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A New Flexible Marshall–Olkin Extended Odd Weibull–G Family of Distributions: Theory, Properties and Applications

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Abstract In this paper, we introduce the Marshall–Olkin Extended Odd Weibull–G (MO–ExOW–G) family of distributions (FoD), as a flexible framework for modeling lifetime data characterized by skewness, heavy tails, and complex hazard rate structures. The proposed family integrates the Marshall–Olkin (MO) generator with the Extended Odd Weibull–G (ExOW–G) family to enhance shape adaptability and tail behavior. This combination enables the new MO-ExOW-G to capture a wide spectrum of hazard rate forms, including increasing, decreasing, unimodal, bimodal, bathtub, and inverse bathtub patterns. We derive some statistical properties of the MO–ExOW–G family. Parameter estimation is addressed through various techniques, and the performance of the estimators is assessed via Monte Carlo simulations. Special cases of the MO-ExOW-G are applied to real-world datasets from clinical studies and engineering contexts, where the proposed models exhibit superior goodness-of-fit compared to several established alternatives. Overall, the MO–ExOW–G FoD provides a powerful and unified approach for analyzing survival and reliability data, offering substantial improvements in flexibility and interpretability.

Keywords Lifetime Data, Survival Analysis, Reliability Analysis, Flexible Hazard Rate Functions, Marshall-Olkin, Extended Odd Weibull-G family of Distributions.

AMS 2010 subject classifications 62E99; 62E15; 60E05

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1. Introduction

In reliability theory and survival analysis, data often display complex characteristics that traditional models fail to capture. Real-world datasets in clinical studies and engineering applications commonly exhibit non-monotonic hazard functions, skewness, and heavy-tailed behavior. Assuming a constant or monotonic hazard rate in such cases can lead to serious model misspecification. Classical lifetime distributions such as the exponential and Weibull are limited in this regard. The exponential distribution assumes a constant hazard rate and is therefore unsuitable for systems whose risk changes over time. The Weibull distribution allows only monotonic hazard functions, which limits its ability to model more complex shapes such as bathtub or inverted bathtub hazards commonly observed in practice [20].

Over the years, a variety of generators have been developed to enrich a baseline distribution G. The Marshall–Olkin (MO) generator, introduced in the context of reliability and shock models by Marshall and Olkin [22], has become one of the most influential distributional generators in survival analysis and reliability theory. Its appeal lies in the simplicity of its transformation, which adds a single shape parameter to a baseline survival function while preserving analytical tractability and offering a natural competing-risks interpretation. This feature has motivated a broad range of MO-generated distributions, many of

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which have been successfully applied in modelling lifetime data across engineering, actuarial sciences and several other areas. The Marshall–Olkin–extended Weibull distribution has been widely studied for its ability to model increasing, decreasing, and bathtub-shaped hazard functions, making it useful in reliability engineering [11]. The Marshall–Olkin–Exponential distribution provides a tractable and flexible extension of the exponential model, yielding heavier tails and a clear sudden-failure (shock) interpretation [10]. Likewise, the Marshall–Olkin–Gompertz distribution has proved effective in demography and actuarial science, where early-life and late-life mortality patterns require flexible hazard forms [13]. Oluyede and Chipepa [25] introduced the Marshall–Olkin Odd Exponential Half Logistic–G distribution, designed to effectively model heavy-tailed data, with applications in fields such as reliability and insurance. Similarly, Sengweni et al. [28] developed the Marshall–Olkin Topp–Leone Half-Logistic–G distribution by integrating Topp–Leone and half-logistic components with the Marshall–Olkin parameter, resulting in a flexible family of distributions. Moakofi et al. [23] proposed the Marshall–Olkin Lindley–Log–Logistic distribution by combining the Lindley and log–logistic distributions through the Marshall–Olkin framework, enabling the model to accommodate a broader range of shapes and tail behaviors than either distribution alone. Beyond these classical variants, hybrid generator models such as the Marshall–Olkin–Kumaraswamy–G family [17] and the Marshall-Olkin Topp-Leone-G family of distributions [19] illustrate the adaptability of the Marshall-Olkin framework when combined with other distributional generators.

Recent studies further demonstrate the vitality of the MO approach. Alsadat et al. [4] proposed the Marshall–Olkin Weibull–Burr XII distribution, which unifies the Burr XII and Weibull models under the MO framework, offering enhanced flexibility for modeling physics data with heavy tails and diverse hazard shapes. Sherwani et al. [29] introduced the Marshall–Olkin Exponentiated Dagum distribution, extending the Dagum model to better capture skewness and tail behavior, with applications in reliability and income data analysis. More recently, El-Ghannam [14] developed the Marshall–Olkin Weibull–Lomax distribution, a five-parameter lifetime model that accommodates complex hazard rate structures and provides superior fit to engineering and biomedical datasets. These works highlight the continuous evolution of MO generated models and their ability to address practical modeling challenges across disciplines. Parallel to these developments, new distributions have also been introduced to capture more flexible hazard structures. For instance, Alizadeh et al. [3] proposed the Extended Odd Weibull–G (ExOW–G) family of distributions, which augments a baseline distribution G by introducing an "odd" Weibull transformation. This mechanism allows the hazard function to assume increasing, decreasing, unimodal, or bimodal forms, thereby accommodating a wide range of hazard behaviors.

The primary motivation is to develop a highly flexible model capable of capturing complex data characteristics such as skewness, heavy tails, and a wide variety of hazard rate shapes (increasing, decreasing, bathtub, unimodal, etc.). Motivated by the inadequacies of classical distributions and the flexibility of both the MO generator and ExOW-G distribution , we propose the Marshall–Olkin Extended Odd Weibull–G (MO–ExOW–G) family of distributions. The primary objective of this work is to develop and analyze the MO–ExOW–G family for modeling skewed, heavy-tailed data with complex hazard rates.

By merging the Marshall–Olkin (MO) generator with the Extended Odd Weibull–G (ExOW–G) family, the resulting MO–ExOW–G distribution inherits complementary strengths from both components. The MO generator introduces an additional parameter that enhances tail behavior and dependence in the survival function, allowing for improved modeling of heavy-tailed data and extreme observations. Meanwhile, the ExOW–G framework substantially enriches the hazard rate structure, enabling a wide variety of shapes. Consequently, the MO–ExOW–G hazard rate can be increasing, decreasing, unimodal, bimodal, bathtub, bathtub followed by upside-down bathtub, or upside-down bathtub followed by bathtub, making the model suitable for complex lifetime data exhibiting both diverse failure mechanisms and heavy tail behavior.

1.1. Contributions

The main contribution of this paper is a flexible family of distributions called Marshall-Olkin Extended Odd Weibull-G, proposed by merging the Marshall-Olkin generator with the Extended Odd Weibull-G family of distributions thus resulting in a model that captures lifetime data with heavy tails, skewness and produce diverse hazard rate functions. This new family consistently provide better fits than other generalized distributions with the same baseline distributions.

1.2. Organization

The remainder of this paper is organized as follows. In Section 2, we derive the MO-ExOW-G family of distributions and examine its statistical properties. Special cases are presented in Section 3. Section 4 addresses parameter estimation and inference. Section 5 presents simulation studies, and Section 6 illustrates real data applications and model comparisons. Finally, Section 7 offers concluding remarks and discusses potential avenues for future research.

2. The Proposed MO–ExOW–G Family of Distributions

In this section, the new family of distributions and some of its statistical properties are developed and presented.

Marshall-Olkin family of distributions [22] has a cumulative distribution function (cdf) defined by:

$$F_{MO-G}(x;\sigma,\xi) = 1 - \frac{\sigma\left(1 - G(x;\xi)\right)}{1 - (1-\sigma)\left(1 - G(x;\xi)\right)}, \tag{1}$$

where $\sigma > 0$ and $G(x;\xi)$ is the cdf of the baseline distribution with the parameter vector ξ.
The Extended Odd Weibull family of distributions [3] has a cdf defined by:

$$F_{ExOW-G}(x;\alpha,\beta,\xi) = 1 - \left(1 + \beta\left(\frac{G(x;\xi)}{1 - G(x;\xi)}\right)^{\alpha}\right)^{-\frac{1}{\beta}}, \tag{2}$$

where $\alpha, \beta > 0$ and $G(x;\xi)$ is the cdf of the baseline distribution with the parameter vector ξ.
The cdf of the new Marshall-Olkin-Extended Odd Weibull-G family of distributions is constructed by substituting Equation (2) on Equation (1), and is given by

$$F_{MO-ExOW-G}(x;\alpha,\beta,\sigma,\xi) = 1 - \frac{\sigma\left(1 + \beta\left(\frac{G(x;\xi)}{1-G(x;\xi)}\right)^{\alpha}\right)^{-\frac{1}{\beta}}}{1 - (1-\sigma)\left(1 + \beta\left(\frac{G(x;\xi)}{1-G(x;\xi)}\right)^{\alpha}\right)^{-\frac{1}{\beta}}}, \tag{3}$$

where $\alpha, \beta, \sigma > 0$ and $G(x;\xi)$ is the cdf of the baseline distribution with the parameter vector ξ. Note that σ is the tilt parameter.
The **probability distribution function (pdf) of MO-ExOW-G** family of distributions is

$$\begin{aligned} f_{MO-ExOW-G}(x;\alpha,\beta,\sigma,\xi) &= \frac{\frac{\sigma\alpha g(x;\xi)G(x;\xi)^{\alpha-1}}{[\bar{G}(x;\xi)]^{\alpha+1}}\left(1 + \beta\left(\frac{G(x;\xi)}{1-G(x;\xi)}\right)^{\alpha}\right)^{-\frac{1}{\beta}-1}}{\left[1 - (1-\sigma)\left(1 + \beta\left(\frac{G(x;\xi)}{1-G(x;\xi)}\right)^{\alpha}\right)^{-\frac{1}{\beta}}\right]^2} \\ &= \frac{\sigma f_{ExOW-G}(x;\alpha,\beta,\xi)}{\left[1 - (1-\sigma)\bar{F}_{ExOW-G}(x;\alpha,\beta,\xi)\right]^2} \end{aligned} \tag{4}$$

for $\alpha, \beta, \sigma > 0$, where $\bar{F}_{ExOW-G}(x;\alpha,\beta,\sigma,\xi) = 1 - F_{ExOW-G}(x;\alpha,\beta,\sigma,\xi)$. Note that, $G(x;\xi)$ is the cdf of the baseline distribution with the parameter vector ξ and $\bar{G}(x;\xi) = 1 - G(x;\xi)$.

The **hazard rate function (hrf)** is

$$h_{MO\text{-}ExOW\text{-}G}(x;\alpha,\beta,\sigma,\xi) = \frac{\alpha g(x;\xi)(G(x;\xi))^{\alpha-1}\left(1+\beta\left(\frac{G(x;\xi)}{1-G(x;\xi)}\right)^{\alpha}\right)^{-1}}{[1-G(x;\xi)]^{\alpha+1}\left[1-(1-\sigma)\left(1+\beta\left(\frac{G(x;\xi)}{1-G(x;\xi)}\right)^{\alpha}\right)^{-\frac{1}{\beta}}\right]}, \tag{5}$$

for $\alpha, \beta, \sigma > 0$ and the baseline parameter vector ξ.

The **reliability function** of the new family of distributions follows from Equation (3) and is given by

$$\begin{aligned} R_{MO-ExOW-G}(x;\alpha,\beta,\sigma,\xi) &= 1 - F_{MO-ExOW-G}(x;\alpha,\beta,\sigma,\xi) \\ &= \frac{\sigma\left(1+\beta\left(\frac{G(x;\xi)}{1-G(x;\xi)}\right)^{\alpha}\right)^{-\frac{1}{\beta}}}{1-(1-\sigma)\left(1+\beta\left(\frac{G(x;\xi)}{1-G(x;\xi)}\right)^{\alpha}\right)^{-\frac{1}{\beta}}}, \end{aligned} \tag{6}$$

for $\alpha, \beta, \sigma > 0$ and parameter vector ξ.

The **cumulative hazard function (chf)** of the new family of distributions follows from Equation (6) and is given by

$$\mathcal{H}_{MO-ExOW-G}(x;\alpha,\beta,\sigma,\xi) = -log[R_{MO-ExOW-G}(x;\alpha,\beta,\sigma,\xi)]. \tag{7}$$

The **quantile function (Q)** of the new Marshall-Olkin Extended Odd Weibull-G family of distributions follows from inverting Equation (3), as

$Q_{MO-ExOW-G}(x;\alpha,\beta,\sigma,\xi) = F^{-1}_{MO-ExOW-G}(x;\alpha,\beta,\sigma,\xi)$.

Let $q = 1 - \frac{\sigma\left(1+\beta\left(\frac{G(x;\xi)}{1-G(x;\xi)}\right)^{\alpha}\right)^{-\frac{1}{\beta}}}{1-(1-\sigma)\left(1+\beta\left(\frac{G(x;\xi)}{1-G(x;\xi)}\right)^{\alpha}\right)^{-\frac{1}{\beta}}}$ where $0 \le q \le 1$,

then $q = 1 - \frac{\sigma A}{1-(1-\sigma)A}$, where $A = \left(1+\beta\left(\frac{G(x;\xi)}{1-G(x;\xi)}\right)^{\alpha}\right)^{-\frac{1}{\beta}}$.

Consequently, the quantile function is given by

$$Q_{\text{MO-ExOW-G}}(q;\alpha,\beta,\sigma,\xi) = G^{-1}\left(\frac{\left(\frac{1}{\beta}\left[\left(\frac{(q-1)(1-\sigma)-\sigma}{q-1}\right)^{\beta}-1\right]\right)^{\frac{1}{\alpha}}}{1+\left(\frac{1}{\beta}\left[\left(\frac{(q-1)(1-\sigma)-\sigma}{q-1}\right)^{\beta}-1\right]\right)^{\frac{1}{\alpha}}}\right). \tag{8}$$

Refer to the appendix for more details. Note that the solutions to the non-linear Equation (8) gives the quantiles of the new family of distributions, for specified baseline cdf G.

2.1. Sub-families

Sub-families of the new family of distributions are presented in this subsection. The following sub-families are obtained from fixing some of the parameters in Equation (3).

