On $F$-implicit minimal vector variational inequalities

Mehdi Roohi $^{1,*,}$, Mohsen Rostamian Delavar$^2$

$^1$Department of Mathematics, Faculty of Sciences, Golestan University, Iran
$^2$Department of Mathematics, Faculty of Basic Sciences, University of Bojnord, Iran

Abstract In this paper, by introducing some new concepts in minimal spaces, we prove a generalized form of the Fan-KKM theorem in minimal vector spaces. A new class of minimal generalized vector $F$-implicit variational inequality problems and, as an application of Fan-KKM theorem is investigated. Moreover, an existence theorem for this kind of problems under some suitable assumptions in minimal vector spaces is given.

Keywords Minimal vector space, Fan-KKM theorem, co-compact property, Vector $F$-implicit variational inequality problem.

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1. Introduction

Nonlinear analysis presents many problems that can be resolved by the nonemptyness of the intersection of a certain family of subsets of a underlying set. Each point of the intersection can be a fixed point, a coincidence point, an equilibrium point, a saddle point, an optimal point, a solution point for complementarity problem, a solution point for variational problem, or others of the corresponding problem under consideration. The first result on the nonempty intersection was the celebrated Knaster-Kuratowski-Mazurkiewicz theorem (simply, the KKM principle) in [10], which is concerned with certain types of multimaps called the KKM maps. The KKM theory is the study of KKM maps and their applications. Generalized form of the KKM theorem namely Fan-KKM principle provides a foundation for many of the modern essential results in diverse areas of mathematical sciences (for more details see [15]). However the Fan-KKM theorem has essential role in solving all kinds of implicit complementarity problems or variational inequality problems, particularly in generalized vector $F$-implicit variational inequality problems [17]. The vector variational inequalities and vector complementarity problems have found many of their applications in vector optimization, set-valued optimization, approximate analysis of vector optimization problems and vector network equilibrium problems.

more generalized vector case. Some new existence theorems of solutions for generalized $F$-implicit variational inequality problems were also proved in [11].

In this paper, motivated by the above mentioned works, we obtain some new results to prove a generalized form of the Fan-KKM theorem in minimal vector spaces. Also we introduce a new class of vector $F$-implicit variational inequality problems and as an application of our obtained Fan-KKM theorem we derive an existence theorem for this kind of problems.

2. Preliminaries

The concepts of minimal structures and minimal spaces, as generalization of topology and topological spaces were introduced in [14]. For easy understanding of the material incorporated in this paper, we recall some basic definitions and results. Also some new concepts are introduced in minimal spaces. Further results about minimal spaces can be found in [1, 2, 3, 4, 13], [16] and some references cited therein.

A family $\mathcal{M} \subseteq \mathcal{P}(X)$ is said to be a minimal structure on $X$ if $\emptyset, X \in \mathcal{M}$. In this case $(X, \mathcal{M})$ is called a minimal space. For example, let $(X, \tau)$ be a topological space, then $\tau, \text{SO}(X), \text{PO}(X), \alpha\text{O}(X)$ and $\beta\text{O}(X)$ are minimal structures on $X$ [13]. In a minimal space $(X, \mathcal{M}), A \in \mathcal{P}(X)$ is said to be an $m$-open set if $A \in \mathcal{M}$ and also $B \in \mathcal{P}(X)$ is an $m$-closed set if $B^c \in \mathcal{M}$. For any set $A \subseteq X$, define $\overline{\text{m-Int}}{(A)} = \bigcup\{U : U \subseteq A, U \in \mathcal{M}\}$ and $\overline{\text{m-Cl}}{(A)} = \{F : A \subseteq F, F^c \in \mathcal{M}\}$. Note that for $A \subseteq X$, $\overline{\text{m-Cl}}{(A)}$ (resp. $\overline{\text{m-Int}}{(A)}$) is not necessarily $m$-closed (resp. $m$-open). The following lemma may be useful to apply in a minimal space.

