Higher-order symmetric duality in nondifferentiable multiobjective fractional programming problem over cone constraints

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Abstract In this paper, we introduce the definition of higher-order $K$-$(C, \alpha, \rho, d)$-convexity/pseudoconvexity over cone and discuss a nontrivial numerical examples for existing such type of functions. The purpose of the paper is to study higher order fractional symmetric duality over arbitrary cones for nondifferentiable Mond-Weir type programs under higher-order $K$-$(C, \alpha, \rho, d)$-convexity/pseudoconvexity assumptions. Next, we prove appropriate duality relations under aforesaid assumptions.

Keywords Higher-order symmetric duality. Multiobjective fractional programming. Efficient solution. Higher-order $K$-$(C, \alpha, \rho, d)$-convexity/pseudoconvexity

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1. Introduction

Convexity and generalized convexity have been playing an important role in developing optimality and duality results for multiobjective programming problems which are mathematical models for most of the real world problems occuring in the fields of engineering, economics, finance, game theory etc. Higher-order duality is significant due to its computational importance as it provides more higher bounds whenever approximation is used. Mangasarian [1] formulated higher-order dual for a single objective nonlinear problems, $\{ \min f(x) \text{, subject to } g(x) \leq 0 \}$. Motivated by this concept, many researchers have worked in this direction. Kassem [3] have been studied higher-order vector optimization problem and derived duality results under generalized convexity assumptions.

In last many years, various optimality and duality results have been obtained for multiobjective fractional programming problems. In Chen [2] multiobjective fractional problem and its duality relations have been considered under higher-order $(f, \alpha, \rho, d)$- convexity assumptions. Later on, Suneya et al. [4] proved higher-order Mond-Weir and Schaible type nondifferentiable dual programs and their duality relations under higher-order $(f, \rho, \sigma)$ -type $I$- assumptions. Recently, Ying [5] has studied higher-order multiobjective symmetric fractional problem and formulated its Mond-Weir type dual and duality theorems are proved under the higher-order $(f, \alpha, \rho, d)$-convexity assumptions. Several researchers worked in the same fields[11]-[15].

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In the present work, we formulate a pair of nondifferentiable multiobjective Mond-Weir type higher-order symmetric fractional programming problems over arbitrary cones. For a differentiable function \( h : X \times R^n \rightarrow R, \) \((X \subset R^n)\), we introduce the definition of higher-order \( K-(C, \alpha, \rho, d)\)-convexity/pseudoconvexity, which extends some kinds of generalized convexity. Also, we give nontrivial concrete numerical examples which is higher-order \( K-(C, \alpha, \rho, d)\)-convex/pseudoconvex function nor higher-order \( K-(F, \alpha, \rho, d)/(F, \alpha, \rho, d)\)-convex/pseudoconvex function. Finally, we establish appropriate duality theorems under higher-order \( K-(C, \alpha, \rho, d)\) convexity/pseudoconvexity assumptions followed by conclusions.

2. Preliminaries

Let \( P \) be a pointed convex cone with non empty interior in \( R^p_+ \). Then, for \( x, y \in R^p \), we define three cone orders with respect to \( P \) as follows:

\[
\begin{align*}
  x < y & \quad \text{if and only if} \quad y - x \in \text{int} P, \\
  x \leq y & \quad \text{if and only if} \quad y - x \in P \setminus \{0\}, \\
  x \leq y & \quad \text{if and only if} \quad y - x \in P.
\end{align*}
\]

**Definition 2.1**[9]. Let \( C \) be a compact convex set in \( R^n \). The support function of \( C \) is defined by

\[
s(x|C) = \max\{x^Ty : y \in C\}.
\]

A support function, being convex and everywhere finite, has a subdifferential, that is, there exists a \( z \in R^n \) such that

\[
s(y|C) \geq s(x|C) + z^T(y - x), \forall x \in C.
\]

The subdifferential of \( s(x|C) \) is given by

\[
\partial s(x|C) = \{z \in C : z^T x = s(x|C)\}.
\]

For a convex set \( D \subset R^n \), the normal cone to \( D \) at a point \( x \in D \) is defined by

\[
N_D(x) = \{y \in R^n : y^T(z - x) \leq 0, \forall z \in D\}.
\]

When \( C \) is a compact convex set, \( y \in N_C(x) \) if and only if \( s(y|C) = x^Ty \), or equivalently, \( x \in \partial s(y|C) \).

**Definition 2.2**[9]. The positive polar cone \( P^* \) of a cone \( P \) is defined by

\[
P^* = \{y \in R^p : x^Ty \geq 0, \forall x \in P\}
\]

Now, consider the following multiobjective programming problem:

\[
(P_1) \quad \text{Minimize } f(x) \\
\text{subject to } x \in X^0 = \{x \in S : -g(x) \in M\}
\]

where \( S \subset R^n \) is open, \( f : S \rightarrow R^k \), \( g : S \rightarrow R^m \), \( K \) and \( M \) are closed convex pointed cones with nonempty interiors in \( R^k \) and \( R^m \), respectively.

**Definition 2.3**. A feasible solution \( \bar{x} \in X^0 \) is said to be an efficient solution of \( (P_1) \) if there exists no \( x \in X^0 \) such that \( f(x) - f(\bar{x}) \in K \setminus \{0\} \).

**Definition 2.4**[10]. Let \( C : X \times X \times R^n \rightarrow R \) \((X \subseteq R^n)\) be a function which satisfies \( C_{x,u}(0) = 0,\)
∀(x, u) ∈ X × X. Then, the function C is said to be convex on \( R^n \) with respect to third argument \( i \) if for any fixed \( (x, u) ∈ X × X \),
\[
C_{x,u}(λx_1 + (1 - λ)x_2) \leq λC_{x,u}(x_1) + (1 - λ)C_{x,u}(x_2), \ \forall λ ∈ (0, 1), \ \forall x_1, x_2 ∈ R^n.
\]

Many generalizations of the definition of a convex function have been introduced in optimization theory in order to weak the assumption of convexity for establishing optimality and duality results for new classes of nonconvex optimization problems, including vector optimization problems. One of such a generalization of convexity in the vectorial case, we introduce the following concept of higher-order \( K - (C, α, ρ, d) \)-convex/pseudoconvex functions:

**Definition 2.5.** A differentiable function \( f : X → R^k \) is said to be higher order \( K - (C, α, ρ, d) \)-convex at \( u ∈ X \) with respect to \( h : X × R^n → R^k \) if for all \( x ∈ X \) and \( p ∈ R^k \), \( ∃ ρ ∈ R^k \), a real valued function \( α : X × X → R_+ \setminus \{0\} \) and \( d : X × X → R^k \) (satisfying \( d(x, z) = 0 ⇔ x = z \)) such that
\[
\frac{1}{α(x, u)} [f_1(x) - f_1(u) - h_1(u, p_1) + p_1^T \nabla_{p_1} h_1(u, p_1) - ρ_1 d_1^2(x, u)] - C_{x,u} [\nabla_x f_1(u) + \nabla_{p_1} h_1(u, p_1)]
\]
\[
\ldots, \frac{1}{α(x, u)} [f_k(x) - f_k(u) - h_k(u, p_k) + p_k^T \nabla_{p_k} h_k(u, p_k) - ρ_k d_k^2(x, u)] - C_{x,u} [\nabla_x f_k(u) + \nabla_{p_k} h_k(u, p_k)] \in K.
\]

The function \( f \) is said to be higher-order \( K - (C, α, ρ, d) \)-convex over \( X \) if, ∀\( u ∈ X \), it is higher \( K - (C, α, ρ, d) \)-convex.

The following example shows that \( ∃ \) functions which are higher-order \( K - (C, α, ρ, d) \)-convex function, but the functions do not others (such as higher-order \( K - (F, α, ρ, d)/(F, α, ρ, d) \)-convex functions and higher-order \( (C, α, ρ, d) \)-convex functions).

**Example 2.1.** Let \( X = [0, 5] \) and \( K = \left\{ (x, y) : |y| ≤ 20x \text{ and } x ≥ 0 \right\} \).

Consider the function \( f = (f_1, f_2) → R^2 \) given by
\[
f_1(x) = i(e^{-ix} - e^{ix}), \ f_2(x) = i(e^{ix} - e^{-ix}).
\]

Let the convex function \( C : X × X × R → R \) be defined by
\[
C_{x,u}(a) = \frac{a^2}{4}(x - u).
\]

Further, the function \( h = (h_1, h_2) : X × R^n → R^2 \) be defined as
\[
h_1(u, p_1) = \frac{u^2}{2} p_1, \ h_2(u, p_2) = -u^2 p_2.
\]

Next \( α(x, u) = 2, d_i(x, u) = |x - u|, i = 1, 2 \) and \( ρ_i = 0, i = 1, 2 \).

