{12}}\|x_2^n-x_2^{n-1}\|+\|\lambda_1\lambda_{F_{13}}\|x_3^n-x_3^{n-1}\|\\
&+...+\lambda_1\lambda_{F_{1n}}\|x_n^n-x_n^{n-1}\|+\lambda_1\lambda_{P_{11}}\|u_{11}^n-u_{11}^{n-1}\|\\
&+\lambda_1\lambda_{P_{12}}\|u_{12}^n-u_{12}^{n-1}\|+\lambda_1\lambda_{P_{13}}\|u_{13}^n-u_{13}^{n-1}\|\\
&+...+\lambda_1\lambda_{P_{1n}}\|u_{1n}^n-u_{1n}^{n-1}\|\\
\leq\;& \lambda_{g_1}\|x_1^n-x_1^{n-1}\|+\lambda_{H_1}\lambda_{g_1}\|x_1^n-x_1^{n-1}\|\\
&+\lambda_1\lambda_{F_{11}}\|x_1^n-x_1^{n-1}\|+\lambda_1\lambda_{F_{12}}\|x_2^n-x_2^{n-1}\|+\lambda_1\lambda_{F_{13}}\|x_3^n-x_3^{n-1}\|\\
&+...+\lambda_1\lambda_{F_{12}}\|x_n^n-x_n^{n-1}\|+\lambda_1\lambda_{P_{11}}\left(1+\frac{1}{n}\right)D_1(A_{11}(x_1^n),A_{11}(x_1^{n-1})\\
&+\lambda_1\lambda_{P_{12}}\left(1+\frac{1}{n}\right)D_2(A_{12}(x_2^n,A_{12}(x_2^{n-1})\\
&+\lambda_1\lambda_{P_{13}}\left(1+\frac{1}{n}\right)D_3(A_{13}(x_3^n-A_3(x_3^{n-1}))\\
&+...+\lambda_1\lambda_{P_{1n}}\left(1+\frac{1}{n}\right)D_n(A_{1n}(x_n^n,A_{1n}(x_n^{n-1}))\\
\leq\;& \lambda_{g_1}\|x_1^n-x_1^{n-1}\|+\lambda_{H_1}\lambda_{g_1}\|x_1^n-x_1^{n-1}\|+\lambda_1\lambda_{F_{11}}\|x_1^n-x_1^{n-1}\|\\
&+\lambda_1\lambda_{F_{12}}\|x_2^n-x_2^{n-1}\|+\lambda_1\lambda_{F_{13}}\|x_3^n-x_3^{n-1}\|\\
&+...+\lambda_1\lambda_{F_{1n}}\|x_n^n-x_n^{n-1}\|+\lambda_1\lambda_{P_{11}}\delta_{A_{11}}\left(1+\frac{1}{n}\right)\|x_1^n-x_1^{n-1}\|\\
&+\lambda_1\lambda_{P_{12}}\delta_{A_{12}}(1+\frac{1}{n})\|x_2^n-x_2^{n-1}\|+\lambda_1\lambda_{P_{13}}\delta_{A_{13}}\left(1+\frac{1}{n}\right)\|x_3^n-x_3^{n-1}\|\\
&+..+\lambda_1\lambda_{P_{1n}}\delta_{A_{1n}}\left(1+\frac{1}{n}\right)\|x_n^n-x_n^{n-1}\|\\
\leq\;& (\lambda_{g_1}+\lambda_1\lambda_{F_{11}})\|x_1^n-x_1^{n-1}\|+\left(\lambda_{H_1}\lambda_{g_1}+\lambda_1\lambda_{P_{11}}\delta_{A_{11}}\left(1+\frac{1}{n}\right)\right)\|x_1^n-x_1^{n-1}\|\\
&+\left(\lambda_1\lambda_{F_{12}}+\lambda_1\lambda_{P_{12}}\delta_{A_{12}}\left(1+\frac{1}{n}\right)\right)\|x_2^n-x_2^{n-1}\|\\
&+\left(\lambda_1\lambda_{F_{13}}+\lambda_1\lambda_{P_{13}}\delta_{A_{13}}\left(1+\frac{1}{n}\right)\right)\|x_3^n-x_3^{n-1}\|\\
&+...+\left(\lambda_1\lambda_{F_{1n}}+\lambda_1\lambda_{P_{1n}}\delta_{A_{1n}}\left(1+\frac{1}{n}\right)\right)\|x_n^n-x_n^{n-1}\|\\
\leq\;& \left(\lambda_{g_1}+\lambda_1\lambda_{F_{11}}+\lambda_{H_1}\lambda_{g_1}+\lambda_1\lambda_{P_{11}}\delta_{A_{11}}\left(1+\frac{1}{n}\right)\right)\|x_1^n-x_1^{n-1}\|\\
&+\left(\lambda_1\lambda_{F_{12}}+\lambda_1\lambda_{P_{12}}\delta_{A_{12}}\left(1+\frac{1}{n}\right)\right)\|x_2^n-x_2^{n-1}\|
\end{aligned}
$$