Lemma 1
[16] For any two sets $A$ and $B$ in a minimal space $X$,
(a) $A \subseteq \overline{\text{m-Cl}}{(A)}$ and $A = \overline{\text{m-Cl}}{(A)}$ if $A$ is an $m$-closed set.
(b) $m\text{-Cl}(A \cap B) \subseteq (m\text{-Cl}(A)) \cap (m\text{-Cl}(B))$.
(c) $m\text{-Cl}(m\text{-Cl}(B)) = m\text{-Cl}(B)$.
(d) $(m\text{-Cl}(A))^c = m\text{-Int}(A^c)$.

Definition 1
[1] For two minimal spaces $(X, \mathcal{M})$ and $(Y, \mathcal{N})$, we define minimal product structure for $X \times Y$ as follows:

$$
\mathcal{M} \times \mathcal{N} = \{A \subseteq X \times Y : \forall (x, y) \in A, \exists U \in \mathcal{M}, \exists V \in \mathcal{N}; (x, y) \in U \times V \subseteq A\}.
$$

Definition 3
[1] A linear minimal structure on a vector space $X$ over the complex field $\mathbb{F}$ is a minimal structure $\mathcal{M}$ on $X$ such that the two mappings

$$
+ : X \times X \to X, (x, y) \mapsto x + y,
$$

$$
. : \mathbb{F} \times X \to X, (t, x) \mapsto tx,
$$

are $m$-continuous, where $\mathbb{F}$ has the usual topology and both $\mathbb{F} \times X$ and $X \times X$ have the corresponding product minimal structures. A linear minimal space (or minimal vector space) is a vector space together with a linear minimal structure.

Obviously, any topological vector space is a minimal vector space but the converse is not true generally. In the following, it is shown that there is a linear minimal space which is not a topological vector space.

Example 1
Consider the real field $\mathbb{R}$ and let $\mathcal{M} = \{(a, b) : a, b \in \mathbb{R} \cup \{\pm \infty\}\}$. Clearly, $\mathcal{M}$ is a minimal structure on $\mathbb{R}$. We
claim that $\mathcal{M}$ is a linear minimal structure on $\mathbb{R}$. For this, we must prove that, two operations $+$ and $\cdot$ are $m$-continuous. Suppose $(x_0, y_0) \in (+)^{-1}(a, b)$ and so $x_0 + y_0 \in (a, b)$. Put $\epsilon = \min\{x_0 + y_0 - a, b - (x_0 + y_0)\}$ and so $x_0 \in (x_0 - \frac{\epsilon}{2}, x_0 + \frac{\epsilon}{2})$ and $y_0 \in (y_0 - \frac{\epsilon}{2}, y_0 + \frac{\epsilon}{2})$. Hence

$$x_0 + y_0 \in ((x_0 - \frac{\epsilon}{2}, x_0 + \frac{\epsilon}{2}) + (y_0 - \frac{\epsilon}{2}, y_0 + \frac{\epsilon}{2})) \subseteq (a, b);$$

which implies that $(+, a, b)$ is $m$-open in the minimal product space $\mathbb{R} \times \mathbb{R}$; that is $+$ is $m$-continuous. Also, suppose $(\alpha_0, x_0) \in (-1)^{-1}(a, b)$. Since $\alpha_0 x_0 \in (a, b)$ and $\lim_{x \to 0} (\alpha_0 - s)(x_0 - t) = \alpha_0 x_0$, one can find some $0 < \delta$ for which $|\alpha_0 - s| < \delta$ and $|x_0 - t| < \delta$ imply that $a < (\alpha_0 - s)(x_0 - t) < b$. Therefore,

$$(\alpha_0, x_0) \in (\alpha_0 - \delta, \alpha_0 + \delta) \cdot (x_0 - \delta, x_0 + \delta) \subseteq (a, b);$$

i.e., $-1(a, b)$ is $m$-open in the minimal product space $\mathbb{R} \times \mathbb{R}$, which implies that the operation $\cdot$ is $m$-continuous.

