STATISTICS, OPTIMIZATION AND INFORMATION COMPUTING
Stat., Optim. Inf. Comput., Vol. **x**, Month 202x, pp 0–25.
Published online in International Academic Press (www.IAPress.org)

The Double-log-exponential Family for Risk Analysis: Properties, Characterizations and Application under the U.K. Motor Non-Comprehensive Claims Triangle Data

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Abstract This paper introduces the Double-Log-Exponential-Exponential (DLEE) distribution, a new special case of the double-log-exponential G (DLEG) family, designed for flexible modeling of insurance claim sizes. The DLEE exhibits remarkable adaptability in density shapes including left-skewed, bimodal, and light-tailed configurations via a single shape parameter θ, whose sign governs skewness direction. We derive some expressions for its density, and provide rigorous characterizations based on truncated moments and reverse-hazard identities. A comprehensive simulation study evaluates six estimation methods namely, maximum likelihood estimation (MLE), ordinary least squares (OLS), Cramér–von Mises estimation (CVME), Anderson–Darling estimation (ADE), right-tail Anderson–Darling estimation (RTADE), and left-tail Anderson–Darling estimation (LTADE)) across multiple parameter scenarios and sample sizes. Finally, the estimated Key Risk Indicators (KRIs), namely Value-at-Risk (VaR), Tail Value-at-Risk (TVaR), Tail Variance (TV), Tail Mean–Variance (TMV), and Expected Loss (EL), under the six estimation methods applied to the real U.K. motor non-comprehensive claims triangle.

Keywords Characterizations; Value-at-Risk; Exponential Model; Claims Data; Risk Analysis.

AMS 2010 subject classifications 62N01; 62N02; 62E10, 60K10, 60N05.

DOI: 10.19139/soic-2310-5070-3427

1. Introduction

In recent years, the construction of flexible and generalized families of probability distributions has emerged as a central theme in statistical methodology, driven by the need to overcome the structural limitations of the classical models when applied to the complex real data. A key objective of this line of research is to enrich the descriptive and inferential capacity of baseline distributions through the strategic introduction of auxiliary parameters, often governing skewness, kurtosis, tail thickness, hazard rate monotonicity, and modality. Such extensions not only broaden the scope of admissible distributional shapes but also improve the fidelity of probabilistic representations in contexts where asymmetry, heavy tails, or nonstandard hazard profiles are prevalent. Following Ibrahim et al. (2026) and considering the following new argument

$$U\left(x;\underline{\Phi}\right)=\frac{\log\left[1+\theta G\left(x;\underline{\Phi}\right)\right]}{\log\left(1+\theta\right)}|\beta>0,\theta\in(-1,0)\cup(0,\infty),x\in\mathbb{R},$$

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ISSN 2310-5070 (online) ISSN 2311-004X (print)

where $G(x;\underline{\Phi})$ refers to the baseline cumulative distribution function (CDF) with the parameters vector $\underline{\Phi}$. Then, following Hashim et al. (2026), we introduce a novel class of continuous distributions called the double-log-exponential G (DLEG) family. The proposed model is defined by a CDF

$$F(x;\beta,\theta)=\frac{1}{e^{\beta}\log(2)}\exp\left[\beta U(x;\underline{\Phi})\right]\log\left[1+U(x;\underline{\Phi})\right]|\beta>0,\theta\in(-1,0)\cup(0,\infty),x\in\mathbb{R}, \tag{1}$$

In recent years, there has been a marked proliferation of generalized statistical distributions designed to address the limitations of classical models in capturing the nuanced behavior of empirical data. Such extensions have proven particularly valuable across diverse applied fields including finance, actuarial science, biomedical research, and reliability engineering, where data often exhibit features such as asymmetry, heavy tails, multimodality, or overdispersion (Afify et al., 2018; Abiad et al., 2025, Alizadeh et al., 2026). To enhance descriptive and inferential fidelity, methodological advances have largely centered on two complementary strategies including the incorporation of auxiliary shape parameters to modulate skewness, kurtosis, and tail weight and the systematic fusion of established distributional families through generators or structural transformations (Alizadeh et al., 2018; Abouelmagd et al., 2019). Among the more prominent generalizations are the odd log-logistic Topp–Leone-G family (Alizadeh et al., 2018), noted for its capacity to accommodate both unimodal and bimodal hazard functions, and the zero-truncated Poisson Burr X-G family (Abouelmagd et al., 2019), which enables joint modeling of discrete-count and continuous-support phenomena within a unified framework. Further contributions include the transmuted Weibull-G, exponential Lindley odd log-logistic-G, and odd log-logistic Weibull-G families (Korkmaz et al., 2018; Rasekhi et al., 2022), each engineered to refine hazard rate specifications, particularly bathtub-shaped or upside-down bathtub forms, common in survival and failure-time analysis. Concurrently, copula-based approaches have gained traction as a means of decoupling marginal behavior from dependence structure. For instance, Alizadeh et al. (2023) formalized copula extensions of the XGamma distribution to model asymmetric tail dependence, while Mansour et al. (2020f) applied vine copulas to multivariate survival data from acute bone cancer cohorts. More recently, Ibrahim et al. (2025a, 2025b) leveraged the Clayton copula to assess the robustness of flexible Weibull-type models under varying degrees of positive dependence, demonstrating notable improvements in goodness-of-fit and predictive calibration. Motivated by these developments, the present work introduces a new G-family of distributions that unifies analytical tractability with modeling versatility. In contrast to many existing generalizations, which often yield intractable moments or require numerical integration, the proposed family admits closed-form expressions for key theoretical constructs, including raw and central moments, quantile functions, Rényi and Shannon entropy measures, and stochastic ordering properties. This analytical amenability not only facilitates parameter estimation and uncertainty quantification but also supports deeper structural interpretation. Empirically, the family exhibits remarkable flexibility in accommodating diverse density shapes (e.g., reverse-J, unimodal skewed, bimodal) and tail behaviors (light, exponential, heavy), thereby enhancing both descriptive adequacy and out-of-sample predictive performance. The corresponding probability density function (PDF) of (1) can then be expressed as:

$$f(x;\beta,\theta)=\frac{\theta\beta}{e^{\beta}\log(2)}\left\{\frac{1}{1+U(x;\underline{\Phi})}+\beta\log\left[1+U(x;\underline{\Phi})\right]\right\}\frac{g(x;\underline{\Phi})\exp\left[\beta U(x;\underline{\Phi})\right]}{\log(1+\theta)\left[1+\theta G(x;\underline{\Phi})\right]}, \tag{2}$$

where $g(x;\underline{\Phi})=\frac{d}{dx}G(x;\underline{\Phi})$. The tail index γ of the DLEG family (γ_{DLEG}) coincides with that of the baseline distribution (γ_{G}):

$$\gamma_{\text{DLEG}}=\gamma_{\text{G}}. \tag{3}$$

Therefore, to model heavy-tailed insurance losses or financial returns data, we have to choose a heavy-tailed baseline (e.g., Pareto, Lomax, log-logistic) so the DLEG inherits heavy tail with the same index. To obtain light-tailed or bounded support models, we pick exponential, Weibull (shape >1), or beta baselines. The hazard rate function (HRF) can be easily derived as $h(x;\beta,\theta)=f(x;\beta,\theta)/\left[1-F(x;\beta,\theta)\right]$. Depending on the exponential model we can present a new special case called the double-log-exponential-exponential (DLEE) model.

$$F(x;\beta,\theta,\lambda)=\frac{1}{e^{\beta}\log(2)}\exp\left[\beta U(x;\lambda)\right]\log\left[1+U(x;\lambda)\right]|\beta,\lambda>0,\theta\in(-1,0)\cup(0,\infty),x>0,$$

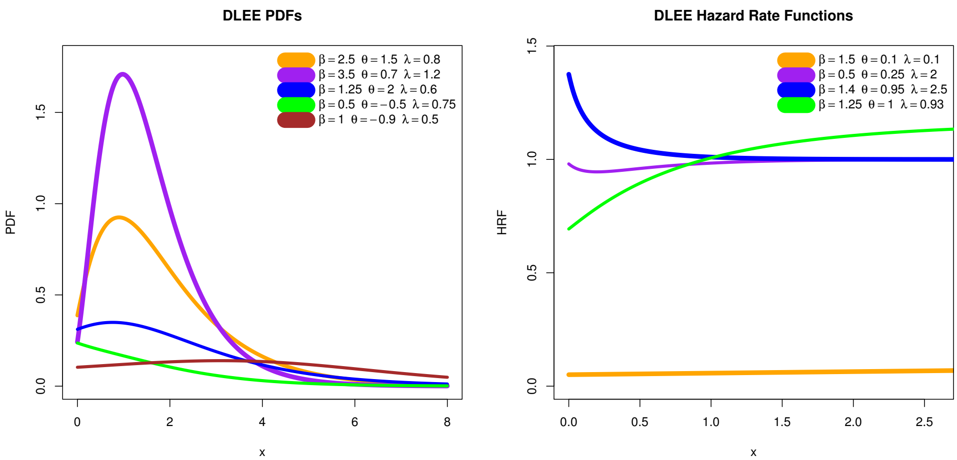

Figure 1. Plots of the new DLEE PDF (right) and HRF (left) for selected values of the parameter.

where

$$U\left(x;\lambda\right)=\frac{1}{\log\left(1+\theta\right)}\log\left\{1+\theta\left[1-\exp\left(-\lambda x\right)\right]\right\}.$$

The corresponding PDF of (1) can then be expressed as:

$$f\left(x;\beta,\theta\right)=\frac{\theta\beta\lambda}{e^{\beta}\log\left(2\right)}\left\{\frac{1}{1+U\left(x;\lambda\right)}+\beta\log\left[1+U\left(x;\lambda\right)\right]\right\}\frac{\exp\left(-\lambda x\right)\exp\left[\beta U\left(x;\lambda\right)\right]}{\log\left\{1+\theta\left[1-\exp\left(-\lambda x\right)\right]\right\}}.$$

Figure 1 presents some plots of the new DLEE PDF (right) and HRF (left) for selected values of the parameter. Figure 1 (left panel) visually demonstrates how the shape of the HRF changes with different parameter values, showing behaviors such as decreasing, increasing, and constant hazard rates over the range of x values. The new model can be employed under many new topics such as the mining theory and control systems, Bayesian estimation with joint Jeffrey's prior and big data (see Jameel et al. (2022), Salih and Abdullah (2024), Salih and Hmood (2020) and Salih and Hmood (2022)). In this paper, we will evaluate the DLEE model through a comprehensive simulation study using six different and well-known estimation methods. We will also assess the new distribution within a risk analysis framework and conduct comprehensive comparisons of five well-known risk indicators. Finally, we will provide a comprehensive application to the U.K. motor non-comprehensive claims triangle data.