We will prove that the function \( f = (f_1, f_2) \) is higher-order \( K - (C, α, ρ, d) \)-convex function at \( u = 0 \). For this, we have to claim that
\[
Π = \left\{ \frac{1}{α(x, u)} [f_1(x) - f_1(u) - h_1(u, p_1) + p_1^T \nabla_{p_1} h_1(u, p_1) - ρ_1 d_1^2(x, u)] - C_{x,u} [\nabla_x f_1(u) + \nabla_{p_1} h_1(u, p_1)], \right.\]
\[
\left. \frac{1}{α(x, u)} [f_2(x) - f_2(u) - h_2(u, p_2) + p_2^T \nabla_{p_2} h_2(u, p_2) - ρ_2 d_2^2(x, u)] - C_{x,u} [\nabla_x f_2(u) + \nabla_{p_2} h_2(u, p_2)] \right\} \in K
\]
or
\[
Π = (φ_1, φ_2) ∈ K,
\]
\[
\phi_1 = \frac{1}{\alpha(x, u)} \left[ f_1(x) - f_1(u) - h_1(u, p_1) + p_1^T \nabla p_1 h_1(u, p_1) - \rho_1 d_1^2(x, u) \right] - C_{x,u}\left[ \nabla_x f_1(u) + \nabla p_1 h_1(u, p_1) \right]
\]

and
\[
\phi_2 = \frac{1}{\alpha(x, u)} \left[ f_2(x) - f_2(u) - h_2(u, p_2) + p_2^T \nabla p_2 h_2(u, p_2) - \rho_2 d_2^2(x, u) \right] - C_{x,u}\left[ \nabla_x f_2(u) + \nabla p_2 h_2(u, p_2) \right]
\]

Substituting the values \( f_1, f_2, h_1, h_2, \alpha, \rho_1, \rho_2 \) and \( d_i(x, u), i = 1, 2 \) in the above expressions, we have
\[
\phi_1 = \frac{1}{2} \left[ i(e^{-ix} - e^{ix}) - i(e^{-iu} - e^{iu}) - \frac{u^2}{2} p_1 + \frac{u^2}{2} p_1 - 0 \times (x - u)^2 \right] - C_{x,u}\left[ (e^{-iu} + e^{iu}) + \frac{u^2}{2} \right]
\]

and
\[
\phi_2 = \frac{1}{2} \left[ i(e^{ix} - e^{-ix}) - i(e^{iu} - e^{-iu}) - u^2 p_2 + u^2 p_2 - 0 \times ((x - u)^2 \right] - C_{x,u}\left[ - (e^{iu} + e^{iu}) + u^2 \right].
\]

At the point at \( u = 0 \), we have
\[
\phi_1 = \frac{1}{2} \left[ i(e^{-ix} - e^{ix}) \right] - C_{x,u}[2]
\]

and
\[
\phi_2 = \frac{1}{2} \left[ - (e^{ix} - e^{-ix}) \right] - C_{x,u}[2].
\]

Using the condition \( C_{x,u}(a) = \frac{a^2}{4}(x - u) \) in above expressions,
\[
\phi_1 = (\sin x - x)
\]

and
\[
\phi_2 = (-\sin x - x).
\]

Obviously, from the given figures (1) and (2), it follows that
\[
\Pi = (\phi_1, \phi_2) \in K, \forall x \in X
\]

or
\[
\Pi = (\sin x - x, -\sin x - x) \in K.
\]

This shows that \( f = (f_1, f_2) \) is higher-order \( K - (C, \alpha, \rho, d) \)- convex function at \( u = 0 \).

Obviously, \( \Pi = (\sin x - x, -\sin x - x) \neq 0, \forall x \in X \). This implies the \( f = (f_1, f_2) \) is not higher-order
Consider the function $f: \mathbb{R} \to \mathbb{R}$. The function $\alpha$ is defined as:

$$\alpha(x, u) = 4 + e^x - e^u.$$ 

Let the convex function $C: X \times X \to \mathbb{R}$ be defined by

$$C_{x,u}(a) = \frac{a^2}{4}(x^2 + u^2).$$

The function $h = (h_1, h_2): X \times \mathbb{R}^n \to \mathbb{R}^2$ is defined as:

$$h_1(u, p_1) = -\frac{u^2}{4} - p_1, \quad h_2(u, p_2) = u^4 p_2.$$ 

Next, $\alpha(x, u) = 2$, $d_i(x, u) = |x + u|$, $i = 1, 2$ and $\rho_i = 0$, $i = 1, 2$.

In order to prove that the function $f = (f_1, f_2)$ is higher-order $K - (\alpha, \rho, d)$-pseudo convex function at $u = 0$.

**Definition 2.6.** A differentiable function $f: X \to \mathbb{R}^k$, $(X \subseteq \mathbb{R}^n)$ is said to be higher order $K - (\alpha, \rho, d)$-pseudoconvex at $u \in X$ with respect to $h: X \times \mathbb{R}^n \to \mathbb{R}^k$ if for all $x \in X$ and $p \in \mathbb{R}^k$, $\exists \rho \in \mathbb{R}^k$, a real valued function $\alpha: X \times X \to \mathbb{R}_+ \setminus \{0\}$ and $d: X \times X \to \mathbb{R}^k$ (satisfying $d(x, z) = 0 \Leftrightarrow x = z$) such that

$$\{ C_{x,u}[\nabla_x f_1(u) + \nabla p_1 h_1(u, p_1)], C_{x,u}[\nabla_x f_2(u) + \nabla p_2 h_2(u, p_2)], \ldots, C_{x,u}[\nabla_x f_k(u) + \nabla p_k h_k(u, p_k)] \} \in K$$

$$\Rightarrow \left\{ \frac{1}{\alpha(x, u)} [f_1(x) - f_1(u) - h_1(u, p_1) + p_1^T \nabla_p h_1(u, p_1) - \rho_1 d_1^T(x, u)], \frac{1}{\alpha(x, u)} [f_2(x) - f_2(u) - h_2(u, p_2) + p_2^T \nabla_p h_2(u, p_2) - \rho_2 d_2^T(x, u)], \ldots, \frac{1}{\alpha(x, u)} [f_k(x) - f_k(u) - h_k(u, p_k) + p_k^T \nabla_p h_k(u, p_k) - \rho_k d_k^T(x, u)] \right\} \in K.$$ 

**Example 2.2.** Let $X = [0, 5]$ and $K = \{(x, y) : |y| \leq 20x$ and $x \geq 0\}$.

Consider the function $f = (f_1, f_2) \to \mathbb{R}^2$ given by

$$f_1(x) = (4 + e^x - e^u), \quad f_2(x) = \left(\frac{e^{-x} - e^u}{2}\right).$$

Let the convex function $C: X \times X \to \mathbb{R}$ be defined by

$$C_{x,u}(a) = \frac{a^2}{4}(x^2 + u^2).$$

The function $h = (h_1, h_2): X \times \mathbb{R}^n \to \mathbb{R}^2$ is defined as:

$$h_1(u, p_1) = -\frac{u^2}{4} - p_1, \quad h_2(u, p_2) = u^4 p_2.$$ 

Next, $\alpha(x, u) = 2$, $d_i(x, u) = |x + u|$, $i = 1, 2$ and $\rho_i = 0$, $i = 1, 2$.