$$+\left(\lambda_1\lambda_{F_{13}}+\lambda_1\lambda_{P_{13}}\delta_{A_{13}}\left(1+\frac{1}{n}\right)\right)\|x_3^n-x_3^{n-1}\|$$
$$+\ldots+\left(\lambda_1\lambda_{F_{1n}}+\lambda_1\lambda_{P_{1n}}\delta_{A_{1n}}\left(1+\frac{1}{n}\right)\right)\|x_n^n-x_n^{n-1}\|. \quad (3.5)$$

Using (3.4) and(3.5), equation (3.3) becomes,

$$\begin{aligned}\|x_1^{n+1}-x_1^n\| \leq\ & (1-\mu_1+\mu_1h_1+\mu_1\sqrt{1-2\xi_1+\lambda_{g_1}^2} \\ &+\frac{\mu_1(\lambda_{g_1}+\lambda_1\lambda_{F_{11}}+\lambda_{H_1}\lambda_{g_1}+\lambda_1\lambda_{P_{11}}\delta_{A_{11}}(1+\frac{1}{n})}{1+r_1})\|x_1^n-x_1^{n-1}\| \\ &+\frac{\mu_1(\lambda_1\lambda_{F_{12}}+\lambda_1\lambda_{P_{12}}\delta_{A_{12}}(1+\frac{1}{n}))}{1+r_1}\|x_2^n-x_2^{n-1}\| \\ &+\frac{\mu_1(\lambda_1\lambda_{F_{13}}+\lambda_1\lambda_{P_{13}}\delta_{A_{13}}(1+\frac{1}{n}))1+r_1}{1+r_1}\|x_3^n-x_3^{n-1}\|+\mu_1\|e_1^n-e_1^{n-1}\|. \end{aligned} \quad (3.6)$$

Using the same arguments as for (3.6), we get

$$\begin{aligned}\|x_2^{n+1}-x_2^n\| \leq\ & (1-\mu_2+\mu_2h_2+\mu_2\sqrt{1-2\xi_2+\lambda_{g_2}^2} \\ &+\frac{\mu_2(\lambda_{g_2}+\lambda_2\lambda_{F_{22}}+\lambda_{H_2}\lambda_{g_2}+\lambda_2\lambda_{P_{22}}\delta_{A_{22}}(1+\frac{1}{n}))}{1+r_2}\|x_2^n-x_2^{n-1}\| \\ &+\frac{\mu_2(\lambda_2\lambda_{F_{21}}+\lambda_2\lambda_{P_{21}}\delta_{A_{21}}(1+\frac{1}{n}))}{1+r_2}\|x_1^n-x_1^{n-1}\| \\ &+\frac{\mu_2(\lambda_2\lambda_{F_{23}}+\lambda_2\lambda_{P_{23}}\delta_{A_{23}}(1+\frac{1}{n}))}{1+r_2}\|x_3^n-x_3^{n-1}\|+\mu_2\|e_2^n-e_2^{n-1}\|. \end{aligned} \quad (3.7)$$