Table 1. Sub-families of the MO-ExOW-G family obtained by fixing parameters

Parameter restriction	**Resulting CDF**	**Name and Reference**
$\alpha = 1$	$F(x;\beta,\sigma,\xi) = 1 - \dfrac{\sigma\left(1+\beta\frac{G(x;\xi)}{1-G(x;\xi)}\right)^{-\frac{1}{\beta}}}{1-(1-\sigma)\left(1+\beta\frac{G(x;\xi)}{1-G(x;\xi)}\right)^{-\frac{1}{\beta}}}$	Reduced MO-ExOW-G (New family of distributions)
$\beta = 1$	$F(x;\alpha,\sigma,\xi) = 1 - \dfrac{\sigma\left(1+\left(\frac{G(x;\xi)}{1-G(x;\xi)}\right)^{\alpha}\right)^{-1}}{1-(1-\sigma)\left(1+\left(\frac{G(x;\xi)}{1-G(x;\xi)}\right)^{\alpha}\right)^{-1}}$	MO Odd log-logistic-G Reduced MO-ExOW-G (New family of distributions)
$\sigma = 1$	$F(x;\alpha,\beta,\xi) = 1 - \left(1+\beta\left(\dfrac{G(x;\xi)}{1-G(x;\xi)}\right)^{\alpha}\right)^{-\frac{1}{\beta}}$	ExOW-G family of distributions (Alizadeh et al. [3])
$\beta = \sigma = 1$	$F(x;\alpha,\xi) = \dfrac{(G(x;\xi))^{\alpha}}{(G(x;\xi))^{\alpha}+[1-G(x;\xi)]^{\alpha}}$	Odd log-logistic-G family of distributions (Gleaton and Lynch [16])
$\alpha = \sigma = 1$	$F(x;\beta,\xi) = 1 - \left(1+\beta\dfrac{G(x;\xi)}{1-G(x;\xi)}\right)^{-\frac{1}{\beta}}$	New sub-family of distributions
$\alpha = \beta = 1$	$F(x;\sigma,\xi) = 1 - \dfrac{\sigma(1-G(x;\xi))}{1-(1-\sigma)(1-G(x;\xi))}$	MO-G family of distributions (Marshall and Olkin [22])
$\alpha = \beta = \sigma = 1$	$F(x;\xi) = G(x;\xi)$	Baseline distribution

2.2. Linear Representation of the Density Function

Theorem 1
Let X be a random variable following the MO-ExOW-G family of distributions with parameters $(\alpha,\beta,\sigma,\xi)$ where the baseline cdf is $G(x;\xi)$ and pdf $g(x;\xi)$. The probability density function of X admits the following linear representation:

$$\begin{aligned} f_{MO-ExOW-G}(x;\alpha,\beta,\sigma,\xi) =& \sigma\alpha \sum_{i,j,k,p,t=0}^{\infty} (-1)^{j+p+t}\frac{\beta^j(1-\sigma)^i}{(t+1)}(1+i)\binom{(\frac{i+1}{\beta})+j}{j}\binom{\alpha j+\alpha+k}{k} \\ &\times \binom{\alpha j+\alpha+k-1}{p}\binom{p}{t}(t+1)g(x)[G(x)]^{t+1-1} \\ =& \sum_{t=0}^{\infty}\gamma_{t+1}g_{t+1}(x;\xi), \end{aligned} \tag{9}$$

where $g_{t+1}(x;\xi) = (t+1)g(x;\xi)[G(x;\xi)]^t$ and

$$\gamma_{t+1} = \sigma\alpha \sum_{i,j,k,p=0}^{\infty} (-1)^{j+p+t} \frac{\beta^j (1-\sigma)^i}{(t+1)} (1+i) \binom{(\frac{i+1}{\beta})+j}{j} \binom{\alpha j+\alpha+k}{k} \binom{\alpha j+\alpha+k-1}{p} \binom{p}{t}. \tag{10}$$

Thus, the mathematical and statistical properties of the MO-ExOW-G family of distributions follow directly from those of the Exp-G family of distributions. See the proof of this theorem on the appendix.

2.3. *Moments*

Statistical properties of probability distributions including moments and conditional expectations have received increasing attention in recent years. In particular, Bakar et al. [6], [5] established characterization results for generalized distributions based on conditional expectations of order statistics. In this section, we present moments, moment generating function, conditional moments and incomplete moments for the new family of distributions.

The $r^{\textbf{th}}$ **ordinary moments** follows from the series expansions in Equation (31) and they are given by:

$$\mu_r' = E(X^r) = \int_{-\infty}^{\infty} x^r\, f_{MO-ExOW-G}(x;\alpha,\beta,\sigma,\xi)\,dx = \sum_{t=0}^{\infty} \gamma_{t+1} E(X_{t+1}^r), \tag{11}$$

where $E(X_{t+1}^r)$ is the r^{th} moment of the exp-G density with power parameter (t+1).

The n^{th} **central moment** is

$$\mu_n = \sum_{r=0}^{n} \binom{n}{r} (-\mu_1')^{n-r} E(X^r) = \sum_{t=0}^{\infty}\sum_{r=0}^{n} \gamma_{t+1} \binom{n}{r} (-\mu_1')^{n-r} E(X_{t+1}^r). \tag{12}$$

Thus, the first four central moments are: $\mu_1 = \mu_1'$, $\mu_2 = \mu_2' - (\mu_1')^2$, $\mu_3 = \mu_3' - 3\mu_1'\mu_2' + 2(\mu_1')^3$ and $\mu_4 = \mu_4' - 4\mu_1'\mu_3' + 6(\mu_1')^2\mu_2' - 3(\mu_1')^4$. The variance ($\sigma^2$), coefficient of variation (CV), coefficient of skewness (CS) and coefficient of kurtosis (CK) are: $\sigma^2 = \mu_2 = E(X-\mu)^2$, CV=$\frac{\sigma}{\mu}$, CS=$\mu_3 {\mu_2}^{-3/2} = \frac{E(X-\mu)^3}{[E(X-\mu)^2]^{3/2}} = \frac{\mu_3' - 3\mu\mu_2' + 2\mu^3}{(\mu_2'-\mu^2)^{3/2}}$, and CK=$\mu_4(\mu_2)^{-2} = \frac{E(X-\mu)^4}{[E(X-\mu)^2]^2} = \frac{\mu_4' - 4\mu\mu_3' + 6\mu^2\mu_2' - 3\mu^4}{(\mu_2'-\mu^2)^2}$.

The **moment generating function** (MGF) of the MO-ExOW-G family of distributions is:

$$M_X(t) = E[e^{sX}] = \sum_{t=0}^{\infty} \gamma_{t+1} \int_{-\infty}^{\infty} e^{sx}(t+1)g(x)[G(x)]^{t+1-1}\,dx. \tag{13}$$

The r^{th} **conditional moment** of the MO-ExOW-G family of distributions is:

$$\begin{aligned} E(X^r \mid X > y) &= \frac{1}{\bar{F}(y)} \int_{y}^{\infty} x^r f_{MO-ExOW-G}(x;\alpha,\beta,\sigma,\xi)\,dx \\ &= \frac{1}{\bar{F}(y)} \sum_{t=0}^{\infty} \gamma_{t+1} \int_{y}^{\infty} x^r (t+1)g(x)[G(x)]^t\,dx, \end{aligned} \tag{14}$$

where $\bar{F}(y) = \bar{F}_{MO-ExOW-G}(y;\alpha,\beta,\sigma,\xi) = 1 - F_{MO-ExOW-G}(y;\alpha,\beta,\sigma,\xi)$ is the survival function.

The s^{th} **incomplete moments** of the MO-ExOW-G family of distributions is

$$\varphi_s = \int_{-\infty}^{v} x^s f_{MO-ExOW-G}(x;\alpha,\beta,\sigma,\xi)\,dx = \sum_{t=0}^{\infty} \gamma_{t+1} \int_{-\infty}^{v} x^s (t+1)g(x)[G(x)]^{t+1-1}\,dx. \tag{15}$$

2.4. Order Statistics

Order statistics provide a natural framework for describing the behavior of sample extremes and intermediate failure times, which are central to lifetime and reliability analysis. Recent studies have investigated the distributional and moment properties of generalized order statistics for flexible lifetime distributions; see, for example [1].

Let $X_1, X_2, \dots, X_n$ be a random sample of size n from the MO-ExOW-G family of distributions, with cdf $F_{MO\text{-}ExOW\text{-}G}(x)$ and pdf $f_{MO\text{-}ExOW\text{-}G}(x)$. Let $X_{1:n} \leq X_{2:n} \leq \cdots \leq X_{n:n}$ denote the ordered statistics. The pdf of the m^{th} order statistic $X_{m:n}$ is given by:

$$f_{m:n}(x) = \frac{n!}{(m-1)!(n-m)!} f(x) \left[F(x)\right]^{m-1} \left[1 - F(x)\right]^{n-m}. \tag{16}$$

We apply the generalized binomial series expansion $(1-z)^q = \sum_{r=0}^{\infty} \binom{q}{r} (-1)^r z^r$ for $|z| < 1$, where $z = F(x)$ and $q = n - m$, to Equation (16) to obtain

$$f_{m:n}(x) = \frac{n!}{(m-1)!(n-m)!} \sum_{r=0}^{\infty} \binom{n-m}{r} (-1)^r f_{MO-ExOW-G}(x) [F_{MO-ExOW-G}(x)]^{r+m-1}.$$

Let $f_{MO-ExOW-G}(x)[F_{MO-ExOW-G}(x)]^{r+m-1}$ be denoted by $f(x)[F(x)]^{r+m-1}$, then

$$\begin{aligned} f(x)[F(x)]^{r+m-1} &= \frac{\sigma\alpha g(x;\xi)(G(x;\xi))^{\alpha-1} \left(1+\beta\left(\frac{G(x;\xi)}{1-G(x;\xi)}\right)^{\alpha}\right)^{-\frac{1}{\beta}-1}}{[1-G(x;\xi)]^{\alpha+1}\left[1-(1-\sigma)\left(1+\beta\left(\frac{G(x;\xi)}{1-G(x;\xi)}\right)^{\alpha}\right)^{-\frac{1}{\beta}}\right]^2} \\ &\quad \times \left(1 - \frac{\sigma\left(1+\beta\left(\frac{G(x;\xi)}{1-G(x;\xi)}\right)^{\alpha}\right)^{-\frac{1}{\beta}}}{1-(1-\sigma)\left(1+\beta\left(\frac{G(x;\xi)}{1-G(x;\xi)}\right)^{\alpha}\right)^{-\frac{1}{\beta}}}\right)^{r+m-1}. \end{aligned} \tag{17}$$

Following similar generalized binomial series expansion approach used for the linear representation of the pdf in section 2.2 we obtain the m^{th} order statistic as

$$\begin{aligned} f_{m:n}(x) &= \frac{n!}{(m-1)!(n-m)!} \sum_{r=0}^{\infty} \binom{n-m}{r} (-1)^r f_{MO-ExOW-G}(x) [F_{MO-ExOW-G}(x)]^{r+m-1}, \\ &= \frac{n!\sigma\alpha}{(m-1)!(n-m)!} \sum_{i,j,k,p,r,s,t=0}^{\infty} \binom{n-m}{r}\binom{r+m-1}{i} (-1)^{i+k+r+s+t} \sigma^i \\ &\quad \times \binom{i+j+1}{j} (1-\sigma)^j \frac{\beta^k}{(t+1)} \binom{\alpha k+\alpha+p}{p} \binom{\alpha k+\alpha+p-1}{s} \binom{s}{t} (t+1) g(x) [G(x)]^t \\ &= \sum_{t=0}^{\infty} w^*_{t+1} g_{t+1}(x;\xi), \end{aligned} \tag{18}$$

where

$$w_{t+1}^{*} = \frac{n!\sigma\alpha}{(m-1)!(n-m)!} \sum_{i,j,k,p,r,s=0}^{\infty} \binom{n-m}{r} \binom{r+m-1}{i} (-1)^{i+k+r+s+t} \sigma^{i} \binom{i+j+1}{j} \times (1-\sigma)^{j} \frac{\beta^{k}}{(t+1)} \binom{\alpha k+\alpha+p}{p} \binom{\alpha k+\alpha+p-1}{s} \binom{s}{t} \quad (19)$$

and $g_{t+1}(x;\xi) = (t+1)g(x;\xi)[G(x;\xi)]^{t}$ is the exp-G pdf with power parameter (t+1).

2.5. Probability Weighted Moments

Probability weighted moments (PWMs) can be used to estimate the unknown parameters, offering improved performance compared to the traditional method of moments. The PWMs of the new Marshall-Olkin Extended Odd Weibull-G family of distributions are given by

$$p_{a,b} = \int_{-\infty}^{\infty} x^{a}\, f_{MO-ExOW-G}(x)[F_{MO-ExOW-G}(x)]^{b}\, dx\,,$$

where $f_{MO-ExOW-G}(x)$ is the pdf of the new Marshall-Olkin Extended Odd Weibull-G distribution in Equation ((4)) and $F_{MO-ExOW-G}(x)$ is the cdf of the new Marshall-Olkin Extended Odd Weibull-G distribution in Equation (3). Let $F_{MO-ExOW-G}(x;\alpha,\beta,\sigma,\xi)$ be denoted by $F(x)$, $G(x;\xi)$ by $G(x)$ and $g(x;\xi)$ by $g(x)$. Following the series expansion $f(x)[F(x)]^{r+m-1}$ in Equation (17), then $f_{MO-ExOW-G}(x)[F_{MO-ExOW-G}(x)]^{s}$ can be expanded as

$$f(x)[F(x)]^{b} = \sigma\alpha \sum_{i,j,k,p,s,t=0}^{\infty} \binom{b}{i} (-1)^{i+k+s+t} \sigma^{i} \binom{i+j+1}{j} (1-\sigma)^{j} \frac{\beta^{k}}{(t+1)} \binom{\frac{i+j+1}{\beta}+k}{k} \times \binom{\alpha k+\alpha+p}{p} \binom{\alpha k+\alpha+p-1}{s} \binom{s}{t} (t+1)g(x)[G(x)]^{t}.$$

Consequently, the PWMs of the MO-ExOW-G family of distributions are given by

$$\begin{aligned} p_{a,b} &= \sigma\alpha \sum_{i,j,k,p,s,t=0}^{\infty} \binom{b}{i} (-1)^{i+k+s+t} \sigma^{i} \binom{i+j+1}{j} (1-\sigma)^{j} \frac{\beta^{k}}{(t+1)} \binom{\frac{i+j+1}{\beta}+k}{k} \binom{\alpha k+\alpha+p}{p} \\ &\times \binom{\alpha k+\alpha+p-1}{s} \binom{s}{t} \int_{-\infty}^{\infty} x^{a}(t+1)g(x)[G(x)]^{t}\, dx \\ &= \sum_{t=0}^{\infty} \phi_{t+1} E(X_{t+1}^{a}), \end{aligned} \quad (20)$$

where, $E(X_{t+1}^{a}) = \int_{-\infty}^{\infty} x^{a} g_{t+1}(x;\xi)\, dx\,,$

$$\phi_{t+1} = \sigma\alpha \sum_{i,j,k,p,s=0}^{\infty} \binom{b}{i} (-1)^{i+k+s+t} \sigma^{i} \binom{i+j+1}{j} (1-\sigma)^{j} \frac{\beta^{k}}{(t+1)} \binom{\frac{i+j+1}{\beta}+k}{k} \binom{\alpha k+\alpha+p}{p} \times \binom{\alpha k+\alpha+p-1}{s} \binom{s}{t}, \quad (21)$$

and $g_{t+1}(x;\xi) = (t+1)g(x;\xi)[G(x;\xi)]^{t+1-1}$ is the exp-G density with power parameter (t+1).