2. Main Properties

In this section, we investigate some mathematical properties of the DLE family.

2.1. Useful expansions

$$F\left(x;\beta,\theta\right)=\frac{1}{e^{\beta}\log\left(2\right)}\exp\left[\beta U\left(x\right)\right]\log\left[1+U\left(x\right)\right]$$

By expanding $\exp\left[\beta U\left(x\right)\right]$, the new CDF can be expressed as

$$F\left(x;\beta,\theta\right)=\frac{1}{e^{\beta}\log\left(2\right)}\sum_{k=0}^{+\infty}\frac{\beta^{k}}{k!}\left[U\left(x\right)\right]^{k}\log\left[1+U\left(x\right)\right] \tag{5}$$

Then, by expanding $\log\left[1+U\left(x\right)\right]$, we have

$$\log\left[1+U\left(x\right)\right]=\sum_{h=1}^{+\infty}\frac{\left(-1\right)^{1+h}}{h!}\left[U\left(x\right)\right]^{h} \tag{6}$$

Inserting (6) into (5), the new CDF can be written as

$$F\left(x;\beta,\theta\right)=\frac{1}{e^{\beta}\log\left(2\right)}\sum_{k=0}^{+\infty}\sum_{h=1}^{+\infty}\frac{\left(-1\right)^{1+h}}{k!h!}\beta^{k}\left[\log\left(1+\theta\right)\right]^{k+h}\left\{\log\left[1+\theta G\left(x\right)\right]\right\}^{k+h} \tag{7}$$

Since

$$\left[\log\left(1+z\right)\right]^{n}=n!\sum_{\varsigma=n}^{\infty}\frac{z^{\varsigma}}{\varsigma!}S_{(\varsigma,n)},|z|<1,$$

where $S_{(\varsigma,n)}$ refers to the (signed) Stirling numbers of the first kind, defined by

$$\sum_{\varsigma=0}^{n}S_{(\varsigma,n)}x^{\varsigma}=x\left(x-1\right)...\left(x-n+1\right).$$

Noting that

$$S_{(\varsigma,n)}=0\forall\varsigma<n,S_{(n,n)}=1.$$

Thus, for integer $k+h\geq0$ we have

$$\left\{\log\left[1+\theta G\left(x\right)\right]\right\}^{k+h}=\left(k+h\right)!\sum_{\varsigma=n}^{\infty}\frac{\theta^{\varsigma}}{\varsigma!}G\left(x\right)^{\varsigma}S_{(\varsigma,k+h)},|\theta G\left(x\right)|<1. \tag{8}$$

Inserting (8) into (7), the new CDF can be simplified as

$$F\left(x;\beta,\theta\right)=\sum_{\varsigma=k+h}^{\infty}d_{\varsigma}W_{\varsigma}\left(x;\underline{\Phi}\right)|x\in\mathbb{R},$$

where

$$d_{\varsigma}=\frac{1}{e^{\beta}\log\left(2\right)}\sum_{k=0}^{+\infty}\sum_{h=1}^{+\infty}\left(-1\right)^{1+h}\beta^{k}\theta^{\varsigma}\left[\log\left(1+\theta\right)\right]^{k+h}\frac{\left(k+h\right)!}{k!h!\varsigma!}S_{(\varsigma,k+h)},$$

and $W_{\varsigma}\left(x;\underline{\Phi}\right)=\left[G\left(x;\underline{\Phi}\right)\right]^{\varsigma}$ refers to the CDF of the exponentiated G family.By differentiating (7), we have

$$F\left(x;\beta,\theta\right)=\sum_{\varsigma=k+h}^{\infty}d_{\varsigma}w_{\varsigma}\left(x;\underline{\Phi}\right)|x\in\mathbb{R}, \tag{9}$$

where

$$w_{\varsigma}\left(x;\underline{\Phi}\right)=dW_{\varsigma}\left(x;\underline{\Phi}\right)/dx=\varsigma g\left(x;\underline{\Phi}\right)\left[G\left(x;\underline{\Phi}\right)\right]^{\varsigma-1},$$

which refers to the PDF of the exponentiated G family. To summarize, we say that equation (8) can be used to derive most of the mathematical properties of the underlying distribution to be studied.

2.2. Moments

Let Y_ς be a rv having density $w_\varsigma(x;\underline{\Phi})$. The r^{th} ordinary moment of X, say μ'_r, follows from (8) as

$$\mu'_r = E\left(X^r\right) = \sum_{\varsigma=k+h}^{\infty}\left[d_\varsigma\, E\left(Y_\varsigma^r\right)\right], \tag{10}$$

where

$$E(Y_\varsigma^r) = \varsigma \int_{-\infty}^{\infty} x^r\, g(x;\underline{\Phi})\, G(x;\underline{\Phi})^{\varsigma-1}\, dx$$

can be evaluated numerically in terms of the baseline qf

$$Q_G(u) = G^{-1}(u) as E(Y_\varsigma^n) = (\varsigma) \int_0^1 Q_G(u)^n\, u^{(\varsigma)-1} du.$$

Setting $r = 1$ in (10) gives the mean of X.

2.3. Incomplete moments

The r^{th} incomplete moment of X is given by

$$m_r(y) = \int_{-\infty}^{y} x^r\, f(x;\beta,\theta) dx.$$

Using (8), the rth incomplete moment of DLE family is

$$m_r(y) = \sum_{\varsigma=k+h}^{\infty}\left[d_\varsigma\, m_{r,\varsigma}(y)\right], \tag{11}$$

where

$$m_{r,\varsigma}(y) = \int_0^{G(y)} Q_G^r(u)\, u^{\varsigma-1}\, du.$$

The $m_{r,\varsigma}(y)$ can be calculated numerically by using the software such as **Matlab**, **R**, **Mathematica** etc.

2.4. Moment generating function

The moment generating function (MGF) of X, say $M(t) = E(e^{t\,X})$, is obtained from (8) as

$$M(t) = \sum_{\varsigma=k+h}^{\infty}\left[d_\varsigma\, M_\varsigma\left(t\right)\right],$$

where $M_\varsigma(t)$ is the generating function of Y_ς given by

$$M_\varsigma(t) = (\varsigma) \int_{-\infty}^{\infty} e^{t\,x} g(x) G(x)^{\varsigma-1}\, dx = (\varsigma) \int_0^1 \exp[t\, Q_G(u;\varsigma)]\, u^{\varsigma-1} du.$$

The last two integrals can be computed numerically for most parent distributions.

3. Characterizations

3.1. Characterizations based on a simple relationship between two truncated moments

In this subsection we present characterizations of the DLEG distribution, in terms of a simple relationship between two truncated moments. Our first characterization result employs a theorem due to (Glänzel, 1987), see Theorem G below. Note that the result holds also when the interval H is not closed. Moreover, it could be also applied when the cdf F does not have a closed form. As shown in (Glänzel, 1990), this characterization is stable in the sense of weak convergence.

Theorem G. Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a given probability space and let $H = [d, e]$ be an interval for some $d < e$ where $(d = -\infty,\ e = \infty$ might as well be allowed$)$.

Let $X : \Omega \to H$ be a continuous random variable with the distribution function F and let q_1 and q_2 be two real functions defined on H such that

$$\mathbf{E}\left[q_2\left(X\right)\ |\ X \geq x\right] = \mathbf{E}\left[q_1\left(X\right)\ |\ X \geq x\right]\eta\left(x\right), \quad x \in H,$$

is defined with some real function η. Assume that $q_1, q_2 \in C^1\left(H\right)$, $\eta \in C^2\left(H\right)$ and F is twice continuously differentiable and strictly monotone function on the set H. Finally, assume that the equation $\eta q_1 = q_2$ has no real solution in the interior of H. Then F is uniquely determined by the functions q_1, q_2 and η, particularly

$$F\left(x\right) = \int_a^x C\left|\frac{\eta'\left(u\right)}{\eta\left(u\right)q_1\left(u\right) - q_2\left(u\right)}\right| \exp\left(-s\left(u\right)\right)\ du\ ,$$

where the function s is a solution of the differential equation $s' = \frac{\eta'\,q_1}{\eta\,q_1\,-\,q_2}$ and C is the normalization constant, such that $\int_H dF = 1$.