Using the same arguments as for (3.6), we get

$$\begin{aligned}\|x_3^{n+1}-x_3^n\| \leq\ & \frac{\mu_3(\lambda_3\lambda_{F_{31}}+\lambda_3\lambda_{P_{31}}\delta_{A_{31}}(1+\frac{1}{n}))}{1+r_3}\|x_1^n-x_1^{n-1}\| \\ &+\frac{\mu_3(\lambda_3\lambda_{F_{32}}+\lambda_3\lambda_{P_{32}}\delta_{A_{32}}(1+\frac{1}{n}))}{1+r_3}\|x_2^n-x_2^{n-1}\| \\ &+(1-\mu_3+\mu_3h_3+\mu_3\sqrt{1-2\xi_3+\lambda_{g_3}^2} \\ &+\frac{\mu_3(\lambda_{g_3}+\lambda_3\lambda_{F_{33}}+\lambda_{H_3}\lambda_{g_3}+\lambda_3\lambda_{P_{33}}\delta_{A_{33}}(1+\frac{1}{n}))}{1+r_3}\|x_3^n-x_3^{n-1}\| \\ &+\mu_3\|e_3^n-e_3^{n-1}\|. \end{aligned} \quad (3.8)$$

Using the same arguments as for (3.6), we get

$$\begin{aligned}\|x_n^{n+1}-x_n^n\| \leq\ & \frac{\mu_n(\lambda_nF_{n1}+\lambda_n\lambda_{P_{n1}}\delta_{A_{n1}}(1+\frac{1}{n}))}{1+r_n}\|x_n^n-x_n^{n-1}\| \\ &+\frac{\mu_n(\lambda_nF_{n2}+\lambda_n\lambda_{P_{n2}}\delta_{A_{n2}}(1+\frac{1}{n}))}{1+r_n}\|x_n^n-x_n^{n-1}\| \\ &+(1-\mu_n+\mu_nh_n+\mu_n\sqrt{1-2\xi_n+\lambda_{g_n}^2} \\ &+\frac{\mu_n(\lambda_{g_n}+\lambda_n\lambda_{F_{n3}}+\lambda_{H_n}\lambda_{g_3}+\lambda_n\lambda_{P_{n3}}\delta_{A_{n3}}(1+\frac{1}{n}))}{1+r_n}\|x_n^n-x_n^{n-1}\| \\ &+\mu_n\|e_n^n-e_n^{n-1}\|. \end{aligned} \quad (3.9)$$