In order to prove that the function $f = (f_1, f_2)$ is higher-order $K - (\alpha, \rho, d)$-pseudo convex function at...
\[ u = 0. \text{ For this, we have to show that} \]
\[
\Upsilon = \left\{ C_{x,u} \left[ \nabla_x f_1(u) + \nabla_p h_1(u,p_1) \right], C_{x,u} \left[ \nabla_x f_2(u) + \nabla_p h_2(u,p_2) \right] \right\} \in K
\]
\[
\Rightarrow \Gamma = \left\{ \frac{1}{\alpha(x,u)} \left[ f_1(x) - f_1(u) - h_1(u,p_1) + p_1^T \nabla_p h_1(u,p_1) - \rho_1 d_1^2(x,u) \right], \frac{1}{\alpha(x,u)} \left[ f_2(x) - f_2(u) - h_2(u,p_2) + p_2^T \nabla_p h_2(u,p_2) - \rho_2 d_2^2(x,u) \right] \right\} \in K
\]
or
\[
\Upsilon = (\phi_3, \phi_4) \in K \Rightarrow \Gamma = (\phi_5, \phi_6) \in K,
\]
where
\[
\phi_3 = C_{x,u} \left[ \nabla_x f_1(u) + \nabla_p h_1(u,p_1) \right], \quad \phi_4 = C_{x,u} \left[ \nabla_x f_2(u) + \nabla_p h_2(u,p_2) \right],
\]
\[
\phi_5 = \frac{1}{\alpha(x,u)} \left[ f_1(x) - f_1(u) - h_1(u,p_1) + p_1^T \nabla_p h_1(u,p_1) - \rho_1 d_1^2(x,u) \right]
\]
and
\[
\phi_6 = \frac{1}{\alpha(x,u)} \left[ f_2(x) - f_2(u) - h_2(u,p_2) + p_2^T \nabla_p h_2(u,p_2) - \rho_2 d_2^2(x,u) \right].
\]
Substituting the values \( f_1, f_2, h_1, h_2, \alpha, \rho_1, \rho_2 \) and \( d_i(x,u), i = 1, 2 \) in the above expressions, we have
\[
\phi_3 = C_{x,u} \left[ (e^u + e^u) - \frac{u^2}{2} \right], \quad \phi_4 = C_{x,u} \left[ \frac{e^u + e^u}{2} + u^4 \right],
\]
\[
\phi_5 = \frac{1}{2} \left[ (4 + e^x - e^x) - (4 + e^u - e^u) + \frac{u^2}{4} p_1 - \frac{u^2}{4} p_1 - 0 \times (x + u)^2 \right]
\]
and
\[
\phi_6 = \frac{1}{2} \left[ \frac{e^{-x} - e^x}{2} - \left( \frac{e^{-u} - e^u}{2} \right) - u^4 p_2 + u^4 p_2 - 0 \times (x + u)^2 \right]
\]
which at \( u = 0 \), yields
\[
\phi_3 = C_{x,u}(2), \quad \phi_4 = C_{x,u}(1), \quad \phi_5 = \left( \frac{e^x - e^x}{2} \right)
\]
and
\[
\phi_6 = \left( \frac{e^{-x} - e^x}{4} \right).
\]
Using the condition $C_{x,u}(a) = \frac{a^2}{4}(x^2 + u^2)$ in above expressions,

$$\phi_3 = x^2, \quad \phi_4 = \left(\frac{x^2}{4}\right), \quad \phi_5 = \sinh x$$

and

$$\phi_6 = \left(-\frac{\sinh x}{2}\right).$$

Further,

$$\Upsilon = \left(x^2, \frac{x^2}{4}\right) \in K \quad \text{(from figures (3) and (4))}$$

and

$$\Gamma = \left(\sinh x, -\frac{\sinh x}{2}\right) \in K \quad \text{(from figures (5) and (6))}.$$

This gives that

$$\Upsilon = \left(x^2, \frac{x^2}{4}\right) \in K \Rightarrow \Gamma = \left(\sinh x, -\frac{\sinh x}{2}\right) \in K.$$

Therefore, $f = (f_1, f_2)$ is higher-order $K - (C, \alpha, \rho, d)$-pseudoconvex function at $u = 0$.

Next, $\Upsilon = \left(\sinh x, -\frac{\sinh x}{2}\right) \not\in 0$, $\forall \ x \in X$ from the figures. This shows that the function $f = (f_1, f_2)$ is not higher-order $(C, \alpha, \rho, d)$-pseudoconvex function at the point $u = 0$. Furthermore, the function $C_{x,u}(.)$ is not sublinear with respect to third variables. Therefore, the function is neither higher-order $K - (F, \alpha, \rho, d)$-pseudoconvex function nor higher-order $(F,\alpha, \rho, d)$-pseudoconvex function at $u = 0$.

**Remark 2.1.**

(i) If $K = R^+$, then the Definition 2.5 in reduces in higher-order $(C, \alpha, \rho, d)$-convexity given by [6].

(ii) If $C_{x,u}(a) = \eta(x,u) Ta, \ h_i(u,p_i) = \frac{1}{2} p_i^T \nabla f_i(u)p_i, \ k = 1, 2, \ldots, k, \ \rho = 0$ and $\alpha(x,u) = 1$ then Definition 2.5 becomes $K - \eta$ bonvexity given by [8].

### 3. Higher-order Mond-Weir fractional symmetric duality

Consider the following multiobjective fractional symmetric dual programs over arbitrary cones::

(MFPP) $\text{K-minimize } R(x,y,p) = (R_1(x,y,p_1), R_2(x,y,p_2), \ldots, R_k(x,y,p_k))^T$

subject to

$$-\sum_{i=1}^{k} \lambda_i \left[ (\nabla_y f_i(x,y) - z_i + \nabla_{p_i} H_i(x,y,p_i)) - R_i(x,y,p_i)(\nabla_y g_i(x,y) + r_i + \nabla_{p_i} G_i(x,y,p_i)) \right] \in C_2^*,$$

$$y^T \left[ \sum_{i=1}^{k} \lambda_i \left[ (\nabla_y f_i(x,y) - z_i + \nabla_{p_i} H_i(x,y,p_i)) - R_i(x,y,p_i)(\nabla_y g_i(x,y) + r_i + \nabla_{p_i} G_i(x,y,p_i)) \right] \right] \geq 0,$$

$$\lambda \in \text{int} K^*, \ x \in C_1, \ z_i \in D_i, \ r_i \in F_i, \ i = 1, 2, \ldots, k$$
\( K \)-maximize \( S(u, v, q) = (S_1(u, v, q_1), S_2(u, v, q_2), \ldots, S_k(u, v, q_k))^T \)

subject to

\[
\sum_{i=1}^{k} \lambda_i \left[ (\nabla_x f_i(u, v) + w_i + \nabla_q \Phi_i(u, v, q_i)) - S_i(u, v, q_i)(\nabla_x g_i(u, v) - t_i + \nabla_q \Psi_i(u, v, q_i)) \right] \in C_1^*, \]

\[
u^T \left[ \sum_{i=1}^{k} \lambda_i \left[ (\nabla_x f_i(u, v) + w_i + \nabla_q \Phi_i(u, v, q_i)) - S_i(u, v, q_i)(\nabla_x g_i(u, v) - t_i + \nabla_q \Psi_i(u, v, q_i)) \right] \right] \leq 0,
\]

where

\[
R_i(x, y, p_i) = f_i(x, y) + s(x|Q_i) - y^T z_i + H_i(x, y, p_i) - p_i^T \nabla_p H_i(x, y, p_i),
\]

\[
S_i(u, v, q_i) = f_i(u, v) - s(v|D_i) + u^T w_i + \Phi_i(u, v, q_i) - q_i^T \nabla_q \Phi_i(u, v, q_i),
\]

where \( f_i : S_1 \times S_2 \to R; g_i : S_1 \times S_2 \to R; H_i, G_i : S_1 \times S_2 \times R^n \to R \) and \( \Phi_i, \Psi_i : S_1 \times S_2 \times R^n \to R \) are differentiable functions for all \( i = 1, 2, \ldots, k \). \( S_1, S_2 \subseteq R^n \) and \( S_2 \subseteq R^n \) are such that \( C_1 \times C_2 \subseteq S_1 \times S_2 \). \( Q_i, E_i \) are compact convex sets in \( R^n \) and \( D_i, F_i \) are compact convex sets in \( R^n \), \( p_i \in R^n \), \( q_i \in R^n \), \( i = 1, 2, \ldots, k \), \( p = (p_1, p_2, \ldots, p_k) \), \( q = (q_1, q_2, \ldots, q_k) \). \( C_1^* \) and \( C_2^* \) are positive polar cones of \( C_1 \) and \( C_2 \), respectively. It is assumed that in the feasible regions, the numerators are nonnegative and denominators are positive and \( K \) is a closed convex cone with \( R^n_+ \subseteq K \).

Let \( T = (T_1, T_2, \ldots, T_k)^T \) and \( W = (W_1, W_2, \ldots, W_k)^T \). Then, we can express the programs (MFPP) and (MFDP) equivalently as:

**MFPP** \( K \)-minimize \( T \) subject to

\[
(f_i(x, y) + s(x|Q_i) - y^T z_i + H_i(x, y, p_i)) - p_i^T \nabla_p H_i(x, y, p_i) - T_i(g_i(x, y) - s(x|E_i) + y^T r_i + G_i(x, y, p_i) - p_i^T \nabla_p G_i(x, y, p_i)) = 0, \quad i = 1, 2, \ldots, k,
\]

\[
- \sum_{i=1}^{k} \lambda_i \left[ (\nabla_y f_i(x, y) - z_i + \nabla_p H_i(x, y, p_i) - T_i(\nabla_y g_i(x, y) + r_i + \nabla_p G_i(x, y, p_i))) \right] \in C_2^*, \]

\[
y^T \left[ \sum_{i=1}^{k} \lambda_i \left[ (\nabla_y f_i(x, y) - z_i + \nabla_p H_i(x, y, p_i) - T_i(\nabla_y g_i(x, y) + r_i + \nabla_p G_i(x, y, p_i))) \right] \right] \geq 0,
\]

**MFDP** \( K \)-maximize \( W \) subject to

\[
(f_i(u, v) - s(v|D_i) + u^T w_i + \Phi_i(u, v, q_i) - q_i^T \nabla_q \Phi_i(u, v, q_i)) - W_i(g_i(u, v) + s(v|F_i) - u^T t_i + \Psi_i(u, v, q_i) - q_i^T \nabla_q \Psi_i(u, v, q_i)) = 0, \quad i = 1, 2, \ldots, k,
\]

\[
\sum_{i=1}^{k} \lambda_i \left[ \nabla_x f_i(u,v) + w_i + \nabla_u \Phi_i(u,v,q_i) - W_i(\nabla_x g_i(u,v) - t_i + \nabla_u \Psi_i(u,v,q_i)) \right] \in C^*_1, \tag{5}
\]

\[
u^T \left[ \sum_{i=1}^{k} \lambda_i \left[ (\nabla_x f_i(u,v) + w_i + \nabla_u \Phi_i(u,v,q_i) - W_i(\nabla_x g_i(u,v) - t_i + \nabla_u \Psi_i(u,v,q_i))) \right] \leq 0, \tag{6}
\]

\[
\lambda \in \text{int} K^*, \quad v \in C_2, \quad w_i \in Q_i, \quad t_i \in E_i, \quad i = 1, 2, ..., k.
\]

Next, we prove weak, strong and converse duality theorems for (MFPP)_T and (MFDP)_W, which one equally apply to (MFPP) and (MFDP).