2.6. Rényi Entropy

Rényi entropy is a generalised measure of uncertainty in probability distributions. It is used to characterise the diversity, complexity and tail behaviour of distributions in information theory and reliability analysis. The Rényi entropy of order $\theta > 0$, $\theta \neq 1$, for a continuous random variable X with density $f_{MO-ExOW-G}(x;\alpha,\beta,\sigma,\xi)$ is given by:

$$
\begin{aligned}
H^{(\theta)}_{MO\text{-}ExOW\text{-}G} &= \frac{1}{1-\theta}\log\left(\int_0^\infty \left[f_{MO\text{-}ExOW\text{-}G}(x;\alpha,\beta,\sigma,\xi)\right]^\theta \, dx\right) \\
&= \frac{1}{1-\theta}\log\left(\int_0^\infty \left[(\sigma\alpha)^\theta (g(x;\xi))^\theta (G(x;\xi))^{\theta(\alpha-1)}[1-G(x;\xi)]^{-\theta(\alpha+1)}\right.\right. \\
&\times \left(1+\beta\left(\frac{G(x;\xi)}{1-G(x;\xi)}\right)^\alpha\right)^{-\theta(\frac{1}{\beta}+1)} \\
&\times \left.\left.\left[1-(1-\sigma)\left(1+\beta\left(\frac{G(x;\xi)}{1-G(x;\xi)}\right)^\alpha\right)^{-\frac{1}{\beta}}\right]^{-2\theta} dx\right).
\end{aligned}
$$

Following the generalized binomial series expansions in section 2.2, we obtain

$$
\begin{aligned}
H^{(\theta)}_{MO\text{-}ExOW\text{-}G} &= \frac{1}{1-\theta}\log\left(\int_0^\infty \left[(\sigma\alpha)^\theta (g(x;\xi))^\theta (G(x;\xi))^{\theta(\alpha-1)}[1-G(x;\xi)]^{-\theta(\alpha+1)}\right.\right. \\
&\times \left(1+\beta\left(\frac{G(x;\xi)}{1-G(x;\xi)}\right)^\alpha\right)^{-\theta(\frac{1}{\beta}+1)} \\
&\times \left.\left.\left[1-(1-\sigma)\left(1+\beta\left(\frac{G(x;\xi)}{1-G(x;\xi)}\right)^\alpha\right)^{-\frac{1}{\beta}}\right]^{-2\theta} dx\right) \\
&= \frac{1}{1-\theta}\log\left((\sigma\alpha)^\theta \sum_{i,j,k,p,t=0}^{\infty}(1-\sigma)^i\binom{2\theta+i-1}{i}(-1)^{j+p+t}\beta^j\binom{\frac{\theta+i}{\beta}+\theta+j-1}{j}\right. \\
&\times \binom{\theta\alpha+\theta+\alpha j+k-1}{k}\binom{\alpha j+\theta(\alpha-1)+k}{p}\binom{p}{t}\left(\frac{1}{\frac{t}{\theta}+1}\right)^\theta \\
&\times \left.\int_0^\infty \left[\left(\frac{t}{\theta}+1\right)^\theta (g(x))^\theta\left([G(x)]^{\frac{t}{\theta}}\right)^\theta\right]dx\right) \\
&= \frac{1}{1-\theta}\log\left[\sum_{t=0}^{\infty}\varrho_t e^{(1-\theta)I_{REG}}\right], \qquad (22)
\end{aligned}
$$

where

$$
\begin{aligned}
\varrho_t &= (\sigma\alpha)^\theta \sum_{i,j,k,p=0}^{\infty}(1-\sigma)^i\binom{2\theta+i-1}{i}(-1)^{j+p+t}\beta^j\binom{\frac{\theta+i}{\beta}+\theta+j-1}{j} \\
&\times \binom{\theta\alpha+\theta+\alpha j+k-1}{k}\binom{\alpha j+\theta(\alpha-1)+k}{p}\binom{p}{t}\left(\frac{1}{\frac{t}{\theta}+1}\right)^\theta, \qquad (23)
\end{aligned}
$$

and

$$
I_{REG} = \frac{1}{1-\theta}\log\left(\int_0^\infty \left[\left(\frac{t}{\theta}+1\right)g(x)\left([G(x)]^{\frac{t}{\theta}}\right)\right]^\theta dx\right) \qquad (24)
$$

is the Rényi entropy of the exp-G with power parameter $\frac{t}{\theta}$.

3. Some Special Cases

In this section, we provide some special cases by specifying the baseline distributions to be the exponential, Weibull and Burr XII distributions, respectively.

Table 2. Baseline distributions and corresponding MO-ExOW-G special cases

Baseline Distribution	$\frac{G(x;\xi)}{1-G(x;\xi)}$	$g(x;\xi)$	MO-ExOW-G
Exponential	$e^{\lambda x}-1$	$\lambda e^{-\lambda x}$	MO-ExOW-E
Weibull	$e^{x^{\lambda}}-1$	$\lambda x^{\lambda-1}e^{-x^{\lambda}}$	MO-ExOW-W
Burr XII	x^c	$ckx^{c-1}(1+x^c)^{-k-1}$	MO-ExOW-Burr XII

3.1. Marshall-Olkin Extended Odd Weibull-Exponential Distribution

The cdf and pdf of the Marshall-Olkin Extended Odd Weibull Exponential (MO-ExOW-E) distribution are given by

$$F_{MO-ExOW-E}(x;\alpha,\beta,\sigma,\lambda)=1-\frac{\sigma\left(1+\beta\left(e^{\lambda x}-1\right)^{\alpha}\right)^{-\frac{1}{\beta}}}{1-(1-\sigma)\left(1+\beta(e^{\lambda x}-1)^{\alpha}\right)^{-\frac{1}{\beta}}}$$

and

$$f_{MO-ExOW-E}(x;\alpha,\beta,\sigma,\lambda)=\frac{\sigma\,\alpha\,\lambda\,e^{\lambda x}\,(e^{\lambda x}-1)^{\alpha-1}\left(1+\beta(e^{\lambda x}-1)^{\alpha}\right)^{-1/\beta-1}}{\left[1-(1-\sigma)\left(1+\beta(e^{\lambda x}-1)^{\alpha}\right)^{-1/\beta}\right]^2}$$

for $\alpha,\beta,\sigma,\lambda>0$ and $x\geq 0$.

The pdf and hrf graphs of the MO-ExOW-E distribution for different parameter values are shown in Figure 1 and Figure 2, respectively. The pdf of MO-ExOW-E distribution accommodates right-skewed, unimodal, heavy-tailed and almost-symmetrical shapes. The hrf of MO-ExOW-E distribution shows increasing, decreasing, increasing-decreasing and bathtub shapes.

Hazard Rate Analysis of the MO-ExOW-E Distribution
The hazard function of the MO-ExOW-E distribution is given by

$$h_{MO-ExOW-E}(x;\alpha,\beta,\sigma,\lambda)=\frac{\alpha\lambda e^{\lambda x}(e^{\lambda x}-1)^{\alpha-1}}{\left(1+\beta(e^{\lambda x}-1)^{\alpha}\right)\left[1-(1-\sigma)\left(1+\beta(e^{\lambda x}-1)^{\alpha}\right)^{-1/\beta}\right]}$$

for $\alpha,\beta,\sigma,\lambda>0$ and $x\geq 0$.

Although the derivative of the hazard rate function $h'(x)$ can be obtained analytically, analytical solutions to $h'(x)=0$ that yield interpretable conditions on the parameters α, β, and σ are intractable. A structured numerical investigation is therefore conducted to examine the qualitative behavior of the hazard rate function. The parameter space is partitioned into eight representative regimes according to whether α, β, and σ are below or above one.

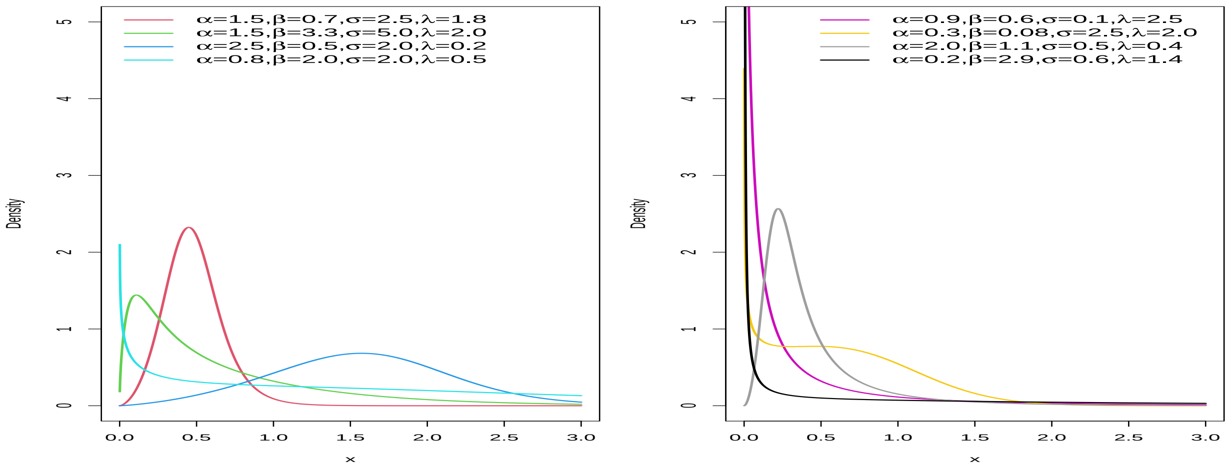


Figure 1. Pdf plots of MO-ExOW-E distribution

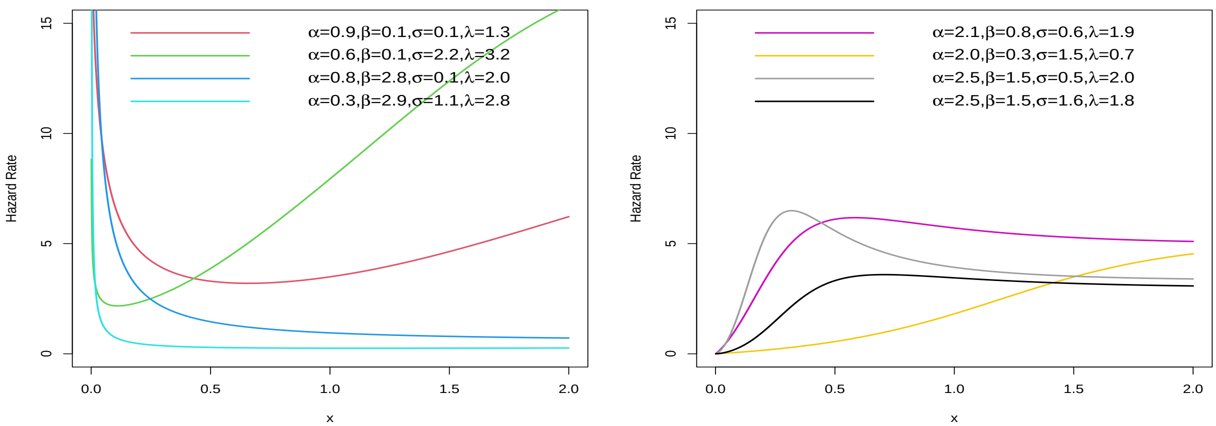


Figure 2. Hrf plots of MO-ExOW-E distribution

Within each regime, $h'(x)$ is evaluated numerically over dense grids to identify various shapes of the hazard rate function. See the appendix for details.

The skewness and kurtosis plots of the MO-ExOW-E distribution are presented in Figure 3. The MO-ExOW-E distribution is right-skewed for the chosen parameter values. High values of skewness are reported at high values of α paired with low values of β when σ is held constant at 2.5 and λ is held constant at 5.0. The MO-ExOW-E distribution is highly leptokurtic for the chosen parameter values. High values of kurtosis are reported at high values of α paired with low values of β when σ is held constant at 2.8 and λ is held constant at 1.5.

3.2. Marshall-Olkin Extended Odd Weibull-Weibull Distribution

The cdf and pdf of the new Marshall-Olkin Extended Odd Weibull-Weibull (MO-ExOW-W) distribution are given by

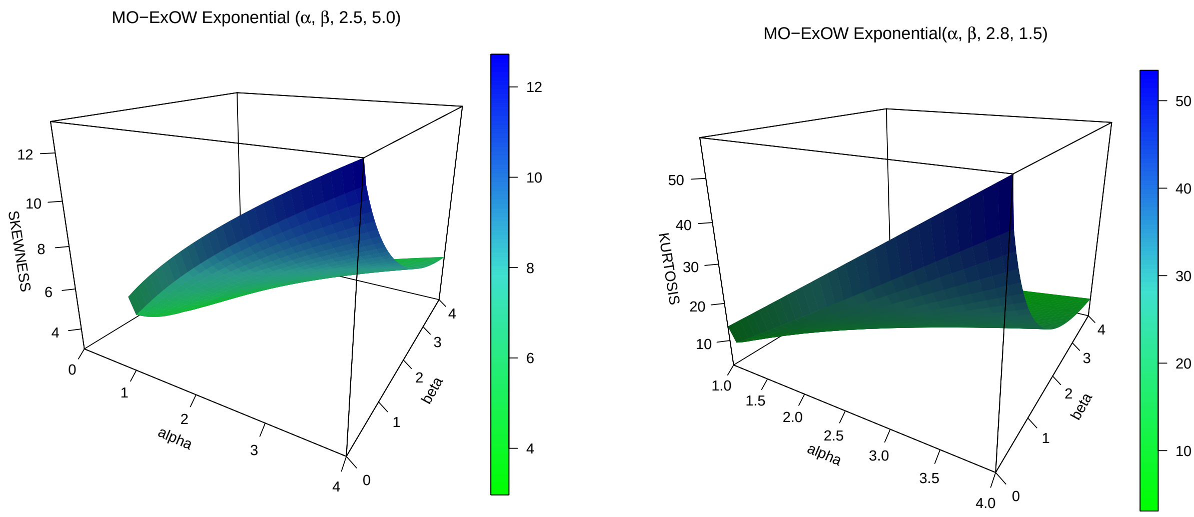


Figure 3. Skewness and kurtosis plots of MO-ExOW-E distribution

$$F_{MO-ExOW-W}(x;\alpha,\beta,\sigma,\lambda) = 1 - \frac{\sigma\left(1+\beta\left(e^{x^\lambda}-1\right)^{\alpha}\right)^{-\frac{1}{\beta}}}{1-(1-\sigma)\left(1+\beta(e^{x^\lambda}-1)^{\alpha}\right)^{-\frac{1}{\beta}}}$$

and

$$f_{MO-ExOW-W}(x;\alpha,\beta,\sigma,\lambda) = \frac{\sigma\,\alpha\lambda\,x^{\lambda-1}e^{x^\lambda}\left(e^{x^\lambda}-1\right)^{\alpha-1}\left(1+\beta\left(e^{x^\lambda}-1\right)^{\alpha}\right)^{-\frac{1}{\beta}-1}}{\left(1-(1-\sigma)\,\left(1+\beta\left(e^{x^\lambda}-1\right)^{\alpha}\right)^{-\frac{1}{\beta}}\right)^{2}}$$

for $\alpha,\beta,\sigma,\lambda > 0$ and $x \geq 0$.