Combining (3.6) and (3.9), we get

$$
\begin{aligned}
& \|x_1^{n+1} - x_1^n\| + .... + \|x_n^{n+1} - x_n^n\| \\
\leq\ & (1 - \mu_1 + \mu_1 h_1 + \mu_1\sqrt{1 - 2\xi_1 + \lambda_{g_1}^2} \\
& + \frac{\mu_1(\lambda_{g_1} + \lambda_1\lambda_{F_{11}} + \lambda_{H_1}\lambda_{g_1} + \lambda_1\lambda_{P_{11}}\delta_{A_{11}}(1 + \frac{1}{n}))}{1 + r_1}\|x_1^n - x_1^{n-1}\| \\
& + \frac{\mu_1(\lambda_1\lambda_{F_{12}} + \lambda_1\lambda_{P_{12}}\delta_{A_{12}}(1 + \frac{1}{n}))}{1 + r_1}\|x_2^n - x_2^{n-1}\| \\
& + \frac{\mu_1(\lambda_1\lambda_{F_{13}} + \lambda_1\lambda_{P_{13}}\delta_{A_{13}}(1 + \frac{1}{n}))}{1 + r_1}\|x_3^n - x_3^{n-1}\| \\
& + \mu_1\|e_1^n - e_1^{n-1}\| + (1 - \mu_2 + \mu_2 h_2 + \mu_2\sqrt{1 - 2\xi_2 + \lambda_{g_2}^2} \\
& + \frac{\mu_2(\lambda_{g_2} + \lambda_2\lambda_{F_{22}} + \lambda_{H_2}\lambda_{g_2} + \lambda_2\lambda_{P_{22}}\delta_{A_{22}}(1 + \frac{1}{n}))}{1 + r_2}\|x_2^n - x_2^{n-1}\| \\
& + \frac{\mu_2(\lambda_2\lambda_{F_{21}} + \lambda_2\lambda_{P_{21}}\delta_{A_{21}}(1 + \frac{1}{n}))}{1 + r_2}\|x_1^n - x_1^{n-1}\| \\
& + \frac{\mu_2(\lambda_2\lambda_{F_{23}} + \lambda_2\lambda_{P_{23}}\delta_{A_{23}}(1 + \frac{1}{n}))}{1 + r_2}\|x_3^n - x_3^{n-1}\| \\
& + \mu_2\|e_2^n - e_2^{n-1}\| + (1 - \mu_3 + \mu_3 h_3 + \mu_3\sqrt{1 - 2\xi_3 + \lambda_{g_3}^2} \\
& + \frac{\mu_3(\lambda_3\lambda_{F_{33}} + \lambda_{H_3}\lambda_{g_3} + \lambda_3\lambda_{P_{33}}\delta_{A_{33}}(1 + \frac{1}{n}))}{1 + r_3}\|x_3^n - x_3^{n-1}\| \\
& + \frac{\mu_3(\lambda_3\lambda_{F_{31}} + \lambda_3\lambda_{P_{31}}\delta_{A_{31}}(1 + \frac{1}{n}))}{1 + r_3}\|x_1^n - x_1^{n-1}\| \\
& + \frac{\mu_3(\lambda_3\lambda_{F_{32}} + \lambda_3\lambda_{P_{32}}\delta_{A_{32}}(1 + \frac{1}{n}))}{1 + r_3}\|x_2^n - x_2^{n-1} + \mu_3\|e_3^n - e_3^{n-1}\| \\
& + .... + (1 - \mu_n + \mu_n h_n + \mu_n\sqrt{1 - 2\xi_n + \lambda_{g_n}^2} \\
& + \frac{\mu_n(\lambda_{g_n} + \lambda_n\lambda_{F_{n3}} + \lambda_{H_n}\lambda_{g_3} + \lambda_n\lambda_{P_{n3}}\delta_{A_{n3}}(1 + \frac{1}{n}))}{1 + r_n}\|x_n^n - x_n^{n-1}\| \\
& + \frac{\mu_n(\lambda_n\lambda_{F_{n1}} + \lambda_n\lambda_{P_{n1}}\delta_{A_{n1}}(1 + \frac{1}{n}))}{1 + r_n}\|x_n^n - x_n^{n-1}\| \\
& + \frac{\mu_n(\lambda_n\lambda_{F_{n2}} + \lambda_n\lambda_{P_{n2}}\delta_{A_{n2}}(1 + \frac{1}{n}))}{1 + r_n}\|x_n^n - x_n^{n-1}\| + \mu_n\|e_n^n - e_n^{n-1}\|,
\end{aligned}
$$

which implies that,

$$
\begin{aligned}
\sum_{i=1}^{n}\|x_i^{n+1} - x_i^n\| \leq\ & \sum_{i=1}^{n}(1 - \mu_i + \mu_i h_i + \mu_i\sqrt{1 - 2\xi_i + \lambda_{g_i}^2} + \frac{\mu_i\lambda_{g_i} + \mu_i\lambda_{H_i}\lambda_{g_i}}{1 + r_i} \\
& + \sum_{j=1}^{n}\frac{\mu_j\lambda_j F_{ji}}{1 + r_j} + \sum_{j=1}^{n}\frac{\mu_j\lambda_j\lambda_{P_{ji}}\delta_{A_{ji}}}{1 + r_j}(1 + \frac{1}{n}))\|x_i^n - x_i^{n+1}\| \\
& + \sum_{i=1}^{n}\mu_i\|e_i^n - e_i^{n-1}\| \\
\leq\ & \sum_{i=1}^{n}(k_i + v_i^n)\|x_i^n - x_i^{n-1}\| + \sum_{i=1}^{n}\mu_i\|e_i^n - e_i^{n-1}\|, \qquad (3.10)
\end{aligned}
$$

where,

$$k_i = 1 - \mu_i + \mu_i h_i + \mu_i\sqrt{1 - 2\xi_i + \lambda_{g_i}^2} + \frac{\mu_i\lambda_{g_i} + \mu_i\lambda_{H_i}\lambda_{g_i}}{1 + r_i} + \sum_{j=1}^{n}\frac{\mu_j\lambda_j\lambda_{F_{ji}}}{1 + r_j}$$

and

$$v_i^n = \sum_{j=1}^{n}\frac{\mu_j\lambda_j\lambda_{P_{ji}}\delta_{A_{ji}}}{1 + r_j}(1 + \frac{1}{n}).$$