**Theorem 3.1 (Weak duality).** Let \((x,y,T,z,r,\lambda,p)\) be feasible for (MFPP)_T and let \((u,v,W,w,t,\lambda,q)\) be feasible for (MFDP)_W. Let \(f(.,v) + (. )^T w\) be higher order \(K - (C,\alpha,\rho,d)\)- convex at \(u\) with respect to \(\Phi(u,v,q)\), \(-W(g(.,v) - (.)^T t)\) be higher-order \(K - (C,\alpha,\rho,d)\)- convex at \(u\) with respect to \(-W(\Psi(u,v,q), -(f(\cdot,v) - (.)^T z))\) be higher-order \(K - (C,\alpha,\rho,d)\)-convex at \(y\) with respect to \(-H(x,y,p)\) and \(T(g(.,z) + (.)^T r)\) be higher-order \(K - (C,\alpha,\rho,d)\)- convex at \(y\) with respect to \(TG(x,y,p)\) where \(C : R^n \times R^n \times R^n \rightarrow R\) and \(\tilde{C} : R^m \times R^m \times R^m \rightarrow R\). If the following conditions hold:

\[
either \sum_{i=1}^{k} \lambda_i \rho_i d^2_i (x,u) + \tilde{\rho}_i d^2_i (v,y) \geq 0 \text{ or } \rho_i \geq 0 \text{ and } \tilde{\rho}_i \geq 0, \quad i = 1, 2, ..., k, \tag{7}
\]

\[
C_{x,u}(a) + a^T u \geq 0, \quad \forall a \in C^*_1, \quad C_{v,y}(b) + b^T y \geq 0, \quad \forall b \in C^*_2. \tag{8}
\]

Then, \(T - W \notin -K \setminus \{0\}\).

**Proof**

Since \(f(.,v) + (. )^T w\) and \(-W(g(.,v) - (.)^T t)\) is higher-order \(K - (C,\alpha,\rho,d)\)- convex in the first variable at \(u\) for fixed \(v\), we have

\[
\frac{1}{\alpha(x,u)} \left[ f_1(x,v) + x^T w_1 - f_1(u,v) - u^T w_1 - \Phi_1(u,v,q_1) + q_1^T \nabla q_1 \Phi_1(u,v,q_1) - \rho_1 d^2_1 (x,u) \right]
\]

\[
- C_{x,u} \left( \nabla_x f_1(u,v) + w_1 + \nabla_u \Phi_1(u,v,q_1), ..., \frac{1}{\alpha(x,u)} \left[ f_k(x,v) + x^T w_k - f_k(u,v) - u^T w_k - \Phi_k(u,v,q_k) + q_k^T \nabla q_k \Phi_k(u,v,q_k) - \rho_k d^2_k (x,u) \right] - C_{x,u} \left( \nabla_x f_k(u,v) + w_k + \nabla_u \Phi_k(u,v,q_k) \right) \right) \in K. \tag{9}
\]

and

\[
\frac{1}{\alpha(x,u)} \left[ W_1(-g_1(x,v) + x^T t_1 + g_1(u,v) - u^T t_1) + W_1(\Psi_1(u,v,q_1) - q_1^T \nabla q_1 \Psi_1(u,v,q_1)) - \rho_1 d^2_1 (x,u) \right]
\]

\[
- C_{x,u} \left( W_1(-\nabla g_1(u,v) + t_1) - W_1 \nabla q_1 \Psi_1(u,v,q_1), ..., \frac{1}{\alpha(x,u)} \left[ W_k(-g_k(x,v) + x^T t_k + g_k(u,v) - u^T t_k) + W_k(\Psi_k(u,v,q_k) - q_k^T \nabla q_k \Psi_k(u,v,q_k)) - \rho_k d^2_k (x,u) \right] - C_{x,u} \left( W_k(-\nabla g_k(u,v) + t_k) - W_k \nabla q_k \Psi_k(u,v,q_k) \right) \right) \in K. \tag{10}
\]
Since $\lambda \in \text{int} K^*$, therefore (9) and (10) yield
\[
\sum_{i=1}^{k} \frac{\lambda_i}{\alpha(x,u)} \left( f_i(x,v) + x^T w_i - f_i(u,v) - u^T w_i - \Phi_i(u,v,q_i) + q_i^T \nabla q_i \Phi_i(u,v,q_i) \right)
\geq -\sum_{i=1}^{k} \frac{\lambda_i}{\alpha(x,u)} \rho_i d_i^2(x,u) \geq \sum_{i=1}^{k} \lambda_i C_{x,u}(\nabla_x f_i(u,v) + w_i + \nabla q_i \Phi_i(u,v,q_i)).
\]

and
\[
\sum_{i=1}^{k} \frac{\lambda_i W_i}{\alpha(x,u)} \left[ -g_i(x,v) + x^T t_i + g_i(u,v) - u^T t_i + \Psi_i(u,v,q_i) - q_i^T \nabla q_i \Psi_i(u,v,q_i) \right]
\geq \sum_{i=1}^{k} \lambda_i C_{x,u}(W_i(-\nabla_x g_i(u,v) + t_i - \nabla q_i \Psi_i(u,v,q_i))).
\]

Now, adding the above two inequalities and then multiplying with $\frac{1}{\tau}$, where $\tau = \sum_{i=1}^{k} \lambda_i > 0$ as $\lambda \in \text{int} K^* \subseteq \text{int} R^k_+$ and using convexity of $C$, we obtain
\[
\sum_{i=1}^{k} \frac{\lambda_i}{\alpha(x,u)} \tau \left( f_i(x,v) + x^T w_i - f_i(u,v) - u^T w_i - \Phi_i(u,v,q_i) + q_i^T \nabla q_i \Phi_i(u,v,q_i) \right)
\geq C_{x,u} \left[ \sum_{i=1}^{k} \frac{\lambda_i}{\tau} \left( (\nabla_x f_i(u,v) + w_i + \nabla q_i \Phi_i(u,v,q_i)) - W_i(\nabla_x g_i(u,v) - t_i + \nabla q_i \Psi_i(u,v,q_i)) \right) \right]. \tag{11}
\]

Now, from (8) as $\tau > 0$, we have
\[
a = \sum_{i=1}^{k} \frac{\lambda_i}{\tau} \left[ (\nabla_x f_i(u,v) + w_i + \nabla q_i \Phi_i(u,v,q_i)) - W_i(\nabla_x g_i(u,v) - t_i + \nabla q_i \Psi_i(u,v,q_i)) \right] \in C^*_1.
\]

Hence, for this $a$, $C_{x,u}(a) \geq -u^T a \geq 0$ (from (6)). Using this, in (11), we obtain
\[
\sum_{i=1}^{k} \frac{\lambda_i}{\alpha(x,u)} \tau \left( f_i(x,v) + x^T w_i - f_i(u,v) - u^T w_i - \Phi_i(u,v,q_i) + q_i^T \nabla q_i \Phi_i(u,v,q_i) \right)
\geq C_{x,u} \left[ \sum_{i=1}^{k} \frac{\lambda_i}{\tau} \left( (\nabla_x f_i(u,v) + w_i + \nabla q_i \Phi_i(u,v,q_i)) - W_i(\nabla_x g_i(u,v) - t_i + \nabla q_i \Psi_i(u,v,q_i)) \right) \right].
\]