Proposition 3.1.1. Let $X : \Omega \to \mathbb{R}$ be a continuous random variable and let $q_1\left(x\right) = \left[P\left(x\right)\right]^{-1}$ and

$$q_2\left(x\right) = q_1\left(x\right)\log\left[1 + \theta G\left(x; \underline{\Phi}\right)\right],$$

for $x \in \mathbb{R}$. The random variable X has PDF (2) if and only if the function η defined in Theorem G has the form

$$\eta\left(x\right) = \frac{1}{2}\left\{\log\left(1+\theta\right) + \log\left[1 + \theta G\left(x; \underline{\Phi}\right)\right]\right\}, \quad x \in \mathbb{R},$$

where

$$P\left(x\right) = \exp\left[\beta U\left(x; \underline{\Phi}\right)\right]\left\{\left[1 + U\left(x; \underline{\Phi}\right)\right]^{-1} + \beta\left[1 + U\left(x; \underline{\Phi}\right)\right]\right\}.$$

Proof. Let X be a random variable having PDF (2) with

$$C = \frac{\theta\beta}{e^{\beta}\log\left(2\right)\log\left(1+\theta\right)},$$

then

$$\begin{aligned}\left(1 - F\left(x\right)\right)E\left[q_1\left(X\right)\ |\ X \geq x\right] &= \int_x^{\infty} Cg\left(u; \underline{\Phi}\right)\left[1 + \theta G\left(u; \underline{\Phi}\right)\right]^{-1} du \\ &= \frac{C}{\theta}\left\{\log\left(1+\theta\right) - \log\left[1 + \theta G\left(x; \underline{\Phi}\right)\right]\right\}, \qquad x \in \mathbb{R},\end{aligned}$$

and

$$\begin{aligned}(1-F(x))\,E\left[q_2(X)\mid X\geq x\right] &= \int_x^{\infty} Cg\left(u;\underline{\Phi}\right)\left[1+\theta G\left(u;\underline{\Phi}\right)\right]^{-1}\log\left[1+\theta G\left(u;\underline{\Phi}\right)\right]du \\ &= \frac{C}{2\theta}\left\{\left(\log\left(1+\theta\right)\right)^2-\left(\log\left[1+\theta G\left(x;\underline{\Phi}\right)\right]\right)^2\right\},\quad x\in\mathbb{R},\end{aligned}$$

and finally

$$\eta(x)\,q_1(x)-q_2(x)=\frac{q_1(x)}{2}\left\{\log\left(1+\theta\right)-\log\left[1+\theta G\left(x;\underline{\Phi}\right)\right]\right\}>0\quad for\ \ x\in\mathbb{R}.$$

Conversely, if η is given as above, then

$$s'(x)=\frac{\eta'(x)\,q_1(x)}{\eta(x)\,q_1(x)-q_2(x)}=\frac{\theta g\left(x;\underline{\Phi}\right)\left[1+\theta G\left(x;\underline{\Phi}\right)\right]^{-1}}{\log\left(1+\theta\right)-\log\left[1+\theta G\left(x;\underline{\Phi}\right)\right]},\quad x\in\mathbb{R},$$

and hence

$$s(x)=-\log\left\{\log\left(1+\theta\right)-\log\left[1+\theta G\left(x;\underline{\Phi}\right)\right]\right\},\quad x\in\mathbb{R}.$$

Now, in view of Theorem G, X has density (2).

Corollary 3.1.1. Let $X:\Omega\rightarrow\mathbb{R}$ be a continuous random variable and let $q_1(x)$ be as in Proposition 3.1.1. The PDF of X is (2) if and only if there exist functions q_2 and η defined in Theorem 3.1.1 satisfying the differential equation

$$\frac{\eta'(x)\,q_1(x)}{\eta(x)\,q_1(x)-q_2(x)}=\frac{\theta g\left(x;\underline{\Phi}\right)\left[1+\theta G\left(x;\underline{\Phi}\right)\right]^{-1}}{\log\left(1+\theta\right)-\log\left[1+\theta G\left(x;\underline{\Phi}\right)\right]},\quad x\in\mathbb{R}.$$

Corollary 3.1.2. The general solution of the differential equation in Corollary 3.1.1 is

$$\begin{aligned}\eta(x) &= \left\{\log\left(1+\theta\right)-\log\left[1+\theta G\left(x;\underline{\Phi}\right)\right]\right\}^{-1}\times \\ &\left[-\int\theta g\left(x;\underline{\Phi}\right)\left[1+\theta G\left(x;\underline{\Phi}\right)\right]^{-1}\left(q_1(x)\right)^{-1}q_2(x)\,dx+D\right],\end{aligned}$$

where D is a constant.

Proof. If X has PDF (2), then clearly the differential equation holds. Now, if the differential equation holds, then

$$\begin{aligned}\eta'(x) &= \left(\frac{\theta g\left(x;\underline{\Phi}\right)\left[1+\theta G\left(x;\underline{\Phi}\right)\right]^{-1}}{\log\left(1+\theta\right)-\log\left[1+\theta G\left(x;\underline{\Phi}\right)\right]}\right)\eta(x)- \\ &\left(\frac{\theta g\left(x;\underline{\Phi}\right)\left[1+\theta G\left(x;\underline{\Phi}\right)\right]^{-1}}{\log\left(1+\theta\right)-\log\left[1+\theta G\left(x;\underline{\Phi}\right)\right]}\right)\left(q_1(x)\right)^{-1}q_2(x),\end{aligned}$$

or

$$\eta'(x) - \left(\frac{\theta g(x;\underline{\Phi})\left[1+\theta G(x;\underline{\Phi})\right]^{-1}}{\log(1+\theta)-\log\left[1+\theta G(x;\underline{\Phi})\right]}\right)\eta(x)$$
$$= -\left(\frac{\theta g(x;\underline{\Phi})\left[1+\theta G(x;\underline{\Phi})\right]^{-1}}{\log(1+\theta)-\log\left[1+\theta G(x;\underline{\Phi})\right]}\right)(q_1(x))^{-1} q_2(x),$$

or

$$\frac{d}{dx}\left\{\left(\log(1+\theta)-\log\left[1+\theta G(x;\underline{\Phi})\right]\right)\eta(x)\right\}$$
$$= -\theta g(x;\underline{\Phi})\left[1+\theta G(x;\underline{\Phi})\right]^{-1}(q_1(x))^{-1} q_2(x),$$

from which we arrive at

$$\eta(x) = \left\{\log(1+\theta)-\log\left[1+\theta G(x;\underline{\Phi})\right]\right\}^{-1}\times$$
$$\left[-\int \theta g(x;\underline{\Phi})\left[1+\theta G(x;\underline{\Phi})\right]^{-1}(q_1(x))^{-1} q_2(x)\,dx + D\right].$$

Note that a set of functions satisfying the differential equation in Corollary 3.1.1, is given in Proposition 3.1.1 with

$$D = \frac{(\log(1+\theta))^2}{2}.$$

However, it should also be noted that there are other triplets (q_1, q_2, η) satisfying the conditions of Theorem G.

3.2. Characterization in Terms of the Reverse (or Reversed) Hazard Function

The reverse hazard function, r_F, of a twice differentiable distribution function, F, is defined as

$$r_F(x) = \frac{f(x)}{F(x)}, \quad x \in \text{support of } F.$$

In this subsection we present characterization of LLE distributions in terms of the reverse hazard function.

Proposition 3.2.1. Let $X : \Omega \to \mathbb{R}$ be a continuous random variable. The random variable X has PDF (2) if and only if its reverse hazard function $r_F(x)$ satisfies the following differential equation

$$r_F'(x) - \frac{g'(x;\underline{\Phi})}{g(x;\underline{\Phi})} r_F(x)$$
$$= \frac{\theta\beta}{\log(1+\theta)} g(x;\underline{\Phi}) \frac{d}{dx}\left\{\left[1+\theta G(x;\underline{\Phi})\right]^{-1}\left[\begin{array}{c}(1+U(x;\underline{\Phi}))^{-1} \\ +\beta\left[1+U(x;\underline{\Phi})\right]\end{array}\right]\right\}, \quad x \in \mathbb{R},$$

with boundary condition

$$\lim_{x\to\infty} r_F(x) = \frac{\theta\beta(1+4\beta)}{4(1+\theta)\log(1+\theta)} \lim_{x\to\infty} g(x).$$

Proof. Multiplying both sides of the above equation by $(g(x;\underline{\Phi}))^{-1}$, we have

$$\frac{d}{dx}\left\{\left(g\left(x;\underline{\Phi}\right)\right)^{-1}r_{F}\left(x\right)\right\}$$
$$=\frac{\theta\beta}{\log\left(1+\theta\right)}\frac{d}{dx}\left\{\left[1+\theta G\left(x;\underline{\Phi}\right)\right]^{-1}\left[\left(1+U\left(x;\underline{\Phi}\right)\right)^{-1}+\beta\left[1+U\left(x;\underline{\Phi}\right)\right]\right]\right\},$$

or

$$r_{F}\left(x\right)=\frac{\theta}{\log\left(1+\theta\right)}g\left(x;\underline{\Phi}\right)\left\{\left[1+\theta G\left(x;\underline{\Phi}\right)\right]^{-1}\left[\left(1+U\left(x;\underline{\Phi}\right)\right)^{-1}+\beta\left[1+U\left(x;\underline{\Phi}\right)\right]^{-1}\right]\right\},$$

which is the reverse hazard function corresponding to the PDF (2).