The graphs of the MO-ExOW-W pdf and hrf for different parameter values are shown in Figure 4 and Figure 5, respectively. The MO-ExOW-W pdf accommodates bimodal, skewed, reverse-J, and heavy-tailed data. The MO-ExOW-W hrf shows increasing, decreasing, bathtub curves, and increasing-decreasing shapes.

Hazard Rate Analysis of the MO-ExOW-W Distribution

The hazard rate function of the MO-ExOW-W distribution is defined as

$$h_{MO-ExOW-W}(x;\alpha,\beta,\sigma,\lambda) = \frac{\alpha\lambda x^{\lambda-1}e^{x^\lambda}\left(e^{x^\lambda}-1\right)^{\alpha-1}}{\left(1+\beta(e^{x^\lambda}-1)^{\alpha}\right)\left[1-(1-\sigma)\left(1+\beta(e^{x^\lambda}-1)^{\alpha}\right)^{-1/\beta}\right]},$$

for $\alpha,\beta,\sigma,\lambda > 0$ and $x \geq 0$.

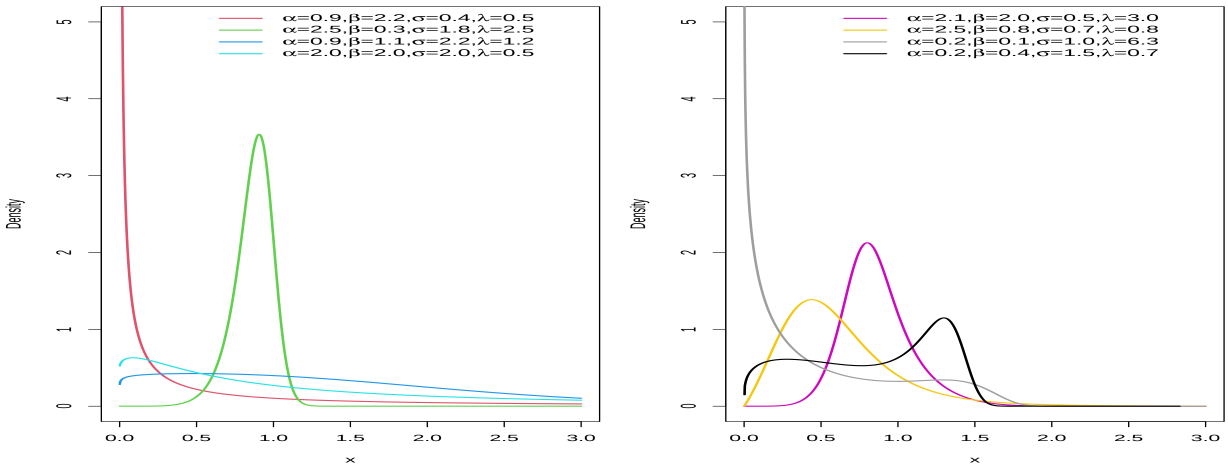


Figure 4. Pdf plots of MO-ExOW-W at different parameter values

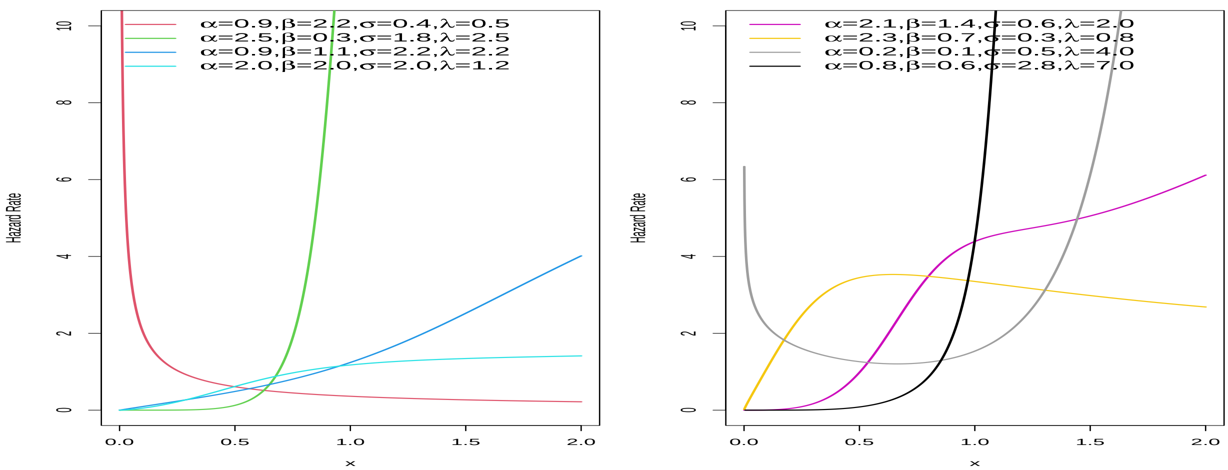


Figure 5. Hrf plots of MO-ExOW-W at different parameter values

A systematic numerical investigation of the derivative $h'(x)$ was conducted to characterize the hazard rate shapes of the MO-ExOW-W. The derivative of the hazard rate function $h'(x)$ was evaluated numerically over dense grids. See appendix for details.

The plots of skewness and kurtosis of the MO-ExOW-W distribution are presented in Figure 6.
The MO-ExOW-W distribution is right-skewed for the chosen parameter values. Highest level of skewness is reported when β is small and σ is high when α is fixed at 1.8 and λ is fixed at 2.5. The MO-ExOW-W distribution is highly leptokurtic for the chosen parameter values. The highest level of kurtosis is reported for large values of α and small values of β when σ is fixed at 3.0 and λ is fixed at 1.5.

3.3. Marshall-Olkin Extended Odd Weibull-Burr XII Distribution

The cdf and pdf of the new Marshall-Olkin Extended Odd Weibull-Burr XII (MO-ExOW-Burr XII) distribution are given by

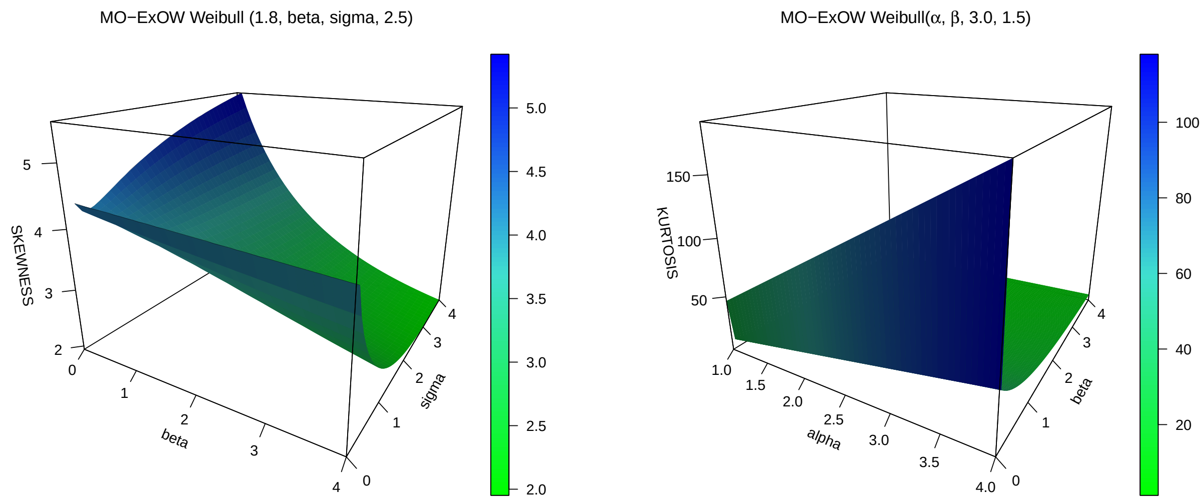


Figure 6. Skewness and kurtosis plots of MO-ExOW-Weibull distribution

$$F_{MO-ExOW-Burr-XII}(x;\alpha,\beta,\sigma,c,k) = 1 - \frac{\sigma\left(1+\beta\left[(1+x^c)^k-1\right]^\alpha\right)^{-\frac{1}{\beta}}}{1-(1-\sigma)\left(1+\beta[(1+x^c)^k-1]^\alpha\right)^{-\frac{1}{\beta}}}$$

and

$$f_{MO-ExOW-Burr-XII}(x;\alpha,\beta,\sigma,c,k) = \left(1+\beta\left[(1+x^c)^k-1\right]^\alpha\right)^{-\frac{1}{\beta}-1} \times \frac{\sigma\alpha ckx^{c-1}(1+x^c)^{k-1}\left[(1+x^c)^k-1\right]^{\alpha-1}}{\left(1-(1-\sigma)\left(1+\beta[(1+x^c)^k-1]^\alpha\right)^{-\frac{1}{\beta}}\right)^2}$$

for $\alpha,\beta,\sigma,c,k>0$ and $x>0$. Note that when k=1, we obtain the MO-ExOW-log-logistic (MO-ExOW-LLoG) distribution and c=1, we obtain the MO-ExOW-Lomax (MO-ExOW-Lom) distribution.

The pdf and hrf graphs of MO-ExOW-Burr XII distribution for different parameter values are shown in Figure 7 and Figure 8, respectively. The MO-ExOW-Burr XII pdf accommodates left-skewed, right-skewed, unimodal and almost symmetric shapes. The MO-ExOW-Burr XII hrf shows increasing, increasing-decreasing, and decreasing-increasing-decreasing shapes. The bathtub followed by upside-down bathtub shape is observed for parameter set $\alpha=0.8,\beta=0.1,\sigma=0.6,c=1.0$ and $k=3.6$.

Hazard Rate Analysis of the MO-ExOW-Burr XII Distribution

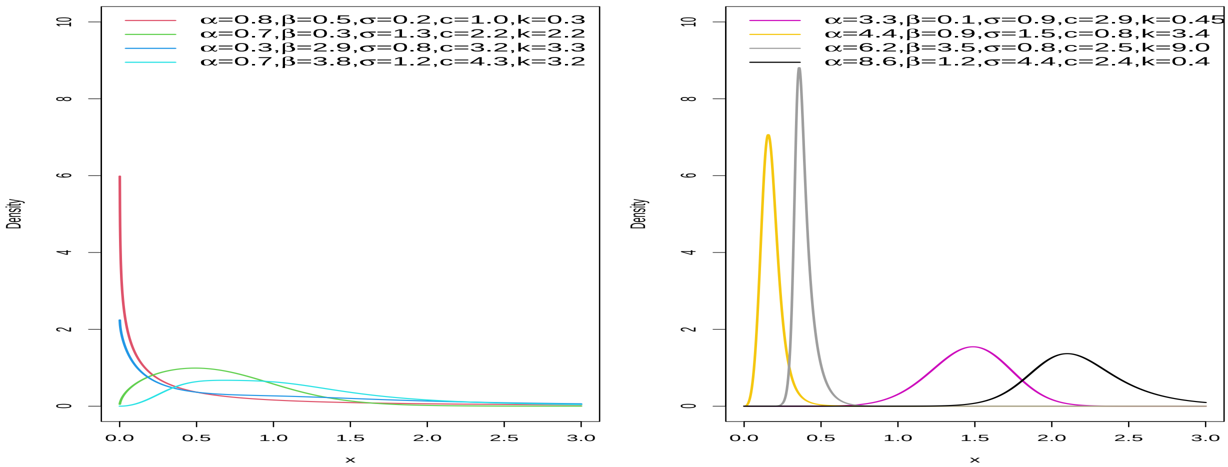


Figure 7. Pdf plots for MO-ExOW-Burr XII at different parameter values

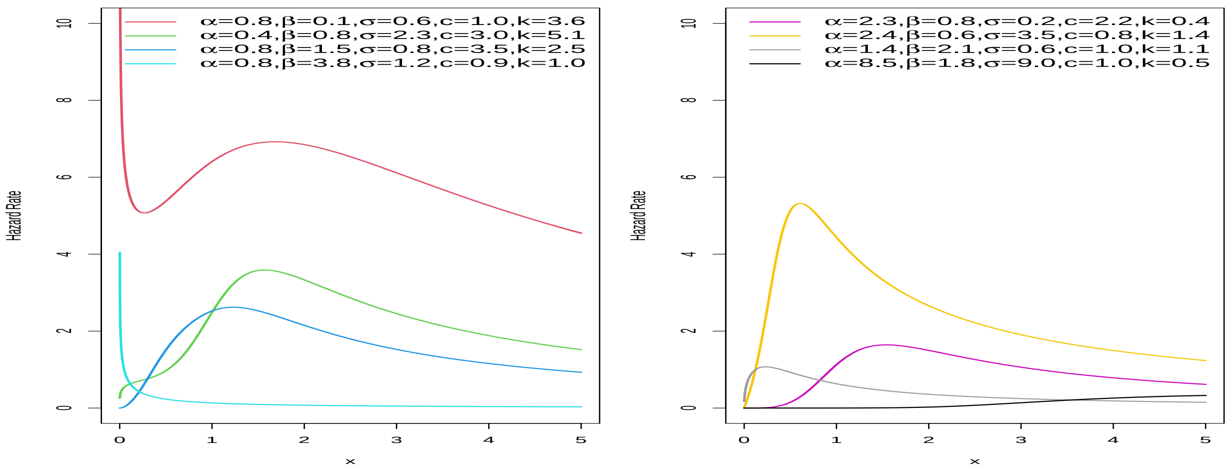


Figure 8. Hrf plots for MO-ExOW-Burr XII at different parameter values

The hazard rate function of the MO-ExOW-Burr XII distribution is defined as

$$h_{MO-ExOW-BurrXII}(x;\alpha,\beta,\sigma,c,k) = \frac{\sigma\,\alpha\,c\,k\,x^{c-1}(1+x^c)^{k-1}\left[(1+x^c)^k-1\right]^{\alpha-1}}{\left(1-(1-\sigma)\left(1+\beta\left[(1+x^c)^k-1\right]^{\alpha}\right)^{-1/\beta}\right)^2} \times\left(1+\beta\left[(1+x^c)^k-1\right]^{\alpha}\right)^{-1/\beta-1},$$

for $\alpha,\beta,\sigma,c,k>0$ and $x\geq 0$.

A systematic numerical investigation of the derivative $h'(x)$ was conducted. The parameter space was partitioned into representative regimes based on whether α, β, and σ are below or above one. See the appendix for details.