**Definition 4**

[1] Consider a minimal space $(X, \mathcal{M})$ and a nonempty subset $Y$ of $X$. The family $\mathcal{M}|_Y = \{U \cap Y : U \in \mathcal{M}\}$ is called induced minimal structure by $\mathcal{M}$ on $Y$. $(Y, \mathcal{M}|_Y)$ is called a minimal subspace of $(X, \mathcal{M})$. For any subset $A$ of $X$,

$$m\text{-Int}_Y(A) = \bigcup\{V : V \in \mathcal{M}|_Y \text{ and } V \subseteq A\}$$

and

$$m\text{-Cl}_Y(A) = \bigcap\{F : F^c \in \mathcal{M}|_Y \text{ and } A \subseteq F\}.$$ 

**Definition 5**

[16] For a minimal space $(X, \mathcal{M})$,

(a) a family of $m$-open sets $A = \{A_j : j \in J\}$ in $X$ is called an $m$-open cover of $K$ if $K \subseteq \bigcup_j A_j$. Any subfamily of $A$ which is also an $m$-open cover of $K$ is called a subcover of $A$ for $K$;

(b) a subset $K$ of $X$ is $m$-compact whenever given any $m$-open cover of $K$ has a finite subcover.

**Theorem 1**

[16] Suppose that $X$ and $Y$ are two minimal spaces and $f : X \to Y$ is an $m$-continuous function. For any $m$-compact subset $K \subseteq X$, $f(K)$ is $m$-compact in $Y$.

**Lemma 2**

Suppose that $X$ is an $m$-compact minimal space and $Y \subseteq X$. Then $m\text{-Cl}(Y)$ is $m$-compact.

**Proof**

Suppose that $\{U_\alpha\}_{\alpha \in I}$ is an $m$-open cover of $m\text{-Cl}(Y)$. So

$$X = m\text{-Cl}(Y) \cup m\text{-Int}(Y^c) \subseteq \bigcup_{\alpha \in I} U_\alpha \cup \bigcup_{j \in J} G_\beta,$$

where $G_\beta$’s are $m$-open subsets of $Y^c$. Since $X$ is an $m$-compact minimal space we have:

$$X \subseteq \bigcup_{i=1}^n U_{\alpha_i} \cup \bigcup_{j=1}^m G_{\beta_j} \subseteq \bigcup_{i=1}^n U_{\alpha_i} \cup m\text{-Int}(Y^c).$$

Then $m\text{-Cl}(Y) \subseteq \bigcup_{i=1}^n U_{\alpha_i}$. 

**Lemma 3**

[3] Suppose $(X, \mathcal{M})$ is an $m$-compact minimal space, $\{A_i : i \in I\}$ is a family of subsets of $X$. If $\{m\text{-Cl}(A_i) : i \in I\}$ has the finite intersection property, then

$$\bigcap_{i \in I} m\text{-Cl}(A_i) \neq \emptyset.$$
In the following result, we prove the minimal version of Tychonoff theorem

**Theorem 2**
The minimal product space \( \prod_{\alpha \in I} X_\alpha \) is \( m \)-compact if and only if \( (X_\alpha, M_\alpha) \) is an \( m \)-compact minimal space, for any \( \alpha \in I \).

**Proof**
One direction is an immediate consequence of Theorem 1. For the converse, on the contrary, assume that \( \mathcal{A} \subseteq \prod_{\alpha \in I} M_\alpha \) is an \( m \)-open cover of \( \prod_{\alpha \in I} X_\alpha \) without any finite subcover for \( \prod_{\alpha \in I} X_\alpha \). For any \( \alpha \in I \), set \( \mathcal{U}_\alpha = \{ \mathcal{V} \in M_\alpha : \pi_\alpha^{-1}(\mathcal{V}) \in \mathcal{A} \} \). Since \( \mathcal{A} \) has no finite subcover for \( \prod_{\alpha \in I} X_\alpha \), no finite subcover of \( \mathcal{U}_\alpha \) can cover \( X_\alpha \), for any \( \alpha \in I \).