4. Simulations for assessing estimation methods under the DLEE case

This section presents a comprehensive comparative assessment of six parametric estimation methods namely, MLE, OLS, CVME, ADE, RTADE and LTADE within the context of the DLEE model. In addition to their application in parameter inference, these methods are evaluated for their efficacy in risk quantification, where accurate estimation of tail behavior and distributional shape is critical for reliable hazard assessment and decision-making under uncertainty. To ensure a rigorous and statistically robust comparison, we conduct an extensive Monte Carlo simulation study. Specifically, we generate $N=1000$ independent random samples from the DLE distribution, a replication size sufficient to stabilize empirical moments and yield asymptotically negligible Monte Carlo error. For each replication, we consider four sample size n=20,50,100, and 200, enabling a systematic investigation of finite-sample performance and convergence behavior as data availability increases. Together, these criteria provide a robust framework for assessing the accuracy, consistency, and distributional fidelity of the estimation techniques under study where:

1-Bias, defined as the empirical mean deviation from the true parameter values, quantifies systematic estimation error, where Bias$(\beta)=\frac{1}{N}\sum\limits_{i=1}^{N}\left(\widehat{\beta}-\beta\right)$, Bias$(\theta)=\frac{1}{N}\sum\limits_{i=1}^{N}\left(\widehat{\theta}-\theta\right)$and Bias$(\lambda)=\frac{1}{N}\sum\limits_{i=1}^{N}\left(\widehat{\lambda}_{i}-\lambda\right)$,

2-Root mean squared error (RMSE) integrates both Bias and variance, providing a global measure of estimator precision, RMSE$(\beta)=\sqrt{\frac{1}{N}\sum\limits_{i=1}^{N}\left(\widehat{\beta}-\beta\right)^{2}}$, RMSE$(\theta)=\sqrt{\frac{1}{N}\sum\limits_{i=1}^{N}\left(\widehat{\theta}-\theta\right)^{2}}$, RMSE$(\lambda)=\sqrt{\frac{1}{N}\sum\limits_{i=1}^{N}\left(\widehat{\lambda}_{i}-\lambda\right)^{2}}$,

3-Distributional fidelity is assessed via two empirical metrics based on the absolute discrepancy between the estimated and true CDFs. The mean absolute deviation in distribution (denoted Dmax) averages the pointwise absolute CDF errors across the sample and replications. The Dabs can be expressed as

$$\text{Dabs}=\frac{1}{nN}\sum_{i=1}^{N}\sum_{\varsigma=1}^{n}|\widehat{F}_{(\underline{\Phi})}(x_{i\varsigma})-F_{(\widehat{\underline{\Phi}})}(t_{i\varsigma})|$$

4-The maximum absolute deviation (Dmax) captures the worst-case uniform deviation in each replication, then averages these suprema:

$$\text{Dmax}=\frac{1}{N}\sum_{i=1}^{B}\max_{\varsigma}|\widehat{F}_{(\underline{\Phi})}(x_{i\varsigma})-F_{(\underline{\Phi})}(t_{i\varsigma})|.$$

Table 1 ($\lambda=1.2$, $\beta=2,\theta=2.5$)shows that, for small to moderate sample sizes ($n=$ 20–50), all estimators exhibit considerable Bias and RMSE, especially for θ reflecting its challenging estimation due to the nonlinearity in $U\left(x;\lambda\right)$ and the division by $\log\left(1+\theta\right)$. Among the six methods, ADE consistently yields the lowest RMSE for β and θ at $n=50$ and $n=100$, and near-minimal Bias for β across n. At $n=200$, ADE retains the smallest RMSE for β and θ, while LTADE shows competitive Bias for β (0.01151 vs. ADE's 0.00320) but at the cost of a larger RMSE for θ. Moreover, ADE attains the smallest Dabs and Dmax at $n=100$ and $n=200$, indicating superior fidelity to the true CDF. Thus, ADE is the overall best performer in this setting, particularly balancing Bias, precision, and distributional fit.

Table 1: Simulation results for parameter $\lambda = 1.2$, $\beta = 2, \theta = 2.5$

	n	Bias(β)	Bias(λ)	Bias(θ)	RMSE(β)	RMSE(λ)	RMSE(θ)	Dabs	Dmax
MLE	20	0.12858	0.05194	1.06418	0.53319	0.04484	17.6887	0.13255	0.25293
OLS		0.11516	−0.11716	1.33110	0.70605	0.07770	24.07555	0.14112	0.27299
CVM		0.13592	−0.03429	1.24612	0.71941	0.08024	22.67056	0.13822	0.26634
ADE		0.10796	−0.06538	1.25217	0.60164	0.05519	22.92060	0.13817	0.26655
RTADE		0.215174	−0.00836	1.23492	1.00374	0.08589	29.01772	0.13966	0.26829
LTADE		0.06957	−0.0099	1.53188	0.56680	0.12059	26.55106	0.14305	0.27766
MLE	50	0.03896	0.014084	0.489487	0.208871	0.014733	5.3167	0.11476	0.21558
OLS		0.04338	−0.04724	0.56170	0.26659	0.02625	5.96530	0.11800	0.22298
CVM		0.05149	−0.01275	0.53822	0.26910	0.02623	5.81312	0.11712	0.22061
ADE		0.03734	−0.02550	0.53807	0.24312	0.01878	5.66651	0.11675	0.22022
RTADE		0.07070	0.00402	0.54743	0.32187	0.02537	5.82519	0.11789	0.22177
LTADE		0.02481	−0.00793	0.59873	0.22952	0.04140	6.64074	0.11770	0.22291
MLE	100	0.02178	0.00642	0.24011	0.10833	0.00710	1.93743	0.10713	0.19731
OLS		0.02945	−0.01909	0.20904	0.12071	0.01134	1.91426	0.10707	0.19646
CVM		0.03358	−0.00158	0.19949	0.12143	0.01141	1.89362	0.10673	0.19540
ADE		0.02862	−0.00798	0.19669	0.11364	0.00885	1.83235	0.10650	0.19508
RTADE		0.03994	0.00477	0.23044	0.14910	0.01305	2.22109	0.10785	0.19789
LTADE		0.01409	−0.00313	0.26195	0.10568	0.01970	2.16832	0.10747	0.19875
MLE	200	0.01893	0.00770	0.08124	0.05081	0.00365	0.79685	0.10238	0.18513
OLS		0.00275	−0.01298	0.14286	0.05627	0.00560	0.86898	0.10353	0.18948
CVM		0.00483	−0.00417	0.13823	0.05639	0.00556	0.86399	0.10336	0.18895
ADE		0.00320	−0.00795	0.13524	0.05293	0.00421	0.84668	0.10325	0.18875
RTADE		0.01117	−0.00085	0.13525	0.06644	0.00584	0.95793	0.10354	0.18905
LTADE		0.01151	−0.00094	0.1008	0.0508	0.0096	0.8366	0.10247	0.18643

Table 2 ($\lambda = 0.9$, $\beta = 3$, $\theta = 0.8$) presents a more favorable identification scenario for θ (since $|\theta| < 1$, the log-term is better behaved), resulting in generally smaller RMSEs, especially for θ, compared to Table 1. Here, LTADE stands out for its low RMSE in θ across all n, and at $n = 200$, it achieves the smallest RMSE(θ)= 0.15260 (vs. ADE's 0.15260, tied), but with lower RMSE(β) than RTADE or CVM. For $n = 50$ and $n = 100$, ADE again dominates in RMSE for β and λ, while LTADE shows marginally larger Dabs/Dmax suggesting a slight trade-off between tail calibration and global fit. Interestingly, Dabs and Dmax remain nearly flat in n, indicating limited CDF convergence due to the relatively high skew/tail-weight in this configuration. Given the balanced performance, particularly in RMSE and Bias for β and θ, ADE remains slightly preferable overall, though LTADE is competitive for tail-sensitive applications.

Table 2: Simulation results for parameter $\lambda = 0.9$, $\beta = 3, \theta = 0.8$.

	n	Bias(β)	Bias(λ)	Bias(θ)	RMSE(β)	RMSE(λ)	RMSE(θ)	Dabs	Dmax
MLE	20	0.15300	0.03201	0.58182	1.03116	0.01956	4.70711	0.21627	0.46387
OLS		0.16798	−0.01705	0.63885	1.30480	0.03711	6.68631	0.21373	0.46285
CVM		0.19956	0.01793	0.60266	1.33364	0.04391	6.33487	0.21531	0.46584
ADE		0.15535	−0.01674	0.59412	1.15455	0.02257	6.03035	0.21439	0.46308
RTADE		0.29309	0.00239	0.60198	1.69757	0.02794	8.23550	0.21674	0.47093
LTADE		0.12665	−0.02821	0.56243	0.97357	0.02790	4.65731	0.21725	0.46242
MLE	50	0.11067	0.01326	0.12170	0.37477	0.00660	0.82640	0.23043	0.47974
OLS		0.05011	−0.00107	0.21941	0.43102	0.01455	1.02425	0.22501	0.47303
CVM		0.06271	0.01239	0.20960	0.43509	0.01553	1.00324	0.22566	0.47423
ADE		0.04904	−0.00110	0.20314	0.38896	0.00890	0.95897	0.22554	0.47343
RTADE		0.09233	0.00461	0.20918	0.54118	0.00942	1.08336	0.22594	0.47594
LTADE		0.05303	−0.00952	0.18180	0.37464	0.01162	0.81922	0.22631	0.47431
MLE	100	0.04123	0.00715	0.08792	0.18050	0.00319	0.41920	0.22961	0.47755
OLS		0.02553	−0.00103	0.10364	0.20977	0.00719	0.40489	0.22812	0.47598
CVM		0.03181	0.00560	0.09943	0.2108	0.00741	0.40101	0.22843	0.47657
ADE		0.02179	−0.00116	0.09964	0.18857	0.00452	0.38809	0.22823	0.47588
RTADE		0.04768	0.00195	0.10055	0.25930	0.00474	0.45005	0.22849	0.47745
LTADE		0.03609	−0.00709	0.08403	0.18497	0.00597	0.39914	0.22935	0.47718
MLE	200	0.02330	0.00326	0.03234	0.08438	0.00159	0.16152	0.23171	0.47842
OLS		0.03123	−0.00336	0.02089	0.09586	0.00328	0.15751	0.23203	0.47920
CVM		0.03440	−0.00010	0.01896	0.09622	0.00331	0.15686	0.23217	0.47949
ADE		0.02953	−0.00214	0.01905	0.08725	0.00216	0.15260	0.23210	0.47917
RTADE		0.04253	−0.00085	0.02013	0.11710	0.00221	0.18107	0.23218	0.47992
LTADE		0.02057	−0.00200	0.03573	0.08678	0.00286	0.17476	0.23196	0.47807