The plots of skewness and kurtosis of the MO-ExOW-Burr XII distribution are presented in Figure 9. The MO-ExOW-Burr XII distribution is right-skewed for the chosen parameter values. High values of skewness are reported at high values of α paired with low values of β when σ is held constant at 1.8, c is held constant at 1.6 and k is held constant at 3.2. The MO-ExOW-Burr XII distribution is leptokurtic for the chosen parameter values. High values of kurtosis are reported at high values of α paired with low values of β when σ is held constant at 1.8, c is held constant at 1.6 and k is held constant at 3.2.

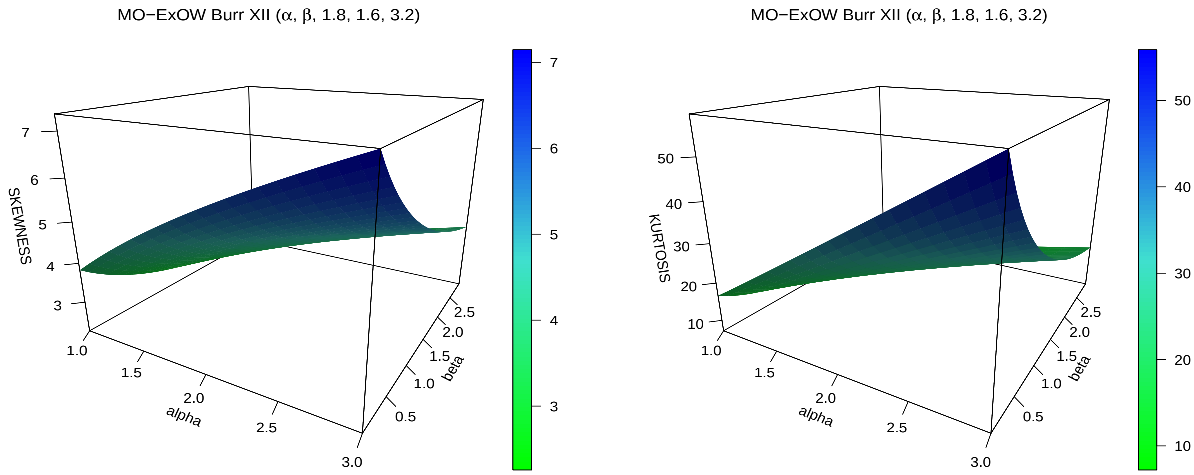


Figure 9. Skewness and kurtosis plots of MO-ExOW-Burr XII distribution

The MO-ExOW-Burr XII distribution is right skewed and highly leptokurtic for the chosen parameter values.

4. Parameter Estimation and Inference

In this section, we estimate the unknown parameters of the MO-ExOW-G family of distributions, using several estimation techniques such as; Maximum likelihood (ML), Least Squares (LS), Weighted Least Squares (WLS), Cramér-von Mises (CVM) and Anderson-Darling (AD). Maximum likelihood estimation is considered alongside several minimum distance–based methods to provide a broad comparative framework for parameter estimation. The LS and WLS estimators are constructed from discrepancies between the empirical and theoretical distribution functions, with WLS assigning greater weight to tail observations. The CVM and AD methods further assess goodness-of-fit by minimizing global discrepancies, with the AD estimator placing additional emphasis on tail behavior. Table 6 provides a concise summary of these estimation methods and their corresponding objective functions, while the details of these techniques are found on the appendix. Let $X_{(i)}, i = 1, 2, ..., n$ denote the i^{th} order statistic and $F(x_{(i)}; \mathbf{\Theta})$ the cdf of the i^{th} order statistic, with the parameter vector $\mathbf{\Theta}$.

5. Simulation Studies

In this section, we present the results of simulation studies that were conducted to examine the performance of several estimation techniques. Note that 3000 independent samples of size, n =50, 100, 200, 400, 800 and 1600 were generated from special cases of the MO-ExOW-G distribution. The chosen parameter values represent plausible scenarios for the MO–ExOW–G family, while other parameter values may also be considered subject to the constraints $\alpha, \beta, \sigma > 0$ and the characteristics of the baseline distribution. The sample sizes cover small to large datasets, allowing assessment of estimator performance across practical settings. The methods of Maximum

Table 3. Summary of Parameter Estimation Methods for MO-ExOW-G Distribution

Method	**Objective**	**Key Formula / Function**
Maximum Likelihood (ML)	Estimate parameters by maximizing the likelihood of the observed data	$\ell(\boldsymbol{\Theta}) = \sum_{i=1}^{n} \ln f(x_i; \boldsymbol{\Theta})$, solved numerically by $\frac{\partial \ell}{\partial \boldsymbol{\Theta}} = \mathbf{0}$
Least Squares (LS)	Minimize the squared differences between empirical and theoretical CDF	$LS(\boldsymbol{\Theta}) = \sum_{i=1}^{n} \left(F(x_{(i)}; \boldsymbol{\Theta}) - \frac{i}{n+1}\right)^2$
Weighted Least Squares (WLS)	Minimize weighted squared differences to give more weight to tails	$WLS(\boldsymbol{\Theta}) = \sum_{i=1}^{n} \frac{(n+1)^2(n+2)}{i(n-i+1)} \left(F(x_{(i)}; \boldsymbol{\Theta}) - \frac{i}{n+1}\right)^2$
Cramér-von Mises (CVM)	Minimize squared differences between empirical and theoretical CDF with scaling	$CVM(\boldsymbol{\Theta}) = \frac{1}{12n} + \sum_{i=1}^{n} \left(F(x_{(i)}; \boldsymbol{\Theta}) - \frac{2i-1}{2n}\right)^2$
Anderson-Darling (AD)	Minimize weighted differences between empirical and theoretical CDF emphasizing tails	$AD(\boldsymbol{\Theta}) = -n - \frac{1}{n} \sum_{i=1}^{n} (2i-1) \Big[\ln F(x_{(i)}; \boldsymbol{\Theta}) + \ln(1 - F(x_{(n+1-i)}; \boldsymbol{\Theta})) \Big]$

Likelihood (ML), Least Squares (LS), Weighted Least Square(WLS), Cramér-von Mises (CVM) and Anderson-Darling (AD) were used to estimate the parameters. These estimation methods were compared based on the average bias (ABias) and root mean square error (RMSE). Lower values of ABias and low values of RMSE indicate better performance. The simulation studies show that the maximum likelihood estimation performs best in terms of average bias and root mean square error. Let Θ represent the true parameter value and $\hat{\Theta}$ represent the estimated parameter value. The average bias (ABias) and root mean square error (RMSE) of an estimate are given by

$$ABias(\hat{\Theta}) = \frac{\sum_{i=1}^{N} \hat{\Theta}_i}{N} - \Theta, \tag{25}$$

and

$$RMSE(\hat{\Theta}) = \sqrt{\frac{\sum_{i=1}^{N} (\hat{\Theta}_i - \Theta)^2}{N}}, \tag{26}$$

respectively.

For the ML estimators, we evaluated the the mean absolute error (MAE), coverage probability (CP) and the confidence interval width (CIW). These metrics provide additional assessment of the inferential validity and robustness of the ML estimation procedure.

Table 4 and Table 7 present the mean estimate, ABias for each estimate with its associated rank in brackets, and the RMSE for each estimate with its associated rank in brackets. The ranks are assigned across different parameters estimation techniques, within the same sample size. The ranks are from 1 to 5, where 1 is the rank for the lowest ABias and 5 is the rank for the highest ABias. Similarly, 1 is the rank for the lowest RMSE and 5 is the rank for the highest RMSE. The ranks are such that lower-is-better (1=best whereas 5=worst). Note that, on Table 1, the first entry for AD estimation at sample size n=50, the mean estimate is 0.8919, the ABias is 0.0919 with a rank of 3 and RMSE is 0.2260 with a rank of 1.

The mean estimate, ABias, and RMSE values for different parameters of the MO-ExOW-Exponential distribution are shown in Table 4. The RMSE and ABias are decreasing as the sample size increases across all

Table 4. Results of Simulations using Different Estimation Techniques for MO-ExOW-E Distribution with parameters $\alpha = 0.8, \beta = 1.2, \lambda = 1.4$ and $\sigma = 0.6$.

Param	n	AD			CVM			LS			ML			WLS		
		Mean	ABias	RMSE	Mean	ABias	RMSE	Mean	ABias	RMSE	Mean	ABias	RMSE	Mean	ABias	RMSE
α	50	0.8919	0.0919$_{(3)}$	0.2260$_{(1)}$	0.9259	0.1259$_{(5)}$	0.2718$_{(5)}$	0.8895	0.0895$_{(2)}$	0.2428$_{(4)}$	0.9114	0.1114$_{(4)}$	0.2416$_{(3)}$	0.8793	0.0793$_{(1)}$	0.2280$_{(2)}$
β	50	2.2629	1.0629$_{(3)}$	2.0852$_{(2)}$	2.5693	1.3693$_{(5)}$	2.2817$_{(5)}$	2.4467	1.2467$_{(4)}$	2.2001$_{(4)}$	2.1217	0.9217$_{(1)}$	2.1975$_{(3)}$	2.1681	0.9681$_{(2)}$	2.0208$_{(1)}$
λ	50	2.3607	0.9607$_{(3)}$	1.8904$_{(3)}$	2.6280	1.2280$_{(5)}$	2.0740$_{(5)}$	2.4733	1.0733$_{(4)}$	1.9901$_{(4)}$	2.2248	0.8248$_{(1)}$	1.8861$_{(2)}$	2.2400	0.8400$_{(2)}$	1.8257$_{(1)}$
σ	50	0.6953	0.0953$_{(3)}$	0.4418$_{(2)}$	0.7702	0.1702$_{(5)}$	0.5648$_{(5)}$	0.7276	0.1276$_{(4)}$	0.5287$_{(4)}$	0.6322	0.0322$_{(1)}$	0.3689$_{(1)}$	0.6901	0.0901$_{(2)}$	0.4703$_{(3)}$
α	100	0.8794	0.0794$_{(2)}$	0.1698$_{(2)}$	0.9099	0.1099$_{(5)}$	0.2048$_{(5)}$	0.8849	0.0849$_{(3)}$	0.1902$_{(4)}$	0.8907	0.0907$_{(4)}$	0.1805$_{(3)}$	0.8664	0.0664$_{(1)}$	0.1680$_{(1)}$
β	100	2.0897	0.8897$_{(3)}$	1.8697$_{(2)}$	2.4579	1.2579$_{(5)}$	2.1420$_{(5)}$	2.3319	1.1319$_{(4)}$	2.0671$_{(4)}$	1.9850	0.7850$_{(1)}$	1.9718$_{(3)}$	2.0230	0.8230$_{(2)}$	1.8069$_{(1)}$
λ	100	2.2567	0.8567$_{(3)}$	1.7723$_{(3)}$	2.6067	1.2067$_{(5)}$	2.0489$_{(5)}$	2.4656	1.0656$_{(4)}$	1.9553$_{(4)}$	2.1063	0.7063$_{(1)}$	1.7571$_{(2)}$	2.1793	0.7793$_{(2)}$	1.7046$_{(1)}$
σ	100	0.6602	0.0602$_{(3)}$	0.3288$_{(2)}$	0.7184	0.1184$_{(5)}$	0.4039$_{(5)}$	0.6901	0.0901$_{(4)}$	0.3857$_{(4)}$	0.5954	-0.0046$_{(1)}$	0.2944$_{(1)}$	0.6539	0.0539$_{(2)}$	0.3299$_{(3)}$
α	200	0.8701	0.0701$_{(3)}$	0.1404$_{(3)}$	0.8954	0.0954$_{(5)}$	0.1655$_{(5)}$	0.8843	0.0843$_{(4)}$	0.1592$_{(4)}$	0.8682	0.0682$_{(2)}$	0.1331$_{(2)}$	0.8557	0.0557$_{(1)}$	0.1290$_{(1)}$
β	200	2.0202	0.8202$_{(3)}$	1.7018$_{(3)}$	2.2953	1.0953$_{(5)}$	1.9782$_{(5)}$	2.1822	0.9822$_{(4)}$	1.9004$_{(4)}$	1.8580	0.6580$_{(1)}$	1.6849$_{(2)}$	1.8801	0.6801$_{(2)}$	1.5725$_{(1)}$
λ	200	2.1887	0.7887$_{(3)}$	1.6163$_{(3)}$	2.4515	1.0515$_{(5)}$	1.8850$_{(5)}$	2.3282	0.9282$_{(4)}$	1.7950$_{(4)}$	2.0402	0.6402$_{(1)}$	1.5899$_{(2)}$	2.0839	0.6839$_{(2)}$	1.5464$_{(1)}$
σ	200	0.6518	0.0518$_{(3)}$	0.2655$_{(3)}$	0.6754	0.0754$_{(5)}$	0.2986$_{(4)}$	0.6565	0.0565$_{(4)}$	0.2995$_{(5)}$	0.6093	0.0093$_{(1)}$	0.2482$_{(1)}$	0.6444	0.0444$_{(2)}$	0.2613$_{(2)}$
α	400	0.8502	0.0502$_{(3)}$	0.1015$_{(3)}$	0.8694	0.0694$_{(5)}$	0.1296$_{(5)}$	0.8643	0.0643$_{(4)}$	0.1263$_{(4)}$	0.8457	0.0457$_{(2)}$	0.0942$_{(1)}$	0.8395	0.0395$_{(1)}$	0.0960$_{(2)}$
β	400	1.7474	0.5474$_{(3)}$	1.3522$_{(3)}$	1.9373	0.7373$_{(5)}$	1.6206$_{(5)}$	1.9195	0.7195$_{(4)}$	1.6050$_{(4)}$	1.6240	0.4240$_{(1)}$	1.2746$_{(2)}$	1.6514	0.4514$_{(2)}$	1.2462$_{(1)}$
λ	400	1.9560	0.5560$_{(3)}$	1.3241$_{(3)}$	2.1313	0.7313$_{(5)}$	1.5540$_{(5)}$	2.1080	0.7080$_{(4)}$	1.5338$_{(4)}$	1.8146	0.4146$_{(1)}$	1.2205$_{(1)}$	1.8628	0.4628$_{(2)}$	1.2284$_{(2)}$
σ	400	0.6304	0.0304$_{(3)}$	0.2192$_{(3)}$	0.6415	0.0415$_{(4)}$	0.2455$_{(5)}$	0.6421	0.0421$_{(5)}$	0.2425$_{(4)}$	0.6026	0.0026$_{(1)}$	0.2036$_{(1)}$	0.6268	0.0268$_{(2)}$	0.2138$_{(2)}$
α	800	0.8320	0.0320$_{(3)}$	0.0724$_{(3)}$	0.8499	0.0499$_{(5)}$	0.0958$_{(5)}$	0.8441	0.0441$_{(4)}$	0.0917$_{(4)}$	0.8270	0.0270$_{(2)}$	0.0604$_{(1)}$	0.8261	0.0261$_{(1)}$	0.0666$_{(2)}$
β	800	1.5537	0.3537$_{(3)}$	1.0340$_{(3)}$	1.6946	0.4946$_{(5)}$	1.2822$_{(5)}$	1.6416	0.4416$_{(4)}$	1.2317$_{(4)}$	1.4660	0.2660$_{(1)}$	0.8762$_{(1)}$	1.4835	0.2835$_{(2)}$	0.9251$_{(2)}$
λ	800	1.7873	0.3873$_{(3)}$	1.0463$_{(3)}$	1.9324	0.5324$_{(5)}$	1.2844$_{(5)}$	1.8618	0.4618$_{(4)}$	1.2116$_{(4)}$	1.6692	0.2692$_{(1)}$	0.8737$_{(1)}$	1.7024	0.3024$_{(2)}$	0.9359$_{(2)}$
σ	800	0.6244	0.0244$_{(4)}$	0.1920$_{(3)}$	0.6300	0.0300$_{(5)}$	0.2203$_{(5)}$	0.6227	0.0227$_{(3)}$	0.2099$_{(4)}$	0.6111	0.0111$_{(1)}$	0.1571$_{(1)}$	0.6169	0.0169$_{(2)}$	0.1751$_{(2)}$
α	1200	0.8240	0.0240$_{(3)}$	0.0580$_{(3)}$	0.8404	0.0404$_{(5)}$	0.0787$_{(5)}$	0.8340	0.0340$_{(4)}$	0.0750$_{(4)}$	0.8181	0.0181$_{(1)}$	0.0446$_{(1)}$	0.8198	0.0198$_{(2)}$	0.0523$_{(2)}$
β	1200	1.4464	0.2464$_{(3)}$	0.8491$_{(3)}$	1.6031	0.4031$_{(5)}$	1.0848$_{(5)}$	1.5269	0.3269$_{(4)}$	1.0539$_{(4)}$	1.3776	0.1776$_{(1)}$	0.6675$_{(1)}$	1.4017	0.2017$_{(2)}$	0.7345$_{(2)}$
λ	1200	1.6758	0.2758$_{(3)}$	0.8664$_{(3)}$	1.8309	0.4309$_{(5)}$	1.0800$_{(5)}$	1.7589	0.3589$_{(4)}$	1.0471$_{(4)}$	1.5844	0.1844$_{(1)}$	0.6816$_{(1)}$	1.6207	0.2207$_{(2)}$	0.7545$_{(2)}$
σ	1200	0.6149	0.0149$_{(4)}$	0.1729$_{(3)}$	0.6265	0.0265$_{(5)}$	0.1953$_{(5)}$	0.6148	0.0148$_{(3)}$	0.1937$_{(4)}$	0.6098	0.0098$_{(1)}$	0.1297$_{(1)}$	0.6133	0.0133$_{(2)}$	0.1552$_{(2)}$
α	1600	0.8211	0.0211$_{(3)}$	0.0498$_{(3)}$	0.8301	0.0301$_{(5)}$	0.0643$_{(4)}$	0.8287	0.0287$_{(4)}$	0.0645$_{(5)}$	0.8111	0.0111$_{(1)}$	0.0317$_{(1)}$	0.8150	0.0150$_{(2)}$	0.0429$_{(2)}$
β	1600	1.3882	0.1882$_{(3)}$	0.7624$_{(3)}$	1.4667	0.2667$_{(5)}$	0.9305$_{(4)}$	1.4631	0.2631$_{(4)}$	0.9361$_{(5)}$	1.3115	0.1115$_{(1)}$	0.4862$_{(1)}$	1.3558	0.1558$_{(2)}$	0.6179$_{(2)}$
λ	1600	1.6167	0.2167$_{(3)}$	0.7836$_{(3)}$	1.7030	0.3030$_{(5)}$	0.9382$_{(4)}$	1.6986	0.2986$_{(4)}$	0.9454$_{(5)}$	1.5160	0.1160$_{(1)}$	0.5027$_{(1)}$	1.5762	0.1762$_{(2)}$	0.6407$_{(2)}$
σ	1600	0.6063	0.0063$_{(1)}$	0.1673$_{(3)}$	0.6121	0.0121$_{(4)}$	0.1854$_{(5)}$	0.6107	0.0107$_{(3)}$	0.1839$_{(4)}$	0.6082	0.0082$_{(2)}$	0.1024$_{(1)}$	0.6128	0.0128$_{(5)}$	0.1389$_{(2)}$