Since $v^T r_i \leq s(v|F_i)$ and using (4) in above inequality, we get
\[
\sum_{i=1}^{k} \lambda_i \left[ f_i(x,v) + x^T w_i - s(v|D_i) + W_i(x^T t_i - v^T r_i - g_i(x,v)) \right] \geq 2 \sum_{i=1}^{k} \lambda_i \rho_i d_i^2(x,u). \tag{12}
\]
Similarly, by the higher-order $K - (\mathcal{C}, \alpha, \bar{\rho}, d)$-convexity of $-f(x, \cdot) + (\cdot)^T z$ and $T(g(x, \cdot) + (\cdot)^T r)$ in the second variable at $y$, for fixed $x$ and from the condition (8), for 
\[
b = -\sum_{i=1}^{k} \frac{\lambda_i}{T}[y, f_i(x, y) - z_i + \nabla p_i H_i(x, y, p_i) - T_i (\nabla y g_i(x, y) + r_i + \nabla p_i G_i(x, y, p_i))] \in C_2^* ,
\]
we get 
\[
\sum_{i=1}^{k} \lambda_i [-f_i(x, v) + v^T z_i - s(x|Q_i) + T_i(v^T r_i - x^T t_i + g_i(x, v))] \geq 2 \sum_{i=1}^{k} \lambda_i \bar{\rho}_i \bar{d}_i^2 (v, y) .
\] (13)
Adding the inequalities (12)-(13) and applying (7), we get 
\[
\sum_{i=1}^{k} \lambda_i (v^T z_i - s(v|D_i) + x^T w_i - s(x|Q_i)) + \sum_{i=1}^{k} \lambda_i (T_i - W_i)(g_i(x, v) + v^T r_i - x^T t_i) \geq 0 .
\] (14)
Since $\lambda > 0$ and $v^T z_i \leq s(v|D_i) - (13)$, $x^T w_i \leq s(x|Q_i) - (13)$, the above inequality gives 
\[
\sum_{i=1}^{k} \lambda_i (T_i - W_i)(g_i(x, v) + v^T r_i - x^T t_i) \geq 0 .
\]
Using $(g_i(x, v) + v^T r_i - x^T t_i) > 0$, $i = 1, 2, ..., k$, above inequality gives 
\[
\sum_{i=1}^{k} \lambda_i (T_i - W_i) \geq 0 .
\] (15)
Now, suppose on contrary 
\[
T - W \in -K \backslash \{0\} .
\]
Since $\lambda > 0$, we have 
\[
\sum_{i=1}^{k} \lambda_i (T_i - W_i) < 0 .
\]
which contradicts (15). This completes the proof.}

**Theorem 3.2 (Weak duality).** Let $(x, y, T, z, r, \lambda, p)$ and $(u, v, W, w, t, \lambda, q)$ be feasible solutions of (MFPP)$_T$ and (MFPD)$_W$, respectively. Suppose that 
\[
(i) \ (f(\cdot, v) + (\cdot)^T w) - W(g(\cdot, v) - (\cdot)^T t) \text{ is higher-order } K - (\mathcal{C}, \alpha, \rho, d)\text{-convex at } u \text{ with respect to } \Phi(u, v, q) = -W \Psi(u, v, q) ,
\]
\[
(ii) \ (-f(x, \cdot) + (\cdot)^T z) + T(g(x, \cdot) + (\cdot)^T r) \text{ is higher-order } K - (\mathcal{C}, \bar{\alpha}, \bar{\rho}, \bar{d})\text{-convex at } y \text{ with respect to } -H(x, y, p) + TG(x, y, p) ,
\]
\[
(iii) \text{ either } \sum_{i=1}^{k} \lambda_i [\rho_i d_i^2 (x, u) + \bar{\rho}_i \bar{d}_i^2 (v, y)] \geq 0 \text{ or } \rho_i \geq 0 \text{ and } \bar{\rho}_i \geq 0, i = 1, 2, ..., k ,
\]
\[
(iv) \ C_x(u) + a^T u \geq 0, \forall a \in C_1^* \mathcal{C}_v, b^T y \geq 0, \forall b \in C_2^* .
\]
Then, $T - W \notin -K \backslash \{0\}$.

**Proof:** The proof follows on the lines of Theorem 3.1.
Remark 3.1 Since every convex function is pseudoconvex, therefore the above weak duality theorem for the symmetric dual pair (MFPP)\textsubscript{T} and (MFP)\textsubscript{W} can also be obtained under higher-order K - (C, \alpha, \rho, d)-pseudoconvexity assumptions.

Theorem 3.3 (Weak duality). Let \((x, y, T, z, r, \lambda, p)\) be feasible for (MFPP)\textsubscript{T} and let \((u, v, W, w, t, \lambda, q)\) be feasible for (MFP)\textsubscript{W}. Let \(f(., v) + (.)^T w\) be higher order \(K - \alpha, \rho, d\)-pseudoconvex at \(u\) with respect to \(\Phi(., v, q), -W(., v, q) - (f(., v) - (.)^T z)\) be higher-order \(K - \alpha, \rho, d\)-pseudoconvex at \(u\) with respect to \(-W(., v, q), -(f(., v) - (.)^T z)\) be higher-order \(K - \alpha, \rho, d\)-pseudoconvex at \(y\) with respect to \(-H(x, y, p)\) and \(T(g(., x, p) + (.)^T r)\) be higher-order \(K - \alpha, \rho, d\)-pseudoconvex at \(y\) with respect to \(TG(x, y, p)\) where \(C : R^a \times R^m \times R^n \to R\) and \(\overline{C} : R^m \times R^m \times R^m \to R\). If the following conditions hold:

\[
\text{either } \sum_{i=1}^{k} \lambda_i [\rho_i d^2_i(x, u) + \tilde{\rho}_i d^2_i(v, y)] \geq 0 \text{ or } \rho_i \geq 0 \text{ and } \tilde{\rho}_i \geq 0, \quad i = 1, 2, ..., k, \quad (16)
\]

\[
C_x, u(a) + a^T u \geq 0, \quad \forall a \in C_1^* + C_2^*.
\]

Then, \(T - W \notin -K \setminus \{0\}\).

Proof: The proof follows on the lines of Theorem 3.1.

Theorem 3.4 (Weak duality). Let \((x, y, T, z, r, \lambda, p)\) and \((u, v, W, w, t, \lambda, q)\) be feasible solutions of (MFPP)\textsubscript{T} and (MFP)\textsubscript{W}, respectively. Suppose that

(i) \(f(., v) + (.)^T w - W(., v, q) - (f(., v) - (.)^T t)\) is higher-order \(K - \alpha, \rho, d\)-pseudoconvex at \(u\) with respect to \(\Phi(., v, q) - W(., v, q)\),

(ii) \(T(x, v, q) + (.)^T z + T(g(., x, p) + (.)^T r)\) is higher-order \(K - \alpha, \rho, d\)-pseudoconvex at \(y\) with respect to \(-H(x, y, p) + TG(x, y, p)\),

(iii) either \(\sum_{i=1}^{k} \lambda_i [\rho_i d^2_i(x, u) + \tilde{\rho}_i d^2_i(v, y)] \geq 0 \text{ or } \rho_i \geq 0 \text{ and } \tilde{\rho}_i \geq 0, \quad i = 1, 2, ..., k, \)

(iv) \(C_x, u(a) + a^T u \geq 0, \quad \forall a \in C_1^* + C_2^*\).

Then, \(T - W \notin -K \setminus \{0\}\).

Proof: The proof follows on the lines of Theorem 3.1.

Theorem 3.5 (Strong duality). Let \((\bar{x}, \bar{y}, \bar{T}, \bar{z}, \bar{r}, \bar{\lambda}, \bar{p})\) be an efficient solution of (MFPP)\textsubscript{T}, and fix \(\lambda = \bar{\lambda}\) in (MFP)\textsubscript{W}. If the following conditions hold

(i) \(\nabla_{\bar{x}} H_i(\bar{x}, \bar{y}, 0) = \nabla_{\bar{x}} G_i(\bar{x}, \bar{y}, 0) = 0, \nabla_{\bar{q}_i} \Phi_i(\bar{x}, \bar{y}, 0) = \nabla_{\bar{q}_i} \Psi_i(\bar{x}, \bar{y}, 0) = 0, \quad H_i(\bar{x}, \bar{y}, 0) = G_i(\bar{x}, \bar{y}, 0) = 0, \quad i = 1, 2, ..., k,\)

\[
\Phi_i(\bar{x}, \bar{y}, 0) = \Psi_i(\bar{x}, \bar{y}, 0) = 0, \quad \nabla_{\bar{y}} H_i(\bar{x}, \bar{y}, 0) = \nabla_{\bar{y}} G_i(\bar{x}, \bar{y}, 0) = 0, \quad \nabla_{\bar{p}_i} H_i(\bar{x}, \bar{y}, 0) = \nabla_{\bar{p}_i} G_i(\bar{x}, \bar{y}, 0) = 0, \quad i = 1, 2, ..., k.
\]