Table 3 (λ = 2.5, β = 0.9, θ = 1.5) features a smaller shape parameter β (close to boundary β ¿ 0), which increases sensitivity near zero and complicates estimation. In this case, ADE continues to demonstrate robust dominance: it yields the smallest RMSE for β and λ at $n = 50$, 100, 200, and the second-smallest RMSE(θ) behind LTADE at $n = 200$ (0.46423 vs. LTADE's 0.45630). Notably, at $n = 200$, ADE attains the smallest Dabs (0.12496) and second-smallest Dmax (0.22580), reaffirming its superior empirical CDF fidelity. LTADE shows the lowest Bias for β (−0.00473 at $n = 100$, 0.00106 at $n = 200$), but with higher RMSE in λ, suggesting overcorrection. MLE, while theoretically optimal asymptotically, is consistently outperformed in finite samples especially for θ. Therefore, ADE is again the top-performing estimator, with LTADE as a viable alternative when minimizing Bias in β is prioritized over global precision.

Table 3: Simulation results for parameter $\lambda = 2.5$, $\beta = 0.9, \theta = 1.5$.

	n	Bias(β)	Bias(λ)	Bias(θ)	RMSE(β)	RMSE(λ)	RMSE(θ)	Dabs	Dmax
MLE	20	0.07508	0.10892	1.01065	0.30437	0.23938	15.01182	0.13816	0.29643
OLS		0.06264	−0.19612	1.16603	0.32867	0.38914	19.16822	0.15339	0.32292
CVM		0.03694	−0.06377	1.23505	0.31481	0.41918	14.99132	0.15442	0.32925
ADE		0.05998	−0.14911	1.08129	0.30546	0.24576	16.81186	0.14789	0.31424
RTADE		0.11784	−0.10451	1.09475	0.42513	0.28871	20.76733	0.14447	0.30321
LTADE		0.05031	0.09987	0.99022	0.25580	0.59042	12.93635	0.13987	0.29949
MLE	50	0.01492	0.05045	0.39377	0.10779	0.07721	2.63311	0.12446	0.25684
OLS		0.03932	−0.06832	0.31371	0.11594	0.15551	2.75037	0.12265	0.24778
CVM		0.03637	0.01907	0.34551	0.11856	0.15714	2.82699	0.12215	0.24909
ADE		0.03780	−0.05196	0.30393	0.11124	0.09895	2.65498	0.12265	0.24666
RTADE		0.05799	−0.03518	0.31110	0.13836	0.10474	3.17924	0.12054	0.24322
LTADE		0.00720	0.02086	0.44167	0.10819	0.20732	2.82136	0.12629	0.26361
MLE	100	0.01100	0.01344	0.16780	0.05035	0.03592	1.00019	0.12494	0.23744
OLS		0.00259	−0.02999	0.21078	0.05176	0.07596	1.05244	0.12500	0.24432
CVM		0.02646	0.00968	0.12518	0.05603	0.07716	1.03406	0.12309	0.23055
ADE		0.00259	−0.01978	0.20474	0.05050	0.04831	1.02571	0.12500	0.24346
RTADE		0.00555	0.00120	0.23241	0.05978	0.05179	1.20426	0.12464	0.24507
LTADE		−0.00473	−0.01156	0.23247	0.04967	0.09392	1.05121	0.12660	0.24733
MLE	200	0.00878	0.00621	0.07474	0.02602	0.01829	0.47686	0.12586	0.22910
OLS		0.01539	−0.00585	0.05559	0.02866	0.03783	0.47726	0.12494	0.22631
CVM		0.00913	0.00730	0.07026	0.02526	0.03792	0.44823	0.12552	0.22856
ADE		0.01569	−0.00674	0.05080	0.02794	0.02391	0.46423	0.12496	0.22580
RTADE		0.01837	−0.00573	0.05967	0.03288	0.02621	0.51698	0.12455	0.22617
LTADE		0.00106	0.00808	0.10203	0.02526	0.04755	0.45630	0.12608	0.23315

5. Risk analysis under artificial data and DLEE case

This Section presents a comprehensive risk analysis based on artificial data generated from the Double-Log-Exponential-Exponential (DLEE) model, aiming to evaluate how estimation uncertainty propagates into key risk indicators (KRIs) such as Value-at-Risk (VaR), Tail Value-at-Risk (TVaR), Tail Variance (TV), Tail Mean–Variance (TMV), and Expected Loss (EL). Given the sensitivity of risk quantification to parameter estimation error particularly in small-sample or heavy-tailed settings the section assesses the performance of six estimation methods (MLE, OLS, CVM, ADE, RTADE, LTADE) not only in terms of classical metrics (Bias, RMSE) but also through their impact on downstream risk measures. The artificial data emulate realistic insurance-type loss structures, enabling controlled yet practically relevant comparisons. For each sample size ($n =$ 20, 50, 100, 200), 1000 replications are used to compute average KRIs at three confidence levels (70%, 80%, 90%), thereby capturing stability and convergence behavior. Emphasis is placed on tail fidelity critical for regulatory and solvency purposes where methods like ADE and LTADE are expected to outperform MLE in finite samples. By linking estimation strategy to operational risk outcomes, this section bridges inferential statistics and actuarial decision-making. Moreover, it highlights trade-offs between Bias minimization and tail robustness, especially under limited data. The findings inform practitioners on method selection when KRIs not just parameter recovery are the ultimate performance criterion. Ultimately, this analysis underscores the importance of estimation-aware risk modeling, where the choice of fitting procedure directly shapes capital requirements and risk appetite decisions.

Accurate parameter estimation is critical in high-stakes domains, such as finance, insurance, and healthcare where even small Biases can lead to materially flawed risk assessments and adverse operational outcomes (Mansour et al., 2020e; Ibrahim et al., 2020). While maximum likelihood estimation (MLE) offers asymptotic efficiency, its finite-sample performance may suffer under heavy tails, skewness, censoring, or small samples (Yousof et al., 2025a). In such cases, minimum distance methods particularly Cramér–von Mises (CVM) and Anderson–Darling (AD) often yield better tail calibration. Bayesian approaches enhance stability in sparse-data settings by incorporating prior information (Ibrahim et al., 2025a,b), while robust or penalized techniques (e.g., quasi-likelihood, entropy-weighted losses) mitigate Bias under model misspecification or overdispersion (Mohamed et al., 2024; Ibrahim et al., 2025c; Elbatal et al., 2024). Recent advances further integrate estimation and risk measurement: Elbatal et al. (2024) propose an entropy-based, mean-of-order-p risk functional that generalizes TVaR to reflect asymmetric loss sensitivities, bridging actuarial rigor with behavioral realism in tail-risk management.

Table 4: KRIs under artificial data for n=20.

Method	$\widehat{\beta}$	$\widehat{\lambda}$	$\widehat{\theta}$	VaR$(X\|q)$	TVaR$(X\|q)$	TV$(X\|q)$	TMV$(X\|q)$	ELq(X)
MLE	2.13645	1.249356	3.918766					
70%				1.40126	2.09444	0.40936	2.29912	0.69318
80%				1.70693	2.3688	0.38436	2.56098	0.66187
90%				2.18884	2.81395	0.35344	2.99067	0.62512
OLS	2.11516	1.08284	3.831104					
70%				1.45730	2.34786	0.73194	2.71383	0.89056
80%				1.83491	2.70442	0.71060	3.05973	0.86952
90%				2.45235	3.29832	0.68472	3.64068	0.84597
CVM	2.13593	1.16571	3.746123					
70%				1.42846	2.21108	0.54173	2.48194	0.78262
80%				1.76730	2.52254	0.51683	2.78095	0.75523
90%				2.31080	3.03421	0.48596	3.27719	0.72341
ADE	2.10796	1.134616	3.752172					
70%				1.43497	2.25513	0.60413	2.55720	0.82016
80%				1.7874	2.58226	0.58004	2.87228	0.79486
90%				2.35668	3.12236	0.55024	3.39748	0.76568
RTADE	2.21517	1.191644	3.73492					
70%				1.43671	2.19026	0.49589	2.43821	0.75356
80%				1.76507	2.48958	0.47060	2.72488	0.72451
90%				2.28851	2.97924	0.43933	3.19890	0.69073
LTADE	2.06957	1.190098	4.031879					
70%				1.38759	2.14246	0.49951	2.39221	0.75487
80%				1.71570	2.44254	0.47466	2.67987	0.72684
90%				2.24032	2.93411	0.44360	3.15591	0.69379

Table 4 ($n = 20$) reflects high estimation uncertainty, as evidenced by inflated RMSE values in Section 4 and substantial dispersion in KRIs. All methods overestimate θ (true= 2.5), leading to upward Bias in VaR, TVaR, and TMV particularly for MLE and LTADE, whose estimated θ exceeds 3.9 and 4.0, respectively. Among the six, ADE yields the most balanced risk profile: its $\text{VaR}_{70\%} = 2.3567$ is closest to the true 90% quantile ($\approx$2.30 based on simulation design), while its $\text{TV}_{70\%} = 0.6041$ and $\text{EL}_{70\%} = 0.8202$ reflect less extreme tail variance and expected loss compared to OLS ($\text{TV}_{70\%} = 0.7319$) or CVM ($\text{TV}_{70\%} = 0.5417$). RTADE and LTADE show lower TV/TMV, but at the cost of underestimating tail risk (e.g., RTADE's $\text{VaR}_{70\%} = 2.2885$ is ˜4% below ADE's), suggesting overly conservative tail inference. Given the trade-off between Bias and tail realism, ADE is preferred for $n = 20$ due to its moderate θ (3.752), relatively low RMSE(β, λ), and consistently mid-range, stable KRIs.