methods of parameter estimation.

Table 5 presents the mean absolute error (MAE), confidence interval width, and coverage probability (CP) of the maximum likelihood estimators for the parameters of the MO-ExOW-E distribution. The MAE and CI widths decrease as the sample size increases, indicating improved accuracy and precision of the estimators. The coverage probabilities remain close to the nominal 95% level across all sample sizes, supporting the asymptotic normality of the MO-ExOW-E MLEs.

Table 6 and Table 9 present the sum of partial ranks (of ABias and RMSE) for each estimation method per sample size with its associated rank in brackets, across the different parameter estimation techniques. The ranks are assigned 1 to 5, where 1 is the rank for the lowest sum of partial ranks and 5 is the rank for the highest sum of partial ranks for the same sample size across various estimation methods. The ranks are such that lower-is-better (1=best whereas 5=worst). Considering Table 2, note that for AD estimation at sample size n=50, the sum of partial ranks is 20, with a rank of 3.

The total sum of ranks is the sum of sums of partial ranks within the same estimation method. The overall rank is the rank associated with the total sum of ranks. Considering Table 3, note that for AD estimation at sample size n=50, the total sum of ranks is 21 and the overall rank is 3.

From Table 6, the MLE has been ranked 1, indicating that it is the best estimation technique for MO-ExOW-E distribution.

The RMSE plots for different parameters of the MO-ExOW-E distribution are shown in Figure 10. The RMSE plots indicate that as sample size increases, the estimates get closer to the true parameter values.

The mean estimate, ABias, and RMSE values for different parameters of the MO-ExOW-Weibull distribution are shown in Table 7. The RMSE and ABias are decreasing as the sample size increases across all methods of parameter estimation.

Table 8 presents the mean absolute error (MAE), confidence interval width, and coverage probability (CP) of the maximum likelihood estimators for parameters of the MO-ExOW-W distribution. The MAE and CI widths

Table 5. MAE, Confidence Interval Width, and Coverage Probability for MLE Simulation of MO-ExOW-E

Parameter	Sample Size	MAE	CIW	CP
α	50	0.1837	0.8106	0.9583
β	50	0.9394	4.9900	0.9701
λ	50	1.4190	4.6971	0.9851
σ	50	0.2747	1.4755	0.9542
α	100	0.1333	0.5896	0.9504
β	100	0.9280	4.0969	0.9680
λ	100	1.3359	4.6833	0.9752
σ	100	0.2337	1.1103	0.9506
α	200	0.0950	0.4575	0.9502
β	200	0.9261	4.0857	0.9606
λ	200	1.1380	4.0642	0.9749
σ	200	0.1916	0.9633	0.9501
α	400	0.0631	0.3257	0.9503
β	400	0.9104	3.9456	0.9659
λ	400	0.8840	3.6203	0.9652
σ	400	0.1588	0.8391	0.9501
α	800	0.0339	0.2129	0.9506
β	800	0.5162	3.4704	0.9580
λ	800	0.5158	3.4228	0.9564
σ	800	0.1014	0.6896	0.9505
α	1200	0.0225	0.1670	0.9503
β	1200	0.3483	2.8387	0.9501
λ	1200	0.3505	2.8272	0.9502
σ	1200	0.0715	0.6144	0.9501
α	1600	0.0153	0.1305	0.9502
β	1600	0.2394	2.2728	0.9501
λ	1600	0.2457	2.4017	0.9502
σ	1600	0.0509	0.5278	0.9501

Table 6. Summed ranks (ABias + RMSE) and overall ranks for different estimation methods of MO-ExOW-E Distribution. Smaller sums indicate better performance.

n	AD	CVM	LS	ML	WLS
50	$20_{(3)}$	$40_{(5)}$	$30_{(4)}$	$16_{(2)}$	$14_{(1)}$
100	$20_{(3)}$	$40_{(5)}$	$31_{(4)}$	$16_{(2)}$	$13_{(1)}$
200	$24_{(3)}$	$39_{(5)}$	$33_{(4)}$	$12_{(1.5)}$	$12_{(1.5)}$
400	$24_{(3)}$	$39_{(5)}$	$33_{(4)}$	$10_{(1)}$	$14_{(2)}$
800	$25_{(3)}$	$40_{(5)}$	$31_{(4)}$	$9_{(1)}$	$15_{(2)}$
1200	$25_{(3)}$	$40_{(5)}$	$31_{(4)}$	$8_{(1)}$	$16_{(2)}$
1600	$22_{(3)}$	$36_{(5)}$	$34_{(4)}$	$9_{(1)}$	$19_{(2)}$
Total sum of ranks	21	35	28	9.5	11.5
Overall ranks	3	5	4	1	2

decrease as the sample size increases, indicating improved accuracy and precision of the estimators. The coverage probabilities remain close to the nominal 95% level across all sample sizes, supporting the asymptotic normality of the MO-ExOW-W MLEs.

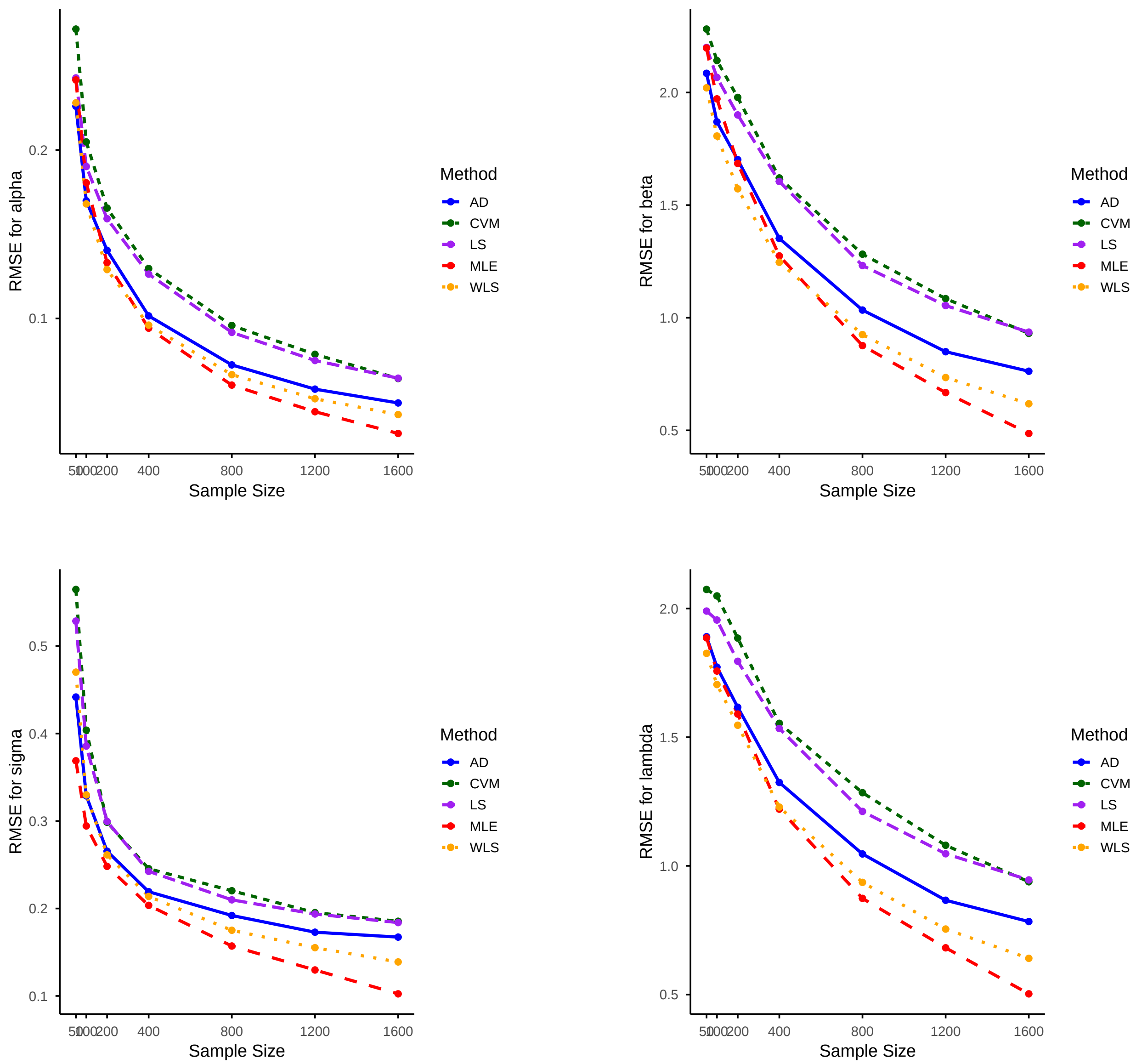


Figure 10. RMSE plots for parameter estimation of the MO-ExOW-Exponential distribution

From Table 9, the MLE has been ranked 1, indicating that it is the best estimation technique for MO-ExOW-W distribution.

The RMSE plots for different parameters of the MO-ExOW-E distribution are shown in Figure 11. The RMSE plots indicate that as sample size increases, the estimates get closer to the true parameter values.

Table 7. Results of Simulations using Different Estimation Techniques for MO-ExOW-W Distribution with parameters $\alpha = 1.8, \beta = 2.4, \lambda = 0.4$ and $\sigma = 1.6$.