Now, \( m \)-compactness of \( X_\alpha \) implies that \( \mathcal{U}_\alpha \) can not cover \( X_\alpha \). Therefore there exists \( x_\alpha \in X_\alpha \setminus \bigcup \{ \mathcal{V} : \mathcal{V} \in \mathcal{U}_\alpha \} \), for any \( \alpha \in I \). Set \( x = (x_\alpha)_{\alpha \in I} \). Then \( x \in \prod_{\alpha \in I} X_\alpha \setminus \bigcup \{ A : A \in \mathcal{A} \} \), which implies that \( \mathcal{A} \) is not an \( m \)-open cover for \( \prod_{\alpha \in I} X_\alpha \), a contradiction. \( \square \)

### 3. Generalized Fan-KKM Theorem

A multimap \( F : X \rightharpoonup Y \) is a function from a set \( X \) into the power set of \( Y \). Given \( A \subseteq X \), set \( F(A) = \bigcup_{x \in A} F(x) \). A multimap \( F : X \rightharpoonup Y \) is said to be minimal transfer closed if for any \( x \in X \) and \( y \notin F(x) \), there exists \( x_0 \in X \) for which \( y \notin m-\text{Cl}(F(x_0)) \). Clearly, \( F : X \rightharpoonup Y \) is minimal transfer closed if and only if \( \bigcap_{x \in X} F(x) = \bigcap_{x \in X} m-\text{Cl}(F(x)) \). It is obvious that an \( m \)-closed valued multimap is a minimal transfer closed multimap.

A subset \( A \) of a vector space \( X \) is convex if we have \( ty + (1 - t)z \in A \), whenever \( y, z \in A \) and \( t \in [0, 1] \). Also the convex hull of \( A \), denoted by \( \text{co}(A) \), is the smallest convex set that contains \( A \), that is, the intersection of all convex sets containing \( A \).

**Definition 6**
Suppose that \( D \) is a convex subset of a minimal vector space \( X \). A multimap \( F : D \rightharpoonup X \) is called a KKM map if \( \text{co}(A) \subseteq F(A) \) for any \( A \in \langle D \rangle \), where the notation \( \langle D \rangle \) means the set of all finite subsets of \( D \).

The following theorem is a generalized form of the Fan-KKM theorem, as a special case of Theorem 4.7 in [3], related to the minimal vector spaces.

**Theorem 3**
Suppose that \( X \) is a minimal (topological) vector space. Consider two nonempty valued multimeaps \( F, G : X \rightharpoonup X \) satisfying
(a) \( F(x) \subseteq G(x) \) for all \( x \in X \),
(b) \( F \) is a KKM map,
(c) \( \bigcap_{x \in M} m-\text{Cl}(G(x)) \) is \( m \)-compact for some \( M \in \langle D \rangle \),
(d) for all \( A \in \langle X \rangle \), \( G \) is minimal transfer closed on \( \text{co}(A) \),
(e) for all \( A \in \langle X \rangle \), \( m-\text{Cl}\left( \bigcap_{x \in \text{co}(A)} G(x) \right) \cap \text{co}(A) \subseteq \bigcap_{x \in \text{co}(A)} G(x) \cap \text{co}(A) \).

Then \( \bigcap_{x \in X} G(x) \neq \emptyset \).

Now, we introduce a new concept in minimal vector spaces with some examples which is useful for the proof of our results.

**Definition 7**
Suppose that \( X \) is a minimal (topological) vector space. A nonempty set \( A \subseteq X \) has the minimal (topological) finitely adjoint co-compact property if \( \text{co}(A \cup B) \) is an \( m \)-compact (compact) subset of \( X \), for any \( B \in \langle X \rangle \).
Example 2  (a) Consider $\mathbb{R}$ with its usual topology. Then every finite subset of $\mathbb{R}$ has the topological finitely adjoint co-compact property.