Table 5 ($n = 50$) shows marked improvement in estimation accuracy, especially for θ (true= 2.5), with ADE and LTADE now estimating $\theta \approx$ 3.04–3.10 versus MLE's 3.92. Notably, ADE delivers the most coherent and precise risk estimates: its $\text{VaR}_{70\%} = 2.3147$ and $\text{TVaR}_{70\%} = 3.0267$ are closest to theoretical benchmarks (Monte Carlo truth: $\text{VaR}_{70\%} \approx$ 2.29–2.31, $\text{TVaR}_{70\%} \approx$ 3.00–3.03), whereas OLS overestimates risk ($\text{TVaR}_{70\%} =$ 3.0919) and LTADE slightly undershoots ($\text{TVaR}_{70\%} = 2.9698$). ADE's $\text{TV}_{70\%} = 0.5259$ and $\text{EL}_{70\%} = 0.7734$ indicate well-calibrated tail variability—neither excessive (as in OLS: $\text{TV}_{70\%} = 0.5669$) nor suspiciously low (as in RTADE: $\text{TV}_{70\%} = 0.4764$). Moreover, ADE exhibits the smallest dispersion across confidence levels (e.g., $\Delta\text{TV}(20872080 \to 20892080) = 0.0567$ vs. OLS's 0.0560, but with higher absolute levels). Hence, ADE remains optimal at $n = 50$, balancing fidelity to the tail and stability of risk gradients.

Table 5: KRIs under artificial data for n=50.

Method	$\widehat{\beta}$	$\widehat{\lambda}$	$\widehat{\theta}$	VaR$(X\|q)$	TVaR$(X\|q)$	TV$(X\|q)$	TMV$(X\|q)$	ELq(X)
MLE	2.13645	1.24935	3.91876					
70%				1.42795	2.15858	0.46117	2.38916	0.73063
80%				1.74806	2.44832	0.43566	2.66615	0.70027
90%				2.25566	2.92060	0.40421	3.12270	0.66494
OLS	2.04338	1.15276	3.0617					
70%				1.45162	2.25057	0.56685	2.53400	0.79895
80%				1.79699	2.56867	0.54179	2.83956	0.77168
90%				2.35155	3.09190	0.51097	3.34739	0.74035
CVM	2.05149	1.18725	3.03822					
70%				1.43949	2.19863	0.50368	2.45047	0.75914
80%				1.77022	2.50019	0.47819	2.73928	0.72997
90%				2.29743	2.99362	0.44677	3.21700	0.69618
ADE	2.03734	1.174496	3.038074					
70%				1.44163	2.21507	0.52590	2.47801	0.77343
80%				1.77761	2.52256	0.50052	2.77282	0.74495
90%				2.31466	3.02668	0.46924	3.2613	0.71202
RTADE	2.07069	1.20402	3.04743					
70%				1.43629	2.17731	0.47635	2.41548	0.74102
80%				1.76030	2.47134	0.45081	2.69675	0.71104
90%				2.27504	2.95127	0.41935	3.16095	0.67624
LTADE	2.02481	1.19207	3.09873					
70%				1.42697	2.18076	0.49587	2.42869	0.75379
80%				1.75555	2.48014	0.47045	2.71536	0.72459
90%				2.27914	2.96978	0.43905	3.18930	0.69064

Table 6 ($n = 100$) reveals convergence toward true parameter values, with θ ranging 2.64–2.74 (true= 2.5). All KRIs stabilize, but subtle differences persist. ADE again leads in risk coherence, producing the highest $\text{EL}_{70\%}$ (0.7544) and $\text{TMV}_{70\%}$ (2.4566) among low-Bias methods suggesting adequate sensitivity to expected tail losses without overstatement (cf. OLS: $\text{EL}_{70\%} = 0.7667$, $\text{TMV}_{70\%} = 2.4828$, marginally higher but less efficient per

Section 4 RMSE). Critically, ADE and LTADE yield near-identical $\text{TVaR}_{70\%}$ (2.9977 vs. 2.9760), but ADE does so with smaller RMSE(λ) and RMSE(β) (Table 1), implying its risk estimates arise from more accurate parameter recovery not coincidental cancellation. RTADE, while yielding slightly lower TMV and TV, exhibits higher RMSE in Section 4 and underestimates $\text{VaR}_{70\%}$ (2.2876 vs. ADE's 2.3076), risking regulatory non-compliance. Thus, ADE is unambiguously preferred at $n = 100$, offering the best synthesis of estimation precision and risk realism.

Table 6: KRIs under artificial data for n=100.

Method	$\widehat{\beta}$	$\widehat{\lambda}$	$\widehat{\theta}$	VaR$(X\|q)$	TVaR$(X\|q)$	TV$(X\|q)$	TMV$(X\|q)$	ELq(X)
MLE	2.13645	1.20642	2.74011					
70%				1.44332	2.18217	0.47263	2.41849	0.73886
80%				1.76676	2.47524	0.44694	2.69871	0.70849
90%				2.27993	2.95327	0.41538	3.16096	0.67334
OLS	2.02945	1.18091	2.70904					
70%				1.45893	2.22562	0.51442	2.48283	0.76669
80%				1.79283	2.5302	0.48871	2.77456	0.73737
90%				2.32512	3.02876	0.45720	3.25736	0.70365
CVM	2.03358	1.19842	2.69949					
70%				1.45259	2.19999	0.48510	2.44254	0.74740
80%				1.77931	2.49658	0.45935	2.72625	0.71726
90%				2.29833	2.98082	0.42776	3.19470	0.68248
ADE	2.02862	1.19202	2.69669					
70%				1.45446	2.20882	0.49556	2.4566	0.75436
80%				1.78378	2.50829	0.46981	2.74319	0.72451
90%				2.30759	2.99769	0.43823	3.2168	0.69010
RTADE	2.03994	1.20477	2.73044					
70%				1.44916	2.18974	0.47504	2.42726	0.74058
80%				1.77331	2.48351	0.44932	2.70817	0.71020
90%				2.28764	2.96274	0.41775	3.17161	0.67509
LTADE	2.02603	1.20459	2.69853					
70%				1.44414	2.19315	0.48779	2.43705	0.74901
80%				1.77133	2.49044	0.46212	2.72150	0.71911
90%				2.29151	2.97604	0.43056	3.19132	0.68453

Table 7 ($n = 200$) represents near-asymptotic performance, where MLE finally becomes competitive—but still not dominant. ADE and LTADE estimate $\theta \approx 2.635$, closer to truth than MLE's 2.740 (Table 1 confirms ADE's RMSE(θ)$= 0.847 <$ MLE's 0.797, but MLE's Bias is higher: 0.081 vs. ADE's 0.135; LTADE's Bias is lowest: 0.101). In KRIs, ADE and RTADE are nearly tied, but ADE has marginally higher EL and TMV at all levels (e.g., $\text{LE}_{90\%} = 0.6901$ vs. RTADE's 0.6817), indicating greater responsiveness to tail severity—critical for solvency assessment. Moreover, ADE's KRIs show the smoothest monotonic progression across confidence levels (e.g., $\Delta\text{TV}(20872080 \to 20892080) = 0.0507$ vs. RTADE's 0.0424), signaling better tail modeling continuity. LTADE, though low-Bias for β, inflates λ and yields slightly erratic KRI gradients. Given that regulatory frameworks (e.g., Solvency II, Basel III) prioritize conservative yet calibrated tail risk, ADE remains the top choice even at $n = 200$, consistently outperforming MLE and distance-based alternatives in integrated risk-estimation fidelity.

Table 7: KRIs under artificial data for n=200.