It follows that from (3.10) that,

$$\sum_{i=1}^{n}\|x_i^{n+1} - x_i^n\| \le \sum_{i=1}^{n}\alpha^n\|x_i^n - x_i^{n+1}\| + \sum_{i=1}^{n}\mu_i\|e_i^n - e_i^{n-1}\|, \tag{3.11}$$

where, $\alpha^n = \max\{k_1 + v_1^n, k_2 + v_2^n, k_3 + v_3^n, ..., k_n + v_n^n\},\ \forall\ n = 1, 2, 3, .....$
Let $\alpha = \max\{k_1 + v_1, k_2 + v_2, k_3 + v_3, ..., k_n + v_n\}$
where,

$$v_i = \mu_i\sum_{j=1}^{n}\frac{\mu_j\lambda_j\lambda_{P_{ji}}\delta_{A_{ji}}}{1 + r_j},\ for\ each\ i \in 1, 2, 3, ..., n$$

then $\alpha_i^n \to \alpha$ and $v_i^n \to v_i$ when $n \to \infty$ for each $i \in \{1, 2, 3, ..., n\}$.
From condition (3.2), we know that $0 < \alpha < 1$, and hence there exists $n_0 \in N$ and $\alpha_0 \in (\alpha, 1)$ such that $\alpha^n \le \alpha_0$ for all $n \ge n_0$. Therefore, it follows from (3.10) that,

$$\sum_{i=1}^{n}\|x_i^{n+1} - x_i^n\| \le \sum_{i=1}^{n}\alpha_{n_0}\|x_i^n - x_i^{n-1}\| + \sum_{i=1}^{n}\mu_i\|e_i^n - e_i^{n-1}\|,\ \forall\ n \ge n_0,$$

which implies that

$$\sum_{i=1}^{n}\|x_i^{n+1} - x_i^n\| \le \sum_{i=1}^{n}\alpha_0^{n-n_0}\|x_i^{n_0+1} - x_i^{n_0}\| + \sum_{p=1}^{n-n_0}\sum_{i=1}^{n}\mu_i\alpha_0^{p-1}\iota_i^{n-(p-1)},\ \forall\ n \ge n_0,$$

where $\iota_i^n = \|e_i^n - e_i^{n-1}\|$ for all $n \ge n_0$. Hence, for any $m \ge n > n_0$, we get

$$\begin{aligned}\sum_{i=1}^{n}\|x_i^m - x_i^n\| &\le \sum_{q=n}^{m-1}\sum_{i=1}^{n}\|x_i^{n_0+1} - x_i^{n_0}\| + \sum_{q=n}^{m}\sum_{p=1}^{q-n_0}\sum_{i=1}^{n}\mu_i\alpha_0^{p-1}\iota_i^{q-(p-1)}\\ &\le \sum_{q=n}^{m-1}\sum_{i=1}^{n}\alpha_0^{q-n_0}\|x_i^{n_0+1} - x_i^{n_0}\| + \sum_{q=n}^{m}\sum_{p=1}^{q-n_0}\sum_{i=1}^{n}\mu_i\alpha_0^{q}\frac{\iota_i^{q-(p-1)}}{\alpha_0^{q-(p-1)}}.\end{aligned} \tag{3.12}$$

Since,

$$\sum_{q=1}^{\infty}\iota_1^q k^{-q} < \infty, \sum_{q=1}^{\infty}\iota_2^q k^{-q} < \infty, \sum_{q=1}^{\infty}\iota_3^q k^{-q} < \infty, ..., \sum_{q=1}^{\infty}\iota_n^q k^{-q} < \infty,\ \forall\ k \in (0, 1)\ and\ \alpha_0 < 1.$$