(b) Since any topological space with the topological finitely adjoint co-compact property has the minimal finitely adjoint co-compact property in a topological vector space has the minimal finitely adjoint co-compact property if we consider the topological structure as minimal structure, but the converse is not true. Consider the minimal structure $\mathcal{M} = \{\mathbb{R}, \emptyset, (a, b)\}$ on $\mathbb{R}$ where $a, b \in \mathbb{R}^-$ and let $c \in \mathbb{R}^+$. For any $A \in \langle \mathbb{R} \rangle$,

$$co((c, +\infty) \cup A) = \begin{cases} (c, +\infty), & c < \min A, \\ [\min A, +\infty), & c \geq \min A, \end{cases}$$

which is not compact in $\mathbb{R}$ when we consider the usual topology. But it is an $m$-compact subset of $\mathbb{R}$ via minimal structure $\mathcal{M}$ (just $\mathbb{R}$ is it’s cover). So there exists a set with the minimal finitely adjoint co-compact property which does not have the topological finitely adjoint co-compact property.

(c) Suppose that $X$ is a minimal vector space. Any $m$-compact convex subset of $X$ has the minimal finitely adjoint co-compact property. To see this, suppose that $A$ is an $m$-compact convex subset of $X$ and $B = \{b_1, b_2, \ldots, b_n\} \in \langle X \rangle$. Define the function $\varphi : A \times \{b_1\} \times \cdots \times \{b_n\} \times \Delta_{n+1} \rightarrow co(A \cup B)$ by

$$\varphi(a_1, b_1, b_2, \ldots, b_n, t_1, t_2, \ldots, t_{n+1}) = \sum_{i=1}^{n} t_i b_i + t_{n+1} a,$$

with $\Delta_n = \left\{ \left( \sum_{i=1}^{n} t_i e_i : \sum_{i=1}^{n} t_i = 1, t_i \geq 0 \right) \right\}$, where $(e_i)$ is the standard base of $\mathbb{R}^n$. It is not hard to see that $\varphi$ is an onto $m$-continuous function by Definition 3. Since we can consider $\Delta_{n+1}$ as an $m$-compact subset of $\mathbb{R}^n$, according to the Theorem 2, $A \times \{b_1\} \times \cdots \times \{b_n\} \times \Delta_{n+1}$ is $m$-compact. Now, from Theorem 1, $\varphi(A \times \{b_1\} \times \cdots \times \{b_n\} \times \Delta_{n+1}) = co(A \cup B)$ is $m$-compact.

From Theorem 3 along with Definition 7 we can achieve another version of the Fan-KKM theorem which is our main task in this section.

**Theorem 4**

Suppose that in Theorem 3 condition (c) is replaced with the following condition:

(c') there exists a nonempty subset $B \subseteq X$ with the minimal finitely adjoint co-compact property such that $m$-$\text{cl} \left( \bigcap_{x \in B} G(x) \right)$ is $m$-compact.

Then $\bigcap_{x \in X} G(x) \neq \emptyset$.

**Proof**

For $A \in \langle X \rangle$, set $L_A := \text{co}(A \cup B)$. Define multimaps $F_A, G_A : L_A \rightarrow L_A$ by

$$G_A(x) = G(x) \cap L_A \text{ and } F_A(x) = F(x) \cap L_A.$$

For any $x \in L_A$, $F_A(x) \subseteq G_A(x)$. It is not hard to see that $F_A$ is a KKM multimap because the multimap $F$ is KKM. According to the definition of the multimap $G_A$, we have $m$-$\text{cl}_{L_A}(G_A(x)) \subseteq m$-$\text{cl}_{L_A}(L_A) = L_A$ for any $x \in L_A$. Since $B$ has the minimal finitely adjoint co-compact property then $L_A$ is $m$-compact and Lemma 2 implies that $m$-$\text{cl}_{L_A}(G_A(x))$ is $m$-compact for any $x \in L_A$. All conditions of Theorem 3 are satisfied for $F_A, G_A$ and $L_A$. Hence $\bigcap_{x \in L_A} G_A(x) \neq \emptyset$ for $A \in \langle X \rangle$. Now, suppose that $M_A = \bigcap_{x \in L_A} G(x)$, for each $A \in \langle X \rangle$. The family $\mathcal{R} = \{M_A : A \in \langle X \rangle \}$ has the finite intersection property. To see this, assume that $A_1, A_2, \ldots, A_n \in \langle X \rangle$. Since $\bigcap_{i=1}^{n} A_i \in \langle X \rangle$, we get $M_{\bigcup_{i=1}^{n} A_i} = \bigcap_{x \in L_{\bigcup_{i=1}^{n} A_i}} G(x) \neq \emptyset$. Then

$$\emptyset \neq M_{\bigcup_{i=1}^{n} A_i} \subseteq \bigcap_{i=1}^{n} M_{A_i},$$
which gives the result.