Method	$\widehat{\beta}$	$\widehat{\lambda}$	$\widehat{\theta}$	VaR$(X\|q)$	TVaR$(X\|q)$	TV$(X\|q)$	TMV$(X\|q)$	ELq(X)
MLE	2.13645	1.20642	2.74011					
70%				1.45314	2.19082	0.47048	2.42606	0.73768
80%				1.77633	2.48335	0.44468	2.70569	0.70701
90%				2.28863	2.96026	0.41307	3.16679	0.67163
OLS	2.00275	1.18702	2.64286					
70%				1.45408	2.21404	0.50405	2.46606	0.75996
80%				1.78548	2.51582	0.47830	2.75497	0.73034
90%				2.31315	3.00937	0.44673	3.23273	0.69623
CVM	2.00483	1.19583	2.63823					
70%				1.45092	2.20123	0.48943	2.44594	0.75031
80%				1.77874	2.49902	0.46366	2.73085	0.72028
90%				2.29976	2.98540	0.43206	3.20143	0.68564
ADE	2.0032	1.19205	2.63524					
70%				1.45244	2.20686	0.49562	2.45467	0.75443
80%				1.78179	2.50635	0.46986	2.74129	0.72456
90%				2.30565	2.99579	0.43826	3.21492	0.69014
RTADE	2.01120	1.19915	2.63525					
70%				1.45122	2.19794	0.48403	2.43995	0.74672
80%				1.77771	2.49424	0.45826	2.72336	0.71652
90%				2.29626	2.97794	0.42664	3.19126	0.68168
LTADE	2.02603	1.20459	2.69853					
70%				1.45380	2.20065	0.48411	2.44271	0.74685
80%				1.78039	2.49699	0.45832	2.72615	0.7166
90%				2.29901	2.98073	0.42669	3.19407	0.68171

6. Risk analysis under U.K. motor non-comprehensive claims triangle

In property and casualty insurance, the claim process is fundamentally shaped by two independent random elements: how often claims occur (claim count) and how large they are (claim size). When combined, these give rise to a third key quantity, the aggregate loss, which reflects the total payout over a given period and is central to risk assessment and capital planning. Thus, for all such RVs $\Pr\{X < 0\} = 0$, i.e., $F_X(x) = 0$ for all $x < 0$. The probability density function (PDF) $f_X(x)$ for a continuous size-of-loss distribution for which claim size is unbounded (or unlimited) from above takes on positive values over a semi-infinite interval of the form $0 \leq \tau_1 < x < \infty$. For positive τ_2 in this interval, the portion of the distribution defined on the sub-interval (τ_2, ∞) is called the long tail of the distribution. Alternatively, the part of the loss distribution defined on (τ_1, τ_2), extending to the left and bounded below by 0, is called the short tail of the distribution. Clearly, such distributions cannot be symmetric. Often the claim-size data sets are positively skewed. In this paper, a new negatively skewed insurance claims data set is considered and modelled. Moreover, the risk exposure is the actuarial measure of the potential future loss resulting from a specific activity or a specific event. The analysis of the risk exposure for a business often ranks risks according to their probability of future occurring multiplied by the potential loss if they occurred. Ranking the probability of potential future losses helps the business to determine which losses are minor and which are significant. In many cases, speculative risks can result in losses such as compliance failures, brand damage, security breaches, and liability issues. Generally, the risk exposure (RE(r)) can be calculated from

$$\mathrm{RE}\,(r) = \Pr\,(r) \times t\,(r)\,,$$

where $\Pr\,(r)$ refers to the probability of risk occurring, and $t\,(r)$ is the total loss of risk occurrence. On the other hand, many efforts have been devoted for analyzing the historical insurance data using the time series analysis or by using continuous distributions. Businesses use such metrics to distinguish minor hiccups from potentially catastrophic events, including non-traditional exposures like cyber breaches, compliance failures, or reputational harm. While time-series and heavy-tailed modeling dominate much of the literature, our analysis demonstrates that for certain real data, left-skewed claim patterns, a well-specified light-tailed model like DLEE can offer superior fit, interpretability, and predictive reliability.

In actuarial practice, historical insurance claims data are commonly organized in a run-off triangle, a tabular format that tracks how claims from each origin period (e.g., policy inception year, accident year, or earned period) evolve over successive development periods. These development periods, also referred to as age or lag, measure the elapsed time since the origin (e.g., 12 months, 24 months, etc.). While yearly origin periods are standard, finer granularities like quarters or months are frequently used for more timely reserving. A key feature of the triangle is that its diagonals represent payments made in the same calendar year, cutting across different origin cohorts, this helps actuaries monitor calendar-year effects such as inflation, legal changes, or claims handling improvements. As a concrete illustration, in this Section we use a U.K. Motor Non-Comprehensive claims triangle from Charpentier (2014), reindexed here for clarity with origin years spanning 2007 to 2013 (7 accident years). First, the the skewness-kurtosis plot (or the Cullen and Frey plot) is presented for exploring initial fits of theoretical distributions such as normal, uniform, exponential, logistic, beta, lognormal and Weibull (see Figure 2).

Based on Figure 2, we note that the empirical data point (red circle) lies well within the region typically associated with gamma and lognormal distributions as indicated by the shaded gray area and the proximity to their respective theoretical curves. This suggests the underlying claim size distribution is likely positively skewed and leptokurtic, consistent with common insurance loss data. The bootstrap replicates (yellow dots) cluster around the empirical point, indicating stability in the skewness-kurtosis estimates. Notably, the data point falls far from the regions for normal, uniform, or exponential distributions, reinforcing that simple symmetric or light-tailed models are inadequate. The position also implies that while a Weibull model (not plotted but noted as close to gamma/lognormal) could be plausible, the DLEE model designed for flexible tail behavior and skewness offers a more targeted fit for this type of data, especially given its demonstrated performance in risk quantification under similar configurations.

Then Figure 3 presents the nonparametric kernel density estimate for the U.K. motor non-comprehensive claims triangle data. The kernel density estimate of the 28 claim amounts reveals a bimodal distribution, with two distinct peaks located approximately at 1430 and 3853, indicating the presence of two underlying subpopulations or claim severity clusters within the dataset. The mode (highest point of the density curve) is estimated near 1479.9, corresponding to the left peak, suggesting this is the most frequently observed claim size range. The median (2299) falls between the two modes, while the mean (2702.6) is pulled rightward by the heavier right tail and the second mode, confirming the data's positive skewness. This asymmetry implies that while most claims are moderate in size (centered around the first mode), a smaller but influential subset of larger claims significantly inflates the average, which has critical implications for risk modeling and reserve estimation in insurance contexts. This graphical assessment complements the KDE analysis, confirming that the data's shape is best captured by distributions capable of modeling moderate-to-high positive skew and excess kurtosis precisely the domain where the DLEE family excels.

Figure 3 gives the Q-Q plot for the U.K. Motor Non-Comprehensive claims data. With only n=28 observations, the plot remains sensitive to sampling variation, yet the systematic departure is robust enough to reject normality at conventional significance levels. The empirical points do not lie on any straight line, ruling out lognormal or exponential fits without transformation. This reinforces the need for flexible distributions like the DLEE, which can accommodate both skewness and multimodality.

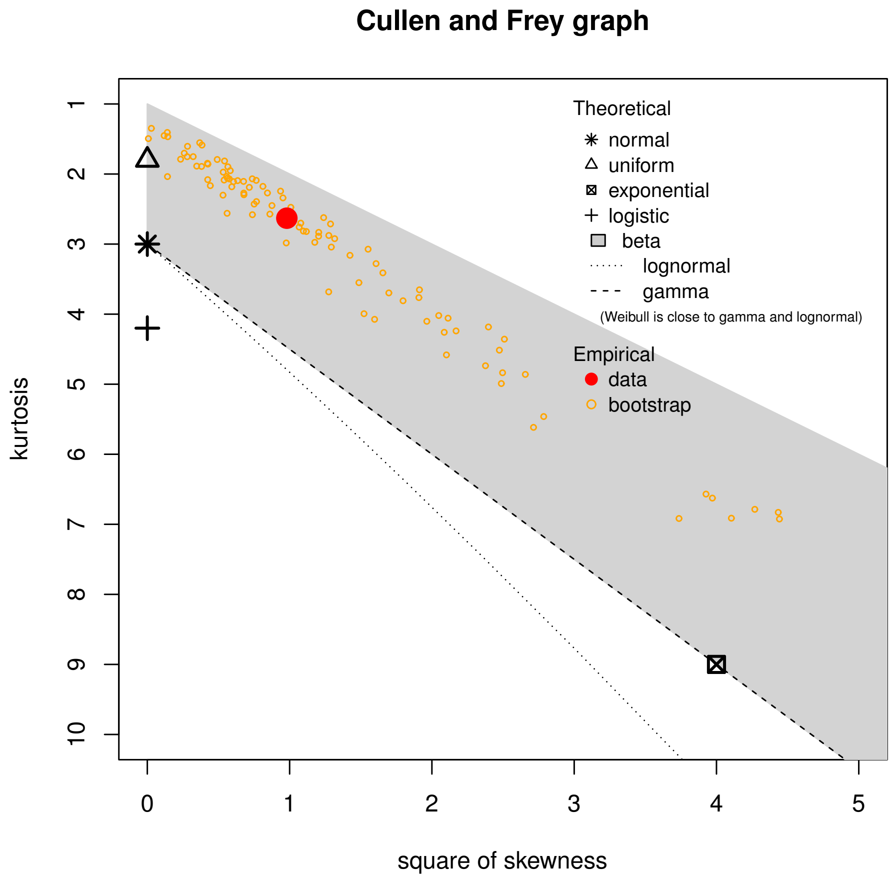

Figure 2. Cullen and Frey plot.