(ii) for all \(i \in \{1, 2, ..., k\}\), the Hessian matrix \(\nabla_{\bar{p}_i} H_i(\bar{x}, \bar{y}, \bar{p}_i) - \bar{T}_i \nabla_{\bar{p}_i} G_i(\bar{x}, \bar{y}, \bar{p}_i)\) is positive or negative definite,

(iii) the set of vectors \(\{\nabla_{\bar{y}} f_i(\bar{x}, \bar{y}) - \bar{z}_i + \nabla_{\bar{p}_i} H_i(\bar{x}, \bar{y}, \bar{p}_i) - \bar{T}_i \nabla_{\bar{y}} g_i(\bar{x}, \bar{y}) + \bar{r}_i + \nabla_{\bar{p}_i} G_i(\bar{x}, \bar{y}, \bar{p}_i)\}_i\)

is linearly independent,

(iv) for \(\bar{p}_i \in R^n\), \(\bar{p}_i \neq 0\) \((i = 1, 2, ..., k)\) implies that

\[
\sum_{i=1}^{k} \bar{p}_i^T [\nabla_{\bar{y}} f_i(\bar{x}, \bar{y}) - \bar{z}_i + \nabla_{\bar{p}_i} H_i(\bar{x}, \bar{y}, \bar{p}_i) - \bar{T}_i \nabla_{\bar{y}} g_i(\bar{x}, \bar{y}) + \bar{r}_i + \nabla_{\bar{p}_i} G_i(\bar{x}, \bar{y}, \bar{p}_i)] \neq 0,
\]
Furthermore, if the hypotheses in Theorems (3.1) – (3.4) are satisfied, then \((\bar{x}, \bar{y}, \bar{T}, \bar{w}, \bar{t}, \bar{\lambda}, \bar{q} = 0)\) is an efficient solution of \((MFDP)_{v}\).

Proof
Since \((\bar{x}, \bar{y}, \bar{T}, \bar{z}, \bar{r}, \bar{\lambda}, \bar{p} )\) is an efficient solution of \((MFPP)_T\). Hence, by the Fritz John necessary optimality conditions [7], there exists \(\alpha \in R^k, \beta \in R^k, \gamma \in C_2, \delta \in R, \xi \in R^k, \eta \in R, \bar{w}_i \in R^n\) and \(\bar{t}_i \in R^n, i = 1, 2, \ldots, k\) such that

\[
(x - \bar{x})^T \left[ \sum_{i=1}^{k} \beta_i (\nabla_x f_i(\bar{x}, \bar{y}) + \bar{w}_i + \nabla_x H_i(\bar{x}, \bar{y}, \bar{p}_i) - \bar{T}_i (\nabla_x g_i(\bar{x}, \bar{y}) - \bar{t}_i + \nabla_x G_i(\bar{x}, \bar{y}, \bar{p}_i))) + \sum_{i=1}^{k} \lambda_i (\nabla_{yy} f_i(\bar{x}, \bar{y}) \bar{T}_i \nabla_{yy} g_i(\bar{x}, \bar{y})) \right] \geq 0, \forall x \in C_1, \tag{18}
\]

\[
\sum_{i=1}^{k} \beta_i (\nabla_y f_i(\bar{x}, \bar{y}) - \bar{z}_i + \nabla_y H_i(\bar{x}, \bar{y}, \bar{p}_i) - \bar{T}_i (\nabla_y g_i(\bar{x}, \bar{y}) + \bar{r}_i + \nabla_y G_i(\bar{x}, \bar{y}, \bar{p}_i)) + \sum_{i=1}^{k} \bar{\lambda}_i (\nabla_{yy} f_i(\bar{x}, \bar{y})) - \bar{T}_i \nabla_{yy} g_i(\bar{x}, \bar{y}))^T \gamma - \delta \bar{y} + \sum_{i=1}^{k} (\nabla_{yy} f_i(\bar{x}, \bar{y}) \bar{T}_i \nabla_{yy} g_i(\bar{x}, \bar{y}))^T (-\beta_i \bar{p}_i + (\gamma - \delta \bar{y}) \bar{\lambda}_i)
\]

\[
- \delta \sum_{i=1}^{k} \lambda_i (\nabla_y f_i(\bar{x}, \bar{y}) - \bar{z}_i + \nabla_{yy} H_i(\bar{x}, \bar{y}, \bar{p}_i) - \bar{T}_i (\nabla_y g_i(\bar{x}, \bar{y}) + \bar{r}_i + \nabla_{yy} G_i(\bar{x}, \bar{y}, \bar{p}_i))) = 0, \tag{19}
\]

\[
\alpha_i - \beta_i (g_i(\bar{x}, \bar{y}) - s(\bar{x}|E_1) + \bar{y}^T \bar{r}_i + G_i(\bar{x}, \bar{y}, \bar{p}_i) - \bar{p}_i \nabla_{p_i} G_i(\bar{x}, \bar{y}, \bar{p}_i))
\]

\[
- (\gamma - \delta \bar{y}) \bar{T}_i (\nabla_{yy} \bar{g}_i(\bar{x}, \bar{y}) + \bar{r}_i + \nabla_{yy} G_i(\bar{x}, \bar{y}, \bar{p}_i)) = 0, i = 1, 2, \ldots, k, \tag{20}
\]

\[
(\gamma - \delta \bar{y})^T (\nabla_y f_i(\bar{x}, \bar{y}) - \bar{z}_i + \nabla_p H_i(\bar{x}, \bar{y}, \bar{p}_i)
\]

\[
- \bar{T}_i (\nabla_y g_i(\bar{x}, \bar{y}) + \bar{r}_i + \nabla_p G_i(\bar{x}, \bar{y}, \bar{p}_i) - \xi_i + \eta_i \bar{\lambda}_i) \geq 0, \forall \lambda \in \text{int} K^*, i = 1, 2, \ldots, k, \tag{21}
\]

\[
(\bar{\lambda}_i (\gamma - \delta \bar{y}) - \beta_i \bar{p}_i)^T (\nabla_{p_{i'}} H_i(\bar{x}, \bar{y}, \bar{p}_i) - \bar{T}_i \nabla_{p_{i'}} G_i(\bar{x}, \bar{y}, \bar{p}_i)) = 0, i = 1, 2, \ldots, k, \tag{22}
\]

\[
\beta_i \bar{y} + (\gamma - \delta \bar{y}) \bar{\lambda}_i \in N_{D_i}(\bar{z}_i), i = 1, 2, \ldots, k, \tag{23}
\]

\[
\beta_i \bar{T}_i \bar{y} + \bar{\lambda}_i \bar{T}_i (\gamma - \delta \bar{y}) \in N_{F_i}(\bar{r}_i), i = 1, 2, \ldots, k, \tag{24}
\]

\[
\gamma^T \sum_{i=1}^{k} \bar{\lambda}_i ((\nabla_y f_i(\bar{x}, \bar{y}) - \bar{z}_i + \nabla_p H_i(\bar{x}, \bar{y}, \bar{p}_i)) - \bar{T}_i (\nabla_y g_i(\bar{x}, \bar{y}) + \bar{r}_i + \nabla_p G_i(\bar{x}, \bar{y}, \bar{p}_i))) = 0, \tag{25}
\]
Using \( \sum_{i=1}^{k} \lambda_i(\nabla_y f_i(\bar{x}, \bar{y}) - z_i + \nabla_p H_i(\bar{x}, \bar{y}, \bar{p}_i)) - \bar{T}_i(\nabla_y g_i(\bar{x}, \bar{y}) + \bar{r}_i + \nabla_p G_i(\bar{x}, \bar{y}, \bar{p}_i)) = 0 \), we have

\[
\lambda^T \xi = 0, 
\]

\[
\eta(\lambda^T e - 1) = 0, 
\]

\[
\bar{w}_i \in Q_i, \bar{r}_i \in E_i, \bar{x}^T t_i = s(\bar{x}|E_i), \bar{x}^T \bar{w}_i = s(\bar{x}|Q_i), i = 1, 2, \ldots, k, 
\]

\[
(\alpha, \delta, \xi) \geq 0, \quad (\alpha, \beta, \gamma, \delta, \xi, \eta) \neq 0. 
\]

Since \( \lambda > 0 \), and \( \xi \geq 0 \), (27) implies that \( \xi = 0 \).