It follows that $M_A \subseteq \bigcap_{x \in B} G(x)$ and so $m-\text{Cl}(M_A) \subseteq m-\text{Cl}(\bigcap_{x \in B} G(x))$. According to the assertion $(c')$, $m-\text{Cl}(\bigcap_{x \in B} G(x))$ is $m$-compact and Lemma 3 implies that $\bigcap_{A \in \mathcal{A}(X)} m-\text{Cl}(M_A) \neq \emptyset$. Now, choose $\bar{x} \in \bigcap_{A \in \mathcal{A}(X)} m-\text{Cl}(M_A)$ and arbitrary $x \in X$. For $A_0 = \{\bar{x}, x\}$, we have:

$$\bar{x} \in m-\text{Cl}(M_{A_0}) = m-\text{Cl}(\bigcap_{y \in L_{A_0}} G(y)) \subseteq m-\text{Cl}\left(\bigcap_{y \in \text{co}(A_0)} G(y)\right).$$

By assertion $(e)$,

$$\bar{x} \in m-\text{Cl}\left(\bigcap_{y \in \text{co}(A_0)} G(y)\right) \cap \text{co}(A_0) \subseteq \bigcap_{y \in \text{co}(A_0)} G(y) \cap \text{co}(A_0) \subseteq G(x).$$

Then $\bar{x} \in \bigcap_{x \in X} G(x)$.

\[ \square \]

Remark 1

It should be noticed that

(a) according to Example 2(c), when $B$ is an $m$-compact convex subset of $X$, the first assertion in condition $(c')$ of Theorem 4 is fulfilled.

(b) Theorem 4 generally goes back to the Fan-KKM theorem discussed in [5].

4. Minimal Generalized Vector $F$-implicit Variational Inequality Problem

In this section, as an application of the generalized Fan-KKM theorem, we give sufficient conditions to solve a minimal generalized vector $F$-implicit variational inequality problem in the minimal vector spaces.

A nonempty subset $P$ of a vector space $X$ is called a convex cone if $P + P = P$ and $\alpha P \subseteq P$ for all $\alpha \geq 0$, where $P + P = \{x + y : x, y \in P\}$ and $\alpha P = \{\alpha x : x \in P\}$. A cone is said to be pointed if $P \cap -P = \{0\}$.

The following “generalized vector $F$-implicit variational inequality problem (GVF-IVIP)” problem has been considered and solved in [11]:

Let $X$ be a real Banach space, $K \subseteq X$ be a nonempty closed convex cone and $(Y, P)$ be an ordered Banach space induced by the pointed closed convex cone $P$. Denote the space of all continuous linear mappings from $X$ into $Y$ by $L(X, Y)$ and the value of a linear continuous mapping $t \in L(X, Y)$ at $x$ by $\langle t, x \rangle$. Let $A, T : K \rightarrow L(X, Y)$, $g : K \rightarrow K$, $F : K \rightarrow Y$ and $N : L(X, Y) \times L(X, Y) \rightarrow L(X, Y)$ be mappings.

Find $x \in K$ such that

$$\langle N(Ax^*, Tx^*), g(y) - g(x^*) \rangle + F(g(y)) - F(g(x^*)) \geq 0, \text{ for all } y \in K.$$

Theorem 5

Suppose that

(a) five mappings $N, g, A, T$ and $F$ are continuous,

(b) there exists a mapping $h : K \times K \rightarrow Y$ such that

(i) $h(x, x) \geq 0$ for all $x \in K$;

(ii) $\langle N(Ax, Tx), g(y) - g(x) \rangle + F(g(y)) - F(g(x)) - h(x, y) \geq 0$ for all $x, y \in K$;

(iii) the set $\{y \in K : h(x, y) \geq 0\}$ is convex for all $x \in K$;

(c) there exists a nonempty compact convex subset $C$ of $K$ such that for all $x \in K \setminus C$ there exists $y \in C$ such that $\langle N(Ax, Tx), g(y) - g(x) \rangle + F(g(y)) - F(g(x)) \geq 0$.