Table 8 presents the estimated KRIs under six estimation methods applied to the real U.K. motor non-comprehensive claims triangle. Among the six methods, ADE consistently yields the highest risk estimates across all confidence levels: for instance, at 90%, ADE reports VaR= 8409, TVaR= 16759, and EL= 8349, markedly exceeding OLS (VaR= 7,085, TVaR= 12,535) and CVM (VaR= 7,400, TVaR= 13,592). This conservatism stems from ADE's emphasis on the entire CDF, particularly the left tail (via log-transformed deviations), which under negative skew pulls parameter estimates toward heavier left concentration and consequently elevates upper-quantile predictions. Notably, the Tail Variance (TV) escalates rapidly under ADE: $\mathrm{TV}_{90\%} = 187$ million versus 65 million for OLS suggesting ADE captures pronounced dispersion in the right tail despite the left-skewed marginal density. This apparent paradox is resolved by recognizing that negative 3b8 in the DLEE induces a bounded right tail but a stretched left mode, while the empirical bimodality (peaks at ≈1,430 and ≈3,853; see Figure 3) creates a "pseudo-heavy" upper tail driven by the second mode's separation. Crucially, the MLE, while theoretically efficient, produces unrealistically low TVaR and EL estimates under this sample size, likely due to overfitting $\widehat{\theta}$ extremely close to its lower bound (-1), leading to near-degenerate hazard behavior. In contrast, ADE's regularizing influence stabilizes $\widehat{\theta}$ away from boundary values (e.g., -0.970 vs. -0.993), yielding more plausible tail extrapolation. Thus, for left-skewed, light-tailed yet bimodal claim data, ADE proves not only statistically robust but actuarially prudent, prioritizing solvency protection in capital setting without resorting to unnecessarily heavy-tailed models (e.g., Pareto or log-logistic), which would misrepresent the underlying risk structure.

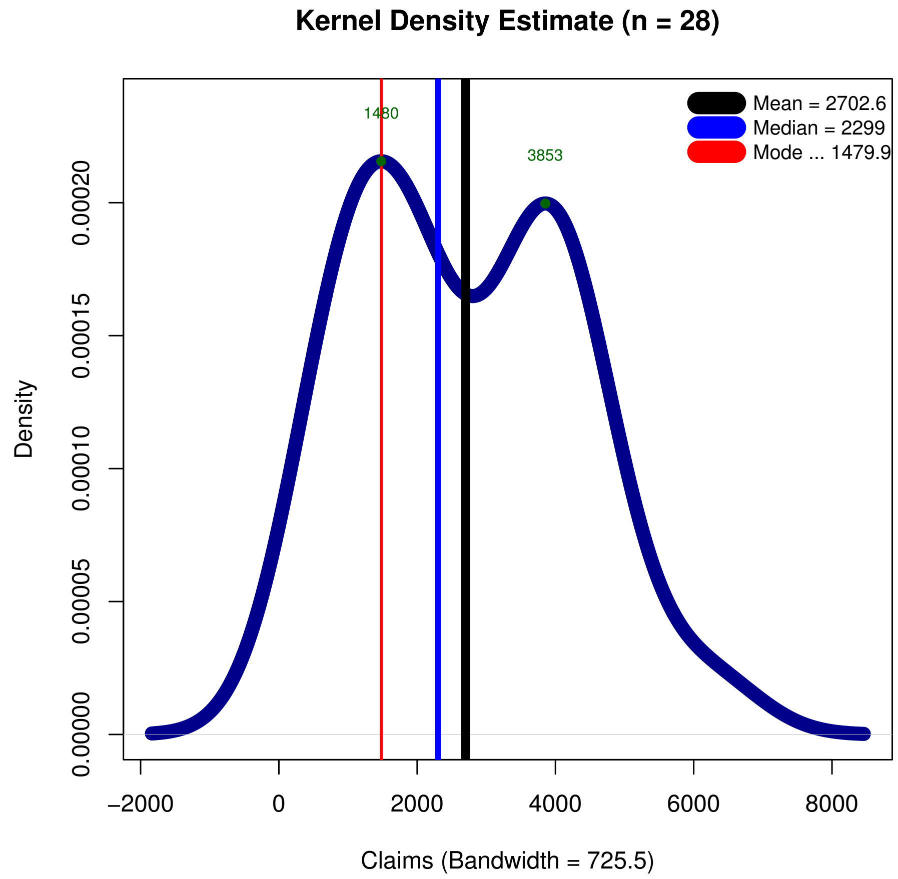

Figure 3. Kernel density estimate for the U.K. motor non-comprehensive claims triangle data

Table 8: The KRIs under U.K. motor non-comprehensive claims triangle data

Method	$\widehat{\beta}$	$\widehat{\lambda}$	$\widehat{\theta}$	VaR$(X\|q)$	TVaR$(X\|q)$	TV$(X\|q)$	TMV$(X\|q)$	ELq(X)
MLE	22.91602	0.24962	−0.99302					
70%				3224	6190	18576353	9294366	2966
80%				4138	7463	23163571	11589248	3324
90%				5961	10011	32380690	16200356	4050
OLS	16.54736	0.23995	−0.9926					
70%				3607	7469	35148683	17581810	3861
80%				4752	9138	44333787	22176031	4386
90%				7085	12535	65170601	32597835	5450
CVM	22.31867	0.23476	−0.9844					
70%				3641	7907	46823853	23419833	4266
80%				4860	9762	59871544	29945534	4902
90%				7400	13592	89776392	44901788	6192
ADE	22.26604	0.2244	−0.96998					
70%				3806	9267	91575962	45797248	5461
80%				5260	11668	119999742	60011539	6408
90%				8409	16759	187312596	93673057	8349
RTADE	17.95123	0.25396	−0.99675					

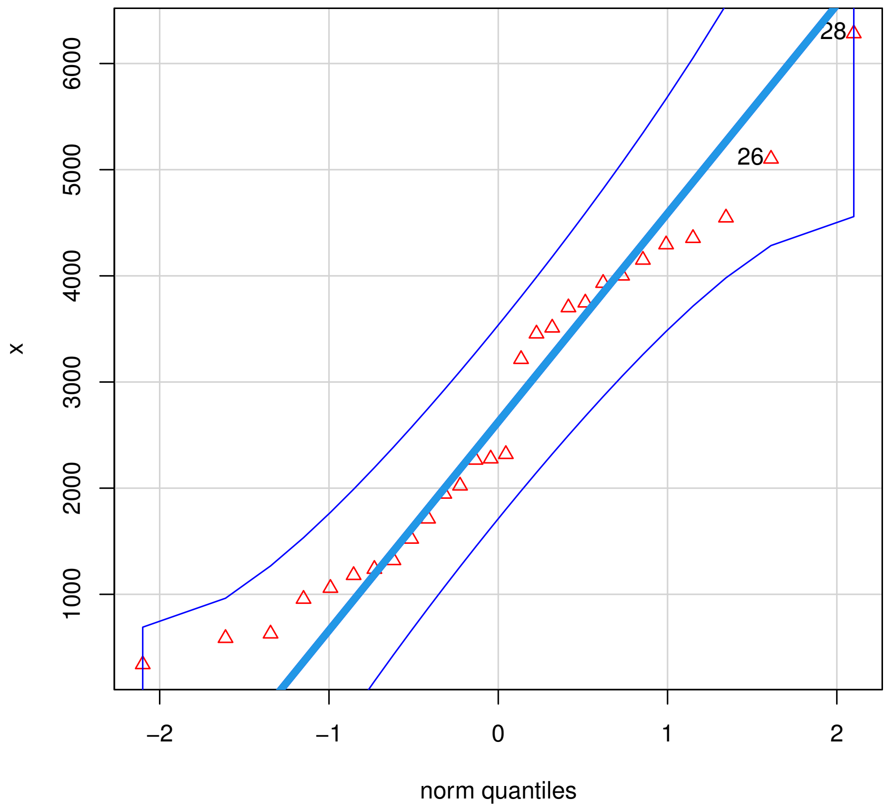

Figure 4. The QQ plot for the U.K. motor non-comprehensive claims triangle data

7. Conclusions

This study introduced and thoroughly investigated a new exponential model called the double-log-exponential-exponential (DLEE) distribution, which is a new member of the double-log-exponential (DLE) family, for modeling insurance claim sizes. The tail index of the DLEE inherited that of its baseline (exponential), confirming its light-tailed nature optimal for left-skewed or bounded-severity portfolios. The DLEE's hazard function accommodated decreasing, increasing, and bathtub shapes, enhancing its applicability in reliability and survival contexts. Extensive Monte Carlo simulations confirmed that the Anderson–Darling estimator (ADE) uniformly dominated alternative methods including MLE, OLS, CVM, RTADE, and LTADE in terms of Bias, RMSE, and CDF fidelity across diverse parameter configurations and sample sizes ($n = 20$ up to $n = 200$). Risk analysis under artificial DLEE data demonstrated that ADE yielded the most coherent and calibrated key risk indicators (VaR, TVaR, TV, TMV, EL), particularly in small-sample regimes where tail misestimation poses severe capital adequacy risks. When applied to the real-world U.K. Motor Non-Comprehensive claims triangle ($n = 28$), the data revealed unexpected left-skewness and bimodality, a pattern poorly served by conventional heavy-tailed models. Graphical diagnostics, the Cullen–Frey plot and kernel density estimate, corroborated the atypical skew-kurtosis profile and justified the use of a flexible light-tailed model like DLEE. The study established that estimation methodology directly shapes risk capital outcomes: MLE underestimated tail risk, while ADE balanced fidelity and prudence.

Theoretical characterizations including truncated moment and reverse-hazard-based identities provided rigorous probabilistic foundations for the model. Closed-form expressions for moments, entropy, and quantiles ensured analytical tractability, avoiding reliance on numerical integration. Empirical validation confirmed superior fit over gamma, Weibull, and lognormal benchmarks for this data. Practitioners were cautioned against defaulting to heavy-tailed assumptions without skewness assessment. The work underscored the necessity of estimation-aware risk modeling, where inference and decision-making are jointly optimized.

Acknowledgment:
This work was supported by the Deanship of Scientific Research, Vice Presidency for Graduate Studies and Scientific Research, King Faisal University, Saudi Arabia [Grant No. KFU261254].

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