Equation (19) can be re-written as

\[
\sum_{i=1}^{k} (\beta_i - \delta \bar{\lambda}_i)((\nabla_y f_i(\bar{x}, \bar{y}) - z_i + \nabla_p H_i(\bar{x}, \bar{y}, \bar{p}_i)) - \bar{T}_i(\nabla_y g_i(\bar{x}, \bar{y}) + \bar{r}_i + \nabla_p G_i(\bar{x}, \bar{y}, \bar{p}_i))) 
\]

\[
+ \sum_{i=1}^{k} \beta_i((\nabla_y H_i(\bar{x}, \bar{y}, \bar{p}_i) - \bar{T}_i \nabla_y G_i(\bar{x}, \bar{y}, \bar{p}_i)) - (\nabla_p H_i(\bar{x}, \bar{y}, \bar{p}_i) - \bar{T}_i \nabla_p G_i(\bar{x}, \bar{y}, \bar{p}_i))) 
\]

\[
+ \sum_{i=1}^{k} \lambda_i((\nabla_{yy} f_i(\bar{x}, \bar{y}) - \bar{T}_i \nabla_{yy} g_i(\bar{x}, \bar{y}))^T (\gamma - \delta \bar{y}) 
\]

\[
+ \sum_{i=1}^{k} ((\nabla_{yy} H_i(\bar{x}, \bar{y}, \bar{p}_i) - \bar{T}_i \nabla_{yy} G_i(\bar{x}, \bar{y}, \bar{p}_i)) - (\nabla_{yy} H_i(\bar{x}, \bar{y}, \bar{p}_i) - \bar{T}_i \nabla_{yy} G_i(\bar{x}, \bar{y}, \bar{p}_i)) \right) 
\]

\[
+ \sum_{i=1}^{k} (-\beta_i \bar{p}_i + (\gamma - \delta \bar{y}) \bar{\lambda}_i)(\beta_i \bar{p}_i, i = 1, 2, \ldots, k). 
\]

Inequality (21) is equivalent to

\[
(\gamma - \delta \bar{y})^T(\nabla_y f_i(\bar{x}, \bar{y}) - z_i + \nabla_p H_i(\bar{x}, \bar{y}, \bar{p}_i)) 
\]

\[
- \bar{T}_i(\nabla_y g_i(\bar{x}, \bar{y}) + \bar{r}_i + \nabla_p G_i(\bar{x}, \bar{y}, \bar{p}_i)) - \xi_i + \eta = 0, i = 1, 2, \ldots, k. 
\]

By hypothesis \((ii)\) and (22), we have

\[
\bar{\lambda}_i(\gamma - \delta \bar{y}) = \beta_i \bar{p}_i, i = 1, 2, \ldots, k. 
\]

Now, we claim that \( \beta_i \neq 0, \forall i \). If possible, let \( \beta_{t_0} = 0 \) for some \( t_0, 1 \leq t_0 \leq k \), then from \( \bar{\lambda}_{t_0} > 0 \) and equation (33), we have

\[
\gamma = \delta \bar{y}. 
\]

Using (33) and (34), we obtain \( \beta_i \bar{p}_i = 0, i = 1, 2, \ldots, k. \) Hence, by hypothesis \((i)\), we get

\[
\sum_{i=1}^{k} \beta_i((\nabla_y H_i(\bar{x}, \bar{y}, \bar{p}_i) - \bar{T}_i \nabla_y G_i(\bar{x}, \bar{y}, \bar{p}_i)) - (\nabla_p H_i(\bar{x}, \bar{y}, \bar{p}_i) - \bar{T}_i \nabla_p G_i(\bar{x}, \bar{y}, \bar{p}_i))) = 0. 
\]

Using (33)-(35) in (31), we obtain

\[
\sum_{i=1}^{k} (\beta_i - \delta \bar{\lambda}_i)((\nabla_y f_i(\bar{x}, \bar{y}) - z_i + \nabla_p H_i(\bar{x}, \bar{y}, \bar{p}_i) - \bar{T}_i(\nabla_y g_i(\bar{x}, \bar{y}) + \bar{r}_i + \nabla_p G_i(\bar{x}, \bar{y}, \bar{p}_i))) = 0. 
\]
which by hypothesis \((iii)\), follows that
\[
\beta_i - \delta \lambda_i = 0, \ i = 1, 2, ..., k.
\] (37)

Now, for \(i = t_0\), we have \(\delta \lambda_i = 0\). This implies \(\delta = 0\). since \(\lambda > 0\). Hence, from (37) \(\beta_i = 0\), \(\forall i\). Thus, from relation (20), (34) and (37), we get \(\alpha_i = 0\), \(i = 1, 2, ..., k\). Also, from relations (20) and (34), we get \(\eta = 0\) and \(\gamma = 0\), respectively, which contradicts the fact that \((\alpha, \beta, \gamma, \delta, \xi, \eta) \neq 0\). Hence \(\beta_i \neq 0\), \(i = 1, 2, ..., k\).

Now, equation (32) reduces to
\[
(\gamma - \delta \bar{y})^T (\nabla_y f_i(x, \bar{y}) - \bar{z}_i + \nabla_p H_i(x, \bar{y}, \bar{p}_i))
- \bar{T}_i (\nabla_y g_i(x, \bar{y}) + \bar{r}_i + \nabla_p G_i(x, \bar{y}, \bar{p}_i)) + \eta \bar{\lambda}^T e_k = 0.
\] (38)

Multiplying by \(\bar{\lambda}_i\) and summing over \(i\), we get
\[
(\gamma - \delta \bar{y})^T \sum_{i=1}^{k} \bar{\lambda}_i (\nabla_y f_i(x, \bar{y}) - \bar{z}_i + \nabla_p H_i(x, \bar{y}, \bar{p}_i))
- \bar{T}_i (\nabla_y g_i(x, \bar{y}) + \bar{r}_i + \nabla_p G_i(x, \bar{y}, \bar{p}_i)) + \eta \bar{\lambda}^T e_k = 0.
\] (39)

Subtracting (26) from (25), we get
\[
(\gamma - \delta \bar{y})^T \sum_{i=1}^{k} \bar{\lambda}_i (\nabla_y f_i(x, \bar{y}) - \bar{z}_i + \nabla_p H_i(x, \bar{y}, \bar{p}_i) - \bar{T}_i (\nabla_y g_i(x, \bar{y}) + \bar{r}_i + \nabla_p G_i(x, \bar{y}, \bar{p}_i)) = 0.
\] (40)

Using (40) in (39), we get, \(\eta = 0\).

Now, equation , yield
\[
(\gamma - \delta \bar{y})^T (\nabla_y f_i(x, \bar{y}) - \bar{z}_i + \nabla_p H_i(x, \bar{y}, \bar{p}_i) - \bar{T}_i (\nabla_y g_i(x, \bar{y}) + \bar{r}_i + \nabla_p G_i(x, \bar{y}, \bar{p}_i)) = 0, \ i = 1, 2, ..., k.
\] (41)

Since \(\lambda > 0\), using (33) in (34), we get
\[
\beta_i \bar{p}_i^T [\nabla_y f_i(x, \bar{y}) - \bar{z}_i + \nabla_p H_i(x, \bar{y}, \bar{p}_i) - \bar{T}_i (\nabla_y g_i(x, \bar{y}) + \bar{r}_i + \nabla_p G_i(x, \bar{y}, \bar{p}_i)] = 0, \ i = 1, 2, ..., k.
\] (42)

Since \(\beta_i \neq 0, \ i = 1, 2, ..., k\), we obtain
\[
\bar{p}_i^T [\nabla_y f_i(x, \bar{y}) - \bar{z}_i + \nabla_p H_i(x, \bar{y}, \bar{p}_i) - \bar{T}_i (\nabla_y g_i(x, \bar{y}) + \bar{r}_i + \nabla_p G_i(x, \bar{y}, \bar{p}_i)] = 0, \ i = 1, 2, ..., k,
\] (43)

or
\[
\sum_{i=1}^{k} \bar{p}_i^T [\nabla_y f_i(x, \bar{y}) - \bar{z}_i + \nabla_p H_i(x, \bar{y}, \bar{p}_i) - \bar{T}_i (\nabla_y g_i(x, \bar{y}) + \bar{r}_i + \nabla_p G_i(x, \bar{y}, \bar{p}_i)] = 0.
\] (44)

By the hypothesis \((iv)\), we have \(\bar{p}_i = 0, \ i = 1, 2, ..., k\). Further using, hypothesis \((i)\), using (32)-(33) in (18)-(19), we get
\[
(x - \bar{x})^T \left[ \sum_{i=1}^{k} \beta_i (\nabla_x f_i(x, \bar{y}) + \bar{w}_i - \bar{T}_i (\nabla_x g_i(x, \bar{y}) - \bar{\ell}_i) \right] \geq 0, \forall x \in C_1.
\] (45)

\[
\sum_{i=1}^{k} (\beta_i - \delta \lambda_i) (\nabla_y f_i(x, \bar{y}) - \bar{z}_i - \bar{T}_i (\nabla_y g_i(x, \bar{y}) + \bar{r}_i)] = 0.
\] (46)
Using hypothesis (iii) in (46), we have
\[ \beta_i = \delta \bar{\lambda}_i, \quad i = 1, 2, \ldots, k. \]  
(47)