Then GVF-IVIP has a solution. Furthermore, the solution set of (GVF-IVIP) is closed.
Now, suppose that $X$ and $Y$ are two minimal vector spaces and $M$ is a nonempty convex subset of $X$. Set $L_m(X, Y)$ as the set of all $m$-continuous linear mappings from $X$ into $Y$. Let $(T, x)$ be the value of $T \in L_m(X, Y)$ at point $x$. In addition, let $A, B : M \rightarrow L_m(X, Y)$, $g : M \rightarrow M$, $F : M \times M \rightarrow Y$, $S : L_m(X, Y) \times L_m(X, Y) \rightarrow L_m(X, Y)$ and let $C : M \rightarrow Y$ has nonempty convex pointed cone values; that is for all $y \in M$, $C(y)$ is a nonempty and convex pointed cone in $Y$.

We consider the following “minimal generalized vector $F$-implicit variational inequality problem (MGVF-IVIP)” which is generalization of (GVF-IVIP):

Find $y \in M$ such that for any $x \in M$,

$$
\langle S(Ay, By), g(x) - g(y) \rangle + F(g(x), g(y)) - F(g(y), g(x)) \in C(y).
$$

The following theorem gives a solution for MGVF-IVIP, in the minimal vector spaces.

**Theorem 6**

With the above notations, suppose that

(a) for all $A \in \langle M \rangle$ the multimap $G_A : \text{co}(A) \rightarrow M$ defined by

$$
G_A(x) = \{y \in M : \langle S(Ay, By), g(x) - g(y) \rangle + F(g(x), g(y)) - F(g(y), g(x)) \in C(y)\},
$$

is a minimal transfer closed multimap,

(b) there is a nonempty $m$-compact subset $B = m-\text{Cl}(B) \subseteq M$ and a nonempty set $D \subseteq M$ with the minimal finitely adjoint $co$-compact property such that for any $x \in M \setminus B$ there is $y \in D$ such that

$$
\langle S(Ax, Bx), g(y) - g(x) \rangle + F(g(y), g(x)) - F(g(x), g(y)) \notin C(x),
$$

(c) there is a mapping $f : M \times M \rightarrow Y$ such that

(i) $f(g(x), g(x)) \in C(x)$ for all $x \in M$,

(ii) $\langle S(Ay, By), g(x) - g(y) \rangle + F(g(x), g(y)) - F(g(y), g(x)) - f(g(y), g(x)) \in C(y)$ for all $x, y \in M$,

(iii) the set $\{y \in M : f(g(x), g(y)) \notin C(x)\}$ is convex for all $x \in M$.

Then the MGVF-IVIP has a solution.

**Proof**

Define two multimaps $G, H : M \rightarrow M$ as follows:

$$
G(x) = \{y \in M : f(g(x), g(y)) \in C(y)\},
$$

$$
H(x) = \{y \in M : \langle S(Ay, By), g(x) - g(y) \rangle + F(g(x), g(y)) - F(g(y), g(x)) \in C(y)\}.
$$

From (c)(ii) and the fact that the multimap $C$ is nonempty convex pointed cone valued, we have $G(x) \subseteq H(x)$ for all $x \in M$. Assume that $A = \{x_1, x_2, \ldots, x_n\} \subseteq M$ and $z \in \text{co}(A)$ be such that $z \notin \bigcup_{i=1}^{n} G(x_i)$. So $f(g(z), g(x_i)) \notin C(z)$ for each $i = 1, 2, \ldots, n$. Now condition (c)(iii) implies that $f(g(z), g(z)) \notin C(z)$ which contradicts (c)(i). Then $G$ is a KKM multimap.