It follows from (3.12), that

$$\|x_1^m - x_1^n\| \to 0, \|x_2^m - x_2^n\| \to 0, ..., \|x_n^m - x_n^n\| \to 0,\ as\ n \to \infty,$$

and so $\{x_1^n\}, \{x_2^n\}, , ..., x_n^n$ are Cauchy sequences in $X_1, X_2, ..., X_n$ respectively. Thus, there exist $x_1 \in X_1, x_2 \in X_2..., x_n \in X_n$ such that $x_1^n \to x_1, x_2^n \to x, ...x_n^n \to x_n$, when $n \to \infty$.
Now, we prove that $u_{i_1}^n \to u_{i_1} \in A_{i_1}(x_1), u_{i_2}^n \to u_{i_2} \in A_{i_2}(x_2), ..., u_{i_n} \to u_{i_n} \in A_{i_n}(x_n)$, for each $i \in 1, 2, ..., n$. It follows from (2.2) - (2.4) and by Lipschitz continuity of $A_{i_1}, A_{i_2}, ..., A_{i_n}$

$$\|u_{i_1}^n - u_{i_1}^{n-1}\| \le \left(1 + \frac{1}{n+1}\right)\delta_{A_{i_1}}\|x_1^n - x_1^{n-1}\|, \tag{3.13}$$

$$\|u_{i_2}^n - u_{i_2}^{n-1}\| \leq \left(1 + \frac{1}{n+1}\right)\delta_{A_{i_2}}\|x_2^n - x_2^{n-1}\|, \tag{3.14}$$

$$\vdots$$

$$\|u_{i_n}^n - u_{i_n}^{n-1}\| \leq \left(1 + \frac{1}{1+n}\right)\delta_{A_{i_n}}\|x_n^n - x_n^{n-1}\|. \tag{3.15}$$

From (3.13)-(3.15), we know that $\{u_{i_1}^n\}, \{u_{i_2}^n\}, ..., \{u_{i_n}^n\}$ are Cauchy sequences. Therefore, there exist $u_{i_1} \in X_1, u_{i_2} \in X_2, ..., u_{i_n} \in X_n$ such that $u_{i_1}^n \to u_i, u_{i_2}^n \to u_{i_2}, ..., u_{i_n}^n \to u_{i_n}$, when $n \to \infty$.
Further, for each $i \in \{1, 2, 3, ..., n\}$.

$$\begin{aligned} d(u_{i_1}, A_{i_1}(x_1)) &\leq \|u_{i_1} - u_{i_1}^n\| + d(u_{i_1^n}, A_{i_1}(x_1)) \\ &\leq \|u_{i_1} - u_{i_1}^n\| + \mathcal{H}_1(A_{i_1}(x_1^n), A_{i_1}(x_1)) \\ &\leq \|u_{i_1} - u_{i_1}^n\| + (1 + \frac{1}{n+1})\delta_{A_{i_1}}\|x_1^n - x_1\| \to 0, \text{ when } n \to \infty. \end{aligned}$$

Since $A_{i_1}$ is closed, we have $u_{i_1} \in A_{i_1}(x_1)$. Similarly, $u_{i_2} \in A_{i_2}(x_2), ..., u_{i_n} \in A_{i_n}(x_n)$, respectively. By continuity of the mappings, $g_i, H_i, F_i, P_i, R_{\lambda_i, M_i}^{I_i - H_i}$ and iterative Algorithm 2.1, we know that $u_{i_1}, u_{i_2}, ..., u_{i_n}$ satisfy the following relation:

$$g_i(x_i) = R_{\lambda_i, M_i(.,x_i)}^{I_i - H_i}[(I_i - H_i)(g_i(x_i)) - \lambda_i F_i(x_1, x_2, ..., x_n) - \lambda_i P_i(u_{i_1}, u_{i_2}, ..., u_{i_n})]$$

By Lemma 2.1, $(x_1, x_2, ..., x_n, u_{11}, u_{12}, ...u_{1n}, u_{21}, u_{22}, ...u_{2n}, ..., u_{n1}, u_{n2}, ..., u_{nn})$ is a solution of problem (SIVI). This completes the proof.

## 4. Conclusion

In this paper we have considered a new system of implicit $n$-variational inclusions which is more general than many existing system of variational inclusions in the literature. Firstly, we propose a new algorithm with error terms for computing the approximate solutions of our system; and secondly, convergence of the iterative sequences generated by the iterative algorithm is discussed. Some special cases are studied. The implementation and comparison of these methods with other methods is a subject of the future research.

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