Param	n	AD			CVM			LS			ML			WLS		
		Mean	Bias	RMSE	Mean	Bias	RMSE	Mean	Bias	RMSE	Mean	Bias	RMSE	Mean	Bias	RMSE
α	50	2.1967	$0.3967_{(2)}$	$1.1122_{(2)}$	2.2580	$0.4580_{(3)}$	$1.2742_{(4)}$	2.2944	$0.4944_{(5)}$	$1.2780_{(5)}$	1.9319	$0.1319_{(1)}$	$0.8759_{(1)}$	2.2731	$0.4731_{(4)}$	$1.1801_{(3)}$
β	50	2.8721	$0.4721_{(2)}$	$1.6526_{(1)}$	3.2164	$0.8164_{(4)}$	$1.8838_{(5)}$	2.9826	$0.5826_{(3)}$	$1.7941_{(4)}$	3.3823	$0.9823_{(5)}$	$1.7826_{(3)}$	2.8283	$0.4283_{(1)}$	$1.6833_{(2)}$
λ	50	0.4014	$0.0014_{(1)}$	$0.1832_{(1)}$	0.4394	$0.0394_{(4)}$	$0.2308_{(5)}$	0.4018	$0.0018_{(2)}$	$0.2118_{(4)}$	0.4784	$0.0784_{(5)}$	$0.1961_{(3)}$	0.3869	$0.0131_{(3)}$	$0.1921_{(2)}$
σ	50	2.4347	$0.8347_{(3)}$	$1.9009_{(3)}$	2.3191	$0.7191_{(2)}$	$1.8802_{(2)}$	2.4818	$0.8818_{(4)}$	$1.9816_{(4)}$	1.7852	$0.1852_{(1)}$	$1.3802_{(1)}$	2.5721	$0.9721_{(5)}$	$2.0074_{(5)}$
α	100	2.0901	$0.2901_{(2)}$	$0.8305_{(2)}$	2.1277	$0.3277_{(4)}$	$0.9884_{(4)}$	2.1652	$0.3652_{(5)}$	$1.0062_{(5)}$	1.9018	$0.1018_{(1)}$	$0.6409_{(1)}$	2.1166	$0.3166_{(3)}$	$0.9075_{(3)}$
β	100	2.6965	$0.2965_{(1)}$	$1.4310_{(2)}$	2.9979	$0.5979_{(4)}$	$1.6772_{(5)}$	2.8741	$0.4741_{(3)}$	$1.6393_{(4)}$	3.1017	$0.7017_{(5)}$	$1.5163_{(3)}$	2.7071	$0.3071_{(2)}$	$1.4169_{(1)}$
λ	100	0.3928	$0.0072_{(2)}$	$0.1568_{(2)}$	0.4194	$0.0194_{(4)}$	$0.1868_{(4)}$	0.3994	$0.0006_{(1)}$	$0.1873_{(5)}$	0.4440	$0.0440_{(5)}$	$0.1459_{(1)}$	0.3908	$0.0092_{(3)}$	$0.1591_{(3)}$
σ	100	2.3336	$0.7336_{(4)}$	$1.7222_{(3)}$	2.3112	$0.7112_{(3)}$	$1.8346_{(4)}$	2.4052	$0.8052_{(5)}$	$1.8861_{(5)}$	1.7632	$0.1632_{(1)}$	$1.1928_{(1)}$	2.3057	$0.7057_{(2)}$	$1.6929_{(2)}$
α	200	1.9846	$0.1846_{(3)}$	$0.6093_{(2)}$	2.0345	$0.2345_{(4)}$	$0.7583_{(4)}$	2.0870	$0.2870_{(5)}$	$0.7957_{(5)}$	1.8477	$0.0477_{(1)}$	$0.4199_{(1)}$	1.9747	$0.1747_{(2)}$	$0.6126_{(3)}$
β	200	2.6675	$0.2675_{(2)}$	$1.2164_{(3)}$	2.8803	$0.4803_{(5)}$	$1.4766_{(5)}$	2.7695	$0.3695_{(3)}$	$1.4424_{(4)}$	2.8380	$0.4380_{(4)}$	$1.1832_{(2)}$	2.6312	$0.2312_{(1)}$	$1.1803_{(1)}$
λ	200	0.3970	$0.0030_{(1)}$	$0.1261_{(2)}$	0.4129	$0.0129_{(4)}$	$0.1602_{(5)}$	0.3953	$0.0047_{(3)}$	$0.1578_{(4)}$	0.4268	$0.0268_{(5)}$	$0.1096_{(1)}$	0.3962	$0.0038_{(2)}$	$0.1267_{(3)}$
σ	200	2.0483	$0.4483_{(3)}$	$1.3461_{(3)}$	2.1362	$0.5362_{(4)}$	$1.5967_{(4)}$	2.2468	$0.6468_{(5)}$	$1.6586_{(5)}$	1.6889	$0.0889_{(1)}$	$0.8751_{(1)}$	2.0418	$0.4418_{(2)}$	$1.3323_{(2)}$
α	400	1.9111	$0.1111_{(3)}$	$0.4251_{(3)}$	1.9912	$0.1912_{(4)}$	$0.6153_{(5)}$	1.9943	$0.1943_{(5)}$	$0.6116_{(4)}$	1.8297	$0.0297_{(1)}$	$0.2776_{(1)}$	1.8871	$0.0871_{(2)}$	$0.4027_{(2)}$
β	400	2.5526	$0.1526_{(1)}$	$0.9274_{(3)}$	2.6702	$0.2702_{(5)}$	$1.1842_{(5)}$	2.6423	$0.2423_{(3)}$	$1.1673_{(4)}$	2.6561	$0.2561_{(4)}$	$0.8173_{(1)}$	2.5848	$0.1848_{(2)}$	$0.9125_{(2)}$
λ	400	0.3970	$0.0030_{(3)}$	$0.0993_{(3)}$	0.3996	$0.0004_{(1)}$	$0.1312_{(4)}$	0.3959	$0.0041_{(4)}$	$0.1320_{(5)}$	0.4145	$0.0145_{(5)}$	$0.0761_{(1)}$	0.4017	$0.0017_{(2)}$	$0.0964_{(2)}$
σ	400	1.8695	$0.2695_{(3)}$	$0.9859_{(3)}$	2.0272	$0.4272_{(4)}$	$1.3269_{(4)}$	2.0608	$0.4608_{(5)}$	$1.3623_{(5)}$	1.6194	$0.0194_{(1)}$	$0.5548_{(1)}$	1.8000	$0.2000_{(2)}$	$0.8704_{(2)}$
α	800	1.8467	$0.0467_{(3)}$	$0.2781_{(3)}$	1.9172	$0.1172_{(4)}$	$0.4500_{(5)}$	1.9243	$0.1243_{(5)}$	$0.4448_{(4)}$	1.8156	$0.0156_{(1)}$	$0.1743_{(1)}$	1.8429	$0.0429_{(2)}$	$0.2554_{(2)}$
β	800	2.4786	$0.0786_{(1)}$	$0.6466_{(3)}$	2.5545	$0.1545_{(5)}$	$0.8781_{(5)}$	2.5137	$0.1137_{(3)}$	$0.8438_{(4)}$	2.5292	$0.1292_{(4)}$	$0.5075_{(1)}$	2.4899	$0.0899_{(2)}$	$0.6306_{(2)}$
λ	800	0.3996	$0.0004_{(2)}$	$0.0720_{(3)}$	0.3994	$0.0006_{(3)}$	$0.1042_{(5)}$	0.3939	$0.0061_{(4)}$	$0.1007_{(4)}$	0.4073	$0.0073_{(5)}$	$0.0500_{(1)}$	0.4002	$0.0002_{(1)}$	$0.0681_{(2)}$
σ	800	1.7172	$0.1172_{(3)}$	$0.6038_{(3)}$	1.8535	$0.2535_{(4)}$	$0.9680_{(5)}$	1.8752	$0.2752_{(5)}$	$0.9582_{(4)}$	1.5965	$-0.0035_{(1)}$	$0.3435_{(1)}$	1.6894	$0.0894_{(2)}$	$0.5341_{(2)}$
α	1200	1.8300	$0.0300_{(2)}$	$0.2055_{(3)}$	1.8763	$0.0763_{(4)}$	$0.3409_{(4)}$	1.8856	$0.0856_{(5)}$	$0.3517_{(5)}$	1.8041	$0.0041_{(1)}$	$0.1284_{(1)}$	1.8315	$0.0315_{(3)}$	$0.2027_{(2)}$
β	1200	2.4507	$0.0507_{(1)}$	$0.5193_{(3)}$	2.5308	$0.1308_{(5)}$	$0.7231_{(5)}$	2.4780	$0.0780_{(3)}$	$0.7148_{(4)}$	2.5003	$0.1003_{(4)}$	$0.3858_{(1)}$	2.4548	$0.0548_{(2)}$	$0.4932_{(2)}$
λ	1200	0.3993	$0.0007_{(2)}$	$0.0580_{(3)}$	0.4010	$0.0010_{(3)}$	$0.0853_{(4)}$	0.3951	$0.0049_{(4)}$	$0.0858_{(5)}$	0.4071	$0.0071_{(5)}$	$0.0387_{(1)}$	0.3996	$0.0004_{(1)}$	$0.0553_{(2)}$
σ	1200	1.6720	$0.0720_{(3)}$	$0.4354_{(3)}$	1.7458	$0.1458_{(4)}$	$0.7353_{(4)}$	1.7879	$0.1879_{(5)}$	$0.7559_{(5)}$	1.5799	$-0.0201_{(1)}$	$0.2489_{(1)}$	1.6599	$0.0599_{(2)}$	$0.4016_{(2)}$
α	1600	1.8265	$0.0265_{(3)}$	$0.1796_{(3)}$	1.8559	$0.0559_{(4)}$	$0.2790_{(4)}$	1.8589	$0.0589_{(5)}$	$0.2851_{(5)}$	1.8052	$0.0052_{(1)}$	$0.1036_{(1)}$	1.8186	$0.0186_{(2)}$	$0.1689_{(2)}$
β	1600	2.4381	$0.0381_{(1)}$	$0.4437_{(3)}$	2.4839	$0.0839_{(5)}$	$0.6092_{(4)}$	2.4725	$0.0725_{(4)}$	$0.6114_{(5)}$	2.4654	$0.0654_{(3)}$	$0.2979_{(1)}$	2.4464	$0.0464_{(2)}$	$0.4219_{(2)}$
λ	1600	0.3991	$0.0009_{(3)}$	$0.0504_{(3)}$	0.3996	$0.0004_{(1)}$	$0.0736_{(4)}$	0.3979	$0.0021_{(4)}$	$0.0749_{(5)}$	0.4042	$0.0042_{(5)}$	$0.0308_{(1)}$	0.4008	$0.0008_{(2)}$	$0.0480_{(2)}$
σ	1600	1.6527	$0.0527_{(3)}$	$0.3529_{(3)}$	1.7086	$0.1086_{(4)}$	$0.5737_{(4)}$	1.7216	$0.1216_{(5)}$	$0.6027_{(5)}$	1.5870	$-0.0130_{(1)}$	$0.2012_{(1)}$	1.6391	$0.0391_{(2)}$	$0.3368_{(2)}$

Table 8. MAE, Confidence Interval Width, and Coverage Probability for MLE Simulation of MO-ExOW-W

Parameter	Sample Size	MAE	CIW	CP
α	50	0.6149	4.0841	0.9760
β	50	1.4958	4.1495	0.9727
λ	50	0.1506	0.6533	0.9498
σ	50	1.0019	4.4526	0.9757
α	100	0.4327	2.3976	0.9505
β	100	1.1789	3.9549	0.9730
λ	100	0.1149	0.5124	0.9509
σ	100	0.8095	4.3526	0.9733
α	200	0.2971	1.6570	0.9503
β	200	0.8668	3.7704	0.9750
λ	200	0.0846	0.4222	0.9507
σ	200	0.5793	3.8017	0.9502
α	400	0.1969	1.1303	0.9501
β	400	0.5889	3.5009	0.9515
λ	400	0.0592	0.3197	0.9506
σ	400	0.3955	2.3178	0.9503
α	800	0.1151	0.7068	0.9504
β	800	0.3223	1.9594	0.9487
λ	800	0.0337	0.2066	0.9501
σ	800	0.2156	1.3653	0.9502
α	1200	0.0820	0.5676	0.9507
β	1200	0.2299	1.5818	0.9504
λ	1200	0.0241	0.1646	0.9503
σ	1200	0.1584	1.1305	0.9501
α	1600	0.0585	0.4460	0.9505
β	1600	0.1743	1.3319	0.9492
λ	1600	0.0182	0.1379	0.9502
σ	1600	0.1177	0.8913	0.9504

Table 9. Summed ranks (Bias + RMSE) and overall ranks for different estimation methods of MO-ExOW-W Distribution. Smaller sums indicate better performance.

n	AD	CVM	LS	ML	WLS
50	$15_{(1)}$	$29_{(4)}$	$31_{(5)}$	$20_{(2)}$	$25_{(3)}$
100	$18_{(1.5)}$	$32_{(4)}$	$33_{(5)}$	$18_{(1.5)}$	$19_{(3)}$
200	$19_{(3)}$	$35_{(5)}$	$34_{(4)}$	$16_{(1.5)}$	$16_{(1.5)}$
400	$22_{(3)}$	$32_{(4)}$	$35_{(5)}$	$15_{(1)}$	$16_{(2)}$
800	$21_{(3)}$	$36_{(5)}$	$33_{(4)}$	$15_{(1.5)}$	$15_{(1.5)}$
1200	$20_{(3)}$	$33_{(4)}$	$36_{(5)}$	$15_{(1)}$	$16_{(2)}$
1600	$22_{(3)}$	$30_{(4)}$	$38_{(5)}$	$14_{(1)}$	$16_{(2)}$
Total sum of ranks	17.5	30	33	9.5	15
Overall ranks	3	4	5	1	2

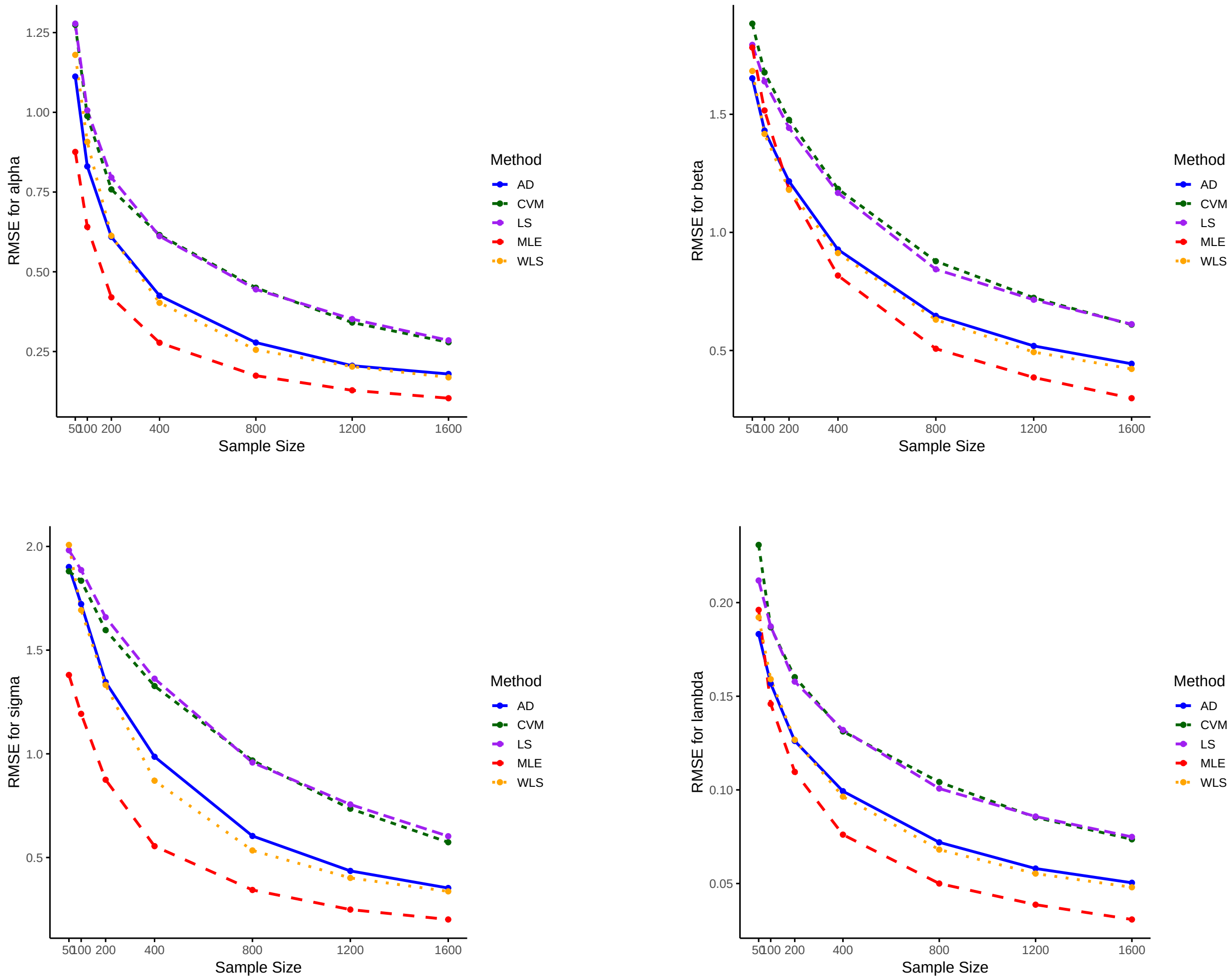


Figure 11. RMSE plots for parameter estimation of the MO-ExOW-Weibull distribution

6. Applications and Model Comparisons

In this section, we assess the practical relevance of some of the special cases of the proposed MO-ExOW-G family of distributions. We analyze real-life data sets and compare the performance of the special cases of MO-ExOW-G family against several existing competing distributions. Model comparisons are based on several goodness-of-fit (GoF) statistics and graphical diagnostics.