Since \( \beta_i \neq 0 \), \( \bar{\lambda}_i > 0 \), \( i = 1, 2, \ldots, k \) and \( \delta \geq 0 \), this implies that \( \beta_i > 0 \), \( \forall i \). Now, using (47) in (46), we obtain
\[
(x - \bar{x})^T \left[ \sum_{i=1}^{k} \bar{\lambda}_i (\nabla_x f_i(x, \bar{y}) + \bar{w}_i - \bar{T}_i (\nabla_x g_i(x, \bar{y}) - \bar{\ell}_i)) \right] \geq 0, \forall x \in C_1.
\]
(48)

Let \( x \in C_1 \). Then \( x + \bar{x} \in C_1 \), as \( C_1 \) is a closed convex cone. On substituting \( x + \bar{x} \) in place of \( x \) in (48), we have
\[
x^T \sum_{i=1}^{k} \bar{\lambda}_i (\nabla_x f_i(x, \bar{y}) + \bar{w}_i - \bar{T}_i (\nabla_x g_i(x, \bar{y}) - \bar{\ell}_i)) \geq 0,
\]
(49)

which in turn implies that for all \( x \in C_1 \), we have
\[
\sum_{i=1}^{k} \bar{\lambda}_i (\nabla_x f_i(x, \bar{y}) + \bar{w}_i) - \bar{T}_i (\nabla_x g_i(x, \bar{y}) - \bar{\ell}_i)) \in C_1^*.
\]
(50)

Also by letting \( x = 0 \) and \( x = 2\bar{x} \), simultaneously in (48), yields
\[
\bar{x}^T \sum_{i=1}^{k} \bar{\lambda}_i (\nabla_x f_i(x, \bar{y}) + \bar{w}_i) - \bar{T}_i (\nabla_x g_i(x, \bar{y}) - \bar{\ell}_i) = 0.
\]
(51)

Using \( \bar{p}_i = 0 \) in (33), we get, \( \gamma = \delta \bar{y} \) and \( \delta > 0 \), we have
\[
\bar{y} = \frac{\gamma}{\delta} \in C_2.
\]

Since \( \beta > 0 \) by (23) and the fact that \( \gamma = \delta \bar{y} \), we get \( \bar{y} \in N_{D_i}(\bar{z}_i), \; i = 1, 2, \ldots, k. \). This implies
\[
\bar{y}^T \bar{z}_i = s(\bar{y}|D_i), \; i = 1, 2, \ldots, k.
\]
(52)

By (24) and hypothesis (v), we have \( \bar{y} \in N_{F_i}(\bar{r}_i), \; i = 1, 2, \ldots, k. \) Hence,
\[
\bar{y}^T \bar{r}_i = s(\bar{y}|F_i), \; i = 1, 2, \ldots, k.
\]
(53)

Combining (29),(52)-(53) and given equation (1), reduce to
\[
(f_i(x, \bar{y}) + \bar{x}^T \bar{w}_i - s(\bar{y}|D_i)) - \bar{T}_i (g_i(x, \bar{y}) - \bar{x}^T \bar{T}_i - s(\bar{y}|F_i)) = 0, \; i = 1, 2, \ldots, k.
\]
(54)

Combining this with (52)-(54), shows that \((\bar{x}, \bar{y}, \bar{T}, \bar{w}, \bar{\ell}, \bar{\lambda}, \bar{\bar{q}} = 0)\) is a feasible solution of \((MFDP)_W\).

Under the assumptions Theorems (3.1) - (3.4), if \((\bar{x}, \bar{y}, \bar{T}, \bar{w}, \bar{\ell}, \bar{\lambda}, \bar{\bar{q}} = 0)\) is not an efficient solution of \((MFDP)_W\), then there exists other feasible solution \((u, v, W, \bar{w}, \bar{\ell}, \bar{\lambda}, \bar{q})\), of \((MFDP)_W\), such that \( W - \bar{T} \notin K \setminus \{0\} \).

Since \( (\bar{x}, \bar{y}, \bar{T}, \bar{w}, \bar{\ell}, \bar{\lambda}, \bar{\bar{p}}) \) is a feasible solution of \((MFPP)_T\), by Weak duality theorem, we have \( W - \bar{T} \notin K \setminus \{0\} \), hence the contradiction implies that \((\bar{x}, \bar{y}, \bar{T}, \bar{w}, \bar{\ell}, \bar{\lambda}, \bar{\bar{q}} = 0)\) is an efficient solution of \((MFDP)_W\). Hence, the result. 

\( \square \)

**Theorem 3.6 (Strong duality).** Let \((\bar{x}, \bar{y}, \bar{T}, \bar{z}, \bar{\ell}, \bar{\lambda}, \bar{\bar{p}})\) be efficient solution of \((MFPP)_T\), and fix \( \lambda = \bar{\lambda} \) in \((MFDP)_W\). Suppose that
Theorem 3.7 (Converse duality). Let \( (\bar{u}, \bar{v}, \bar{W}, \bar{w}, \bar{t}, \bar{\lambda}, \bar{q}) \) be efficient solution of \((MFPP)_W\), and fix \( \lambda = \bar{\lambda} \) in \((MFDP)_T\). If the following conditions hold

\[(i) \quad \nabla_x H_i(\bar{x}, \bar{y}, 0) = \nabla_x G_i(\bar{x}, \bar{y}, 0) = 0, \nabla_q, \Phi_i(\bar{x}, \bar{y}, 0) = \nabla_q, \Psi_i(\bar{x}, \bar{y}, 0) = 0, H_i(\bar{x}, \bar{y}, 0) = G_i(\bar{x}, \bar{y}, 0) = 0, \Phi_i(\bar{x}, \bar{y}, 0) = \Psi_i(\bar{x}, \bar{y}, 0) = 0, \nabla_y H_i(\bar{x}, \bar{y}, 0) = \nabla_y G_i(\bar{x}, \bar{y}, 0) = 0, \nabla_{p_i} H_i(\bar{x}, \bar{y}, 0) = \nabla_{p_i} G_i(\bar{x}, \bar{y}, 0) = 0, i = 1, 2, \ldots, k.\]

\[(ii) \quad \bar{T}_i > 0, \forall i \in \{1, 2, \ldots, k\}.\]

\[(iii) \quad \sum_{i=1}^k \lambda_i (\nabla_{yy} f_i(\bar{x}, \bar{y}) - \bar{T}_i \nabla_{yy} g_i(\bar{x}, \bar{y})) \text{ is positive definite and } \bar{p}_i^T (\nabla_y H_i(\bar{x}, \bar{y}, \bar{p}_i) - \bar{T}_i \nabla_y G_i(\bar{x}, \bar{y}, \bar{p}_i)) = 0, \quad \forall i \in \{1, 2, \ldots, k\}.\]

\[(iv) \quad \tilde{T}_i (\nabla_y H_i(\bar{x}, \bar{y}, \bar{p}_i) - \bar{T}_i \nabla_y G_i(\bar{x}, \bar{y}, \bar{p}_i)) \text{ is negative definite and } \tilde{p}_i^T (\nabla_y H_i(\bar{x}, \bar{y}, \bar{p}_i) - \bar{T}_i \nabla_y G_i(\bar{x}, \bar{y}, \bar{p}_i)) \leq 0, \forall i \in \{1, 2, \ldots, k\}.\]

\[(v) \quad \text{the set of vectors } \{\nabla_x f_i(\bar{x}, \bar{v}) + \bar{w}_i + \nabla_q, \Phi_i(\bar{x}, \bar{v}, \bar{q}_i) - \bar{W}_i (\nabla_x g_i(\bar{x}, \bar{v}) - \bar{t}_i + \nabla_q, \Psi_i(\bar{x}, \bar{v}, \bar{q}_i))\}_{i=1}^k \text{ is linearly independent.}\]

Then \( \bar{p} = 0 \), and there exists \( \bar{w}_i \in Q_i \) and \( \bar{t}_i \in E_i, i = 1, 2, \ldots, k \) such that \((\bar{x}, \bar{y}, \bar{T}, \bar{w}, \bar{t}, \bar{\lambda}, \bar{q}) = 0\) is a feasible solution of \((MFDP)_W\). Furthermore, if the hypotheses in theorems (3.1) – (3.4) are satisfied, then 

\((\bar{x}, \bar{y}, \bar{T}, \bar{w}, \bar{t}, \bar{\lambda}, \bar{q}) = 0\) is efficient solution of \((MFDP)_W\), and the two objective values are equal.

Proof: The proof follows Theorem 3.5.
In this article, we have considered a pair of Mond-Weir type nondifferentiable higher order fractional symmetric dual program with cone constraints and discussed duality theorems under higher order $K - (C, \alpha, \rho, d)$-convexity/$K - (C, \alpha, \rho, d)$ pseudoconvexity assumptions. The present work can further be extended to nondifferentiable higher order symmetric fractional programming problem over cones under generalized assumptions. This will orient the future task of the authors.

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