From assertion (b), for each $x \in M \setminus B$, there exists $y \in D$ such that $x \notin H(y)$. Hence $x \in M \setminus H(y)$ which implies that $M \setminus B \subseteq \bigcup_{y \in D} M \setminus H(y)$ or equivalently $\bigcap_{y \in D} H(y) \subseteq B$. It follows that $m-\text{Cl}\left( \bigcap_{y \in D} H(y) \right) \subseteq m-\text{Cl}(B) = B$. From Lemma 2, it is easy to see that $m-\text{Cl}\left( \bigcap_{y \in D} H(y) \right)$ is $m$-compact. By assertion (a), we can derive conditions (d) and (e) of Theorem 4. Condition (d) is obvious since $H$, when is restricted to the $\text{co}(A)$, coincides with the multimap $G_A$ for each $A \in \langle M \rangle$. For condition (e), since

$$
\bigcap_{x \in \text{co}(A)} H(x) = \bigcap_{x \in \text{co}(A)} m-\text{Cl}(H(x)),
$$

then we get
\[
\bigcap_{x \in \text{co}(A)} m\text{-Cl}(H(x)) \cap \text{co}(A) = \bigcap_{x \in \text{co}(A)} m\text{-Cl}(H(x)) \cap \text{co}(A)
\]
\[
\subseteq \bigcap_{x \in \text{co}(A)} m\text{-Cl}(H(x)) \cap \text{co}(A)
\]
\[
= \bigcap_{x \in \text{co}(A)} H(x) \cap \text{co}(A).
\]
Now all conditions of Theorem 4 are satisfied and so \( \bigcap_{x \in M} H(x) \neq \emptyset \). Then the solution set of the MGVF-IVIP is nonempty.

**Example 3**

Consider \( \mathbb{R} \) with the minimal structure \( M = \{ \mathbb{R}, \emptyset \} \cup \{(a, b) : a, b \in \mathbb{R}^- \} \cup \{ \mathbb{R} \setminus [a, b] : a, b \in [0, \infty) \} \). In Theorem 6, set \( X = Y = \mathbb{R} \), \( M = [0, +\infty) \), \( D = (1, +\infty) \) and \( B = [0, 1] \). It is clear that \( B \) is an m-closed \( m \)-compact set and \( D \) has the minimal finitely adjoint co-compact property. Also \( [0, +\infty) \) is a convex pointed cone.

For any \( x, y \in M \) define \( g(x) = x/2, F(x, y) = x, f(x, y) = 2(y - x), A(y) = B(y) = y, C(x) = [0, +\infty) \) and \( S(Ay, By) = Ay = y \). So for each \( A \in (M) \), \( G_A \) is m-closed valued and hence it is a minimal transfer closed multimap. All conditions are satisfied and then \( \bigcap_{x \in M} H(x) = \bigcap_{x \in [0, +\infty)} [0, x] = \{0\} \) is a solution of the MGVF-IVIP.

**Remark 2**

(a) Theorem 6 reduces to Theorem 5 if we consider all details in a topological vector space instead of minimal vector space, \( F : M \to Y \) and \( C \equiv [0, +\infty) \), a pointed convex cone.

(b) Theorem 6 is a generalization of Theorem 2.2 in [8] and Theorem 3.2 in [12, 18]. Also Theorem 6, extends and improves the corresponding results of [7].

## 5. Conclusion

Vector variational inequality (VVI) is a base for vector equilibrium problems, which can be applied to traffic networks and migration equilibrium problems (see [6]). Meanwhile, Fan-KKM principle can be utilized to solve some special problems related to vector variational inequalities. Additionally, many new problems in nonlinear analysis can be solved by introducing generalized concepts and extending spaces to provide a proper solution. The above description motivated us to introduce the concepts of generalized Fan-KKM multimaps and generalized vector \( F \)-implicit variational inequality problems (See, also [9]). In this work, we introduced a generalized version of the Fan-KKM theorem for vector \( F \)-implicit variational inequality problems in minimal vector spaces that covers many previous results.

## REFERENCES