The GoF statistics that were used include: the -2loglikelihood (-2log(L)), Akaike Information Criterion (AIC), Consistent Akaike Information Criterion (CAIC), Bayesian Information Criterion (BIC), Cramér -von Mises (W^*) and Andersen-Darling (A^*), sum of squares (SS) from the probability plots, Kolmogorov-Smirnov (K-S) statistic and its P-value. The model with the smallest values of these GoF statistics and biggest p-value of K-S statistics is regarded as the best model. In addition, graphical diagnostics that were used in model comparison include density plots and probability plots. Moreover profile likelihood plots where used to ascertain identifiability of parameters. Scaled TTT plots, hazard rate plots, empirical cdf plots, and Kaplan–Meier survival plots where used to validate the suitability of the proposed distribution in reliability and lifetime data analysis. In addition, likelihood ratio (LR) tests are employed to formally assess the significance of the additional parameters introduced by the MO-ExOW-G family and to compare nested models. Furthermore, Cox–Snell residuals are examined to evaluate the overall adequacy of the fitted models, where an approximately linear pattern along the unit exponential reference line indicates a satisfactory fit.

6.1. Chemotherapy Data

The chemotherapy dataset was analysed by Sapkota et al. [27]. It represents the survival times (in years) of 45 patients given chemotherapy alone. The data are as follows: 0.047, 0.115, 0.121, 0.132, 0.164, 0.197, 0.203, 0.260, 0.282, 0.296, 0.334, 0.395, 0.458, 0.466, 0.501, 0.507, 0.529, 0.534, 0.540, 0.641, 0.644, 0.696, 0.841, 0.863, 1.099, 1.219, 1.271, 1.326, 1.447, 1.485, 1.553, 1.581, 1.589, 2.178, 2.343, 2.416, 2.444, 2.825, 2.830, 3.578, 3.658, 3.743, 3.978, 4.003, 4.033.

The chemotherapy dataset has the following descriptive statistics : mean = 1.3414, variance = 1.5540, coefficient of skewness = 0.9721 and coefficient of kurtosis = 2.6638. These data are right-skewed and platykurtic.

The MO-ExOW-E distribution was fitted and compared with other known distributions, namely; Kumaraswamy-Marshall-Olkin Exponential (KMOE) by George and Thobias [15], Exponentiated Kumaraswamy-Exponential (EKwE) by Rodrigues and Silva [26], Exponentiated Generalized Exponential (EGEE) by Bukoye and Oyeyemi [8], Kumaraswamy-Extended Exponential (KwEE) by Hassan et al. [18] and Marshall-Olkin-Kumaraswamy-Exponential (MOKwE) by Handique et al. [17].

Table 10 shows the parameter estimates, standard errors of estimates in brackets for various distributions that were fitted to the chemotherapy data, alongside with goodness-of-fit statistics. The proposed MO-ExOW-E distribution fits the chemotherapy data better than other models as it has the lowest goodness-of-fit statistics and the highest K-S p-value.

The **density and probability plots** of the Mo-ExOW-E distribution against other models are presented in Figure 12 and Figure 13. These plots indicate that the MO-ExOW-E distribution fits the chemotherapy data.

The **profile log-likelihood** plots for chemotherapy data under the MO-ExOW-E distribution are given in Figure 14. Each parameter was identifiable.

The **hazard rate** and **scaled total time on test (TTT)** plots for chemotherapy data under the assumption of MO-ExOW-E distribution are given in Figure 15. They indicate an increasing-decreasing-increasing failure rate.

Table 10. Parameter Estimates and GoF Statistics for Chemotherapy Data

									GoF Statistics			
Distribution	Estimates and SEs				-2log(L)	AIC	CAIC	BIC	W^*	A^*	K-S	p-value
	α	β	σ	λ								
MO ExOW-E	2.2297	7.7231	1.5722	2.7021	113.1998	121.1998	122.1998	128.4264	0.0376	0.2895	0.0735	0.9534
	(1.13329)	(7.35307)	(0.75624)	(1.45728)								
	α	β	σ	λ								
ExOW-E	3.4111	10.3656	-	2.4461	118.7768	124.7768	125.3622	130.1968	0.0488	0.3878	0.1401	0.3102
	(0.4175)	(1.2132)	-	(0.7446)								
	α	β	σ	λ								
MO-E	-	-	1.01901	0.5523	120.3581	124.3581	124.6438	127.9714	0.0752	0.5064	0.1738	0.1167
	-	-	(0.5099)	(0.2133)								
	a	b	α	λ								
KMOE	1.3325	1.8782	1.0255×10^{5}	1.0583×10^{-4}	118.4665	126.4668	127.4668	133.6934	0.0749	0.5107	0.1026	0.6923
	(1.2206×10^{-8})	(1.0069×10^{-9})	(1.3299×10^{-14})	(1.7247×10^{-5})								
	α	θ	β	λ								
EKwE	1.5554	1.4933×10^{-2}	3.4642×10^{2}	3.8285×10^{-1}	115.3939	123.3939	124.3939	130.6205	0.0588	0.4103	0.0894	0.8328
	(8.0345×10^{-1})	(2.0793×10^{-2})	(2.4861×10^{-4})	(3.7298×10^{-1})								
	α	β	k	θ								
EGEE	1.5548	6.3444×10^{2}	1.0002×10^{-2}	2.6205	115.3825	123.3825	124.3825	130.6092	0.0585	0.4088	0.0867	0.8585
	(7.1412×10^{-1})	(1.3215×10^{-3})	(1.4014×10^{-2})	(2.3335)								
	ν	δ	ϕ	β								
KwEE	1.5396	0.1248	6.2741	2.6060×10^{-4}	115.3952	123.3952	124.3952	130.6219	0.0655	0.4481	0.1130	0.5752
	(0.6449)	(0.0199)	(0.0578)	(5.7088×10^{-4})								
	a	b	α	λ								
MOKwE	1.2268	4.9273×10^{1}	4.6907×10^{-1}	2.1278×10^{-2}	115.6785	123.6785	124.6785	130.9052	0.0678	0.4618	0.0961	0.7644
	(2.5547×10^{-1})	(1.2997×10^{-4})	(4.6930×10^{-1})	(9.2906×10^{-3})								

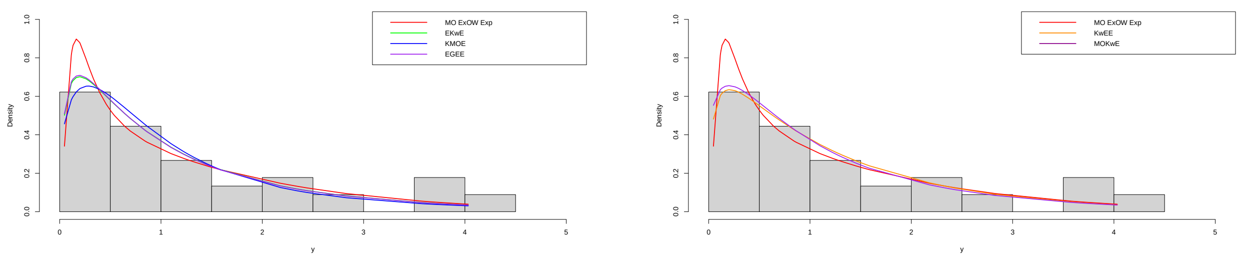


Figure 12. Density plots for the MO-ExOW-E distribution and competing models for chemotherapy data.

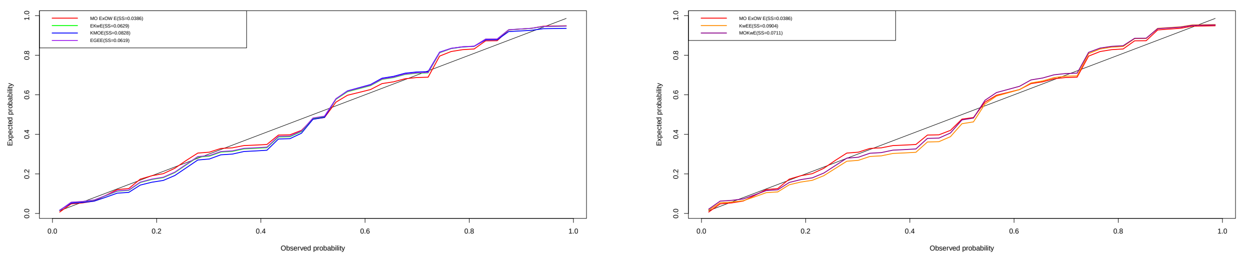


Figure 13. Probability plots for the MO-ExOW-E distribution and competing models for chemotherapy data.

The **Kaplan–Meier (K-M) Survival** and **Empirical cdf (ECDF)** plots for chemotherapy data following the MO-ExOW-E distribution are given in Figure 16. The ECDF plot is in agreement with the corresponding theoretical cdf of the MO-ExOW-E distribution and the K–M survival curve closely aligns with the fitted survival function of the MO-ExOW-E distribution. This supports the adequacy of the proposed MO-ExOW-E distribution for the chemotherapy data.

Figure 17 shows the Cox–Snell residual plot for the chemotherapy data, where the estimated cumulative hazard closely follows the 45° reference line, indicating an adequate overall fit of the MO-ExOW-E distribution.

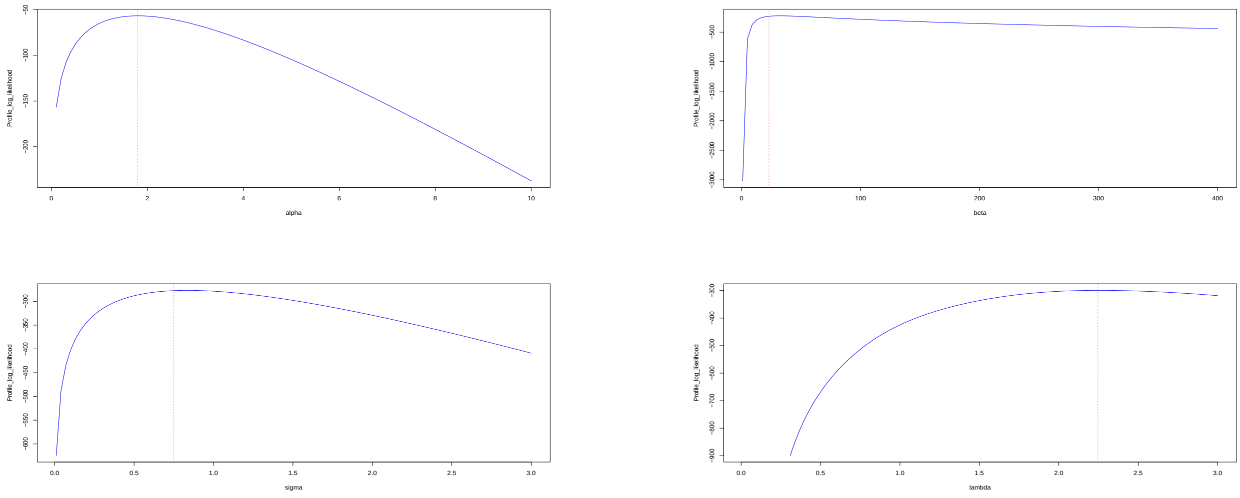
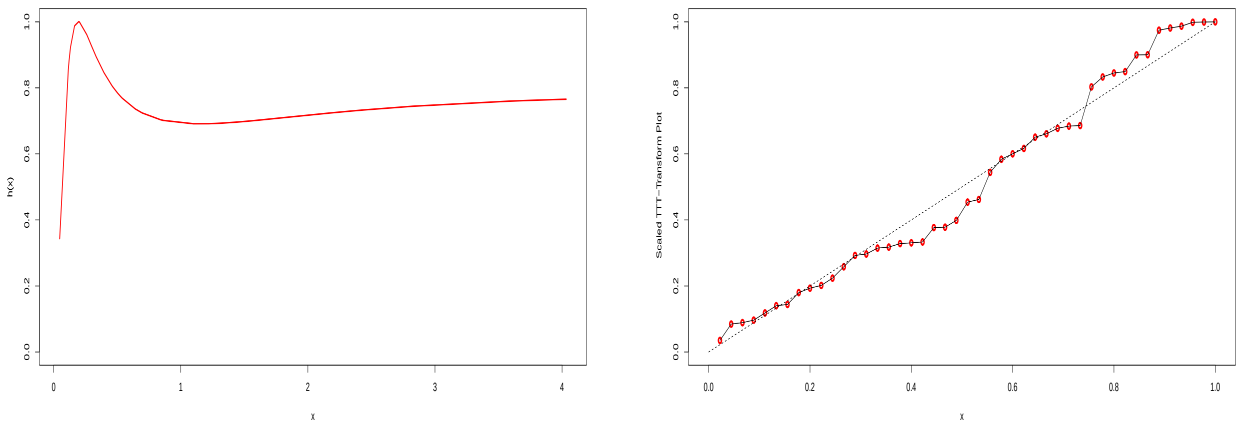


Figure 14. Profile log-likelihood plots for the parameters $(\alpha, \sigma, \beta, \lambda)$ of the MO-ExOW-E distribution fitted to the chemotherapy data.



(a) Hazard rate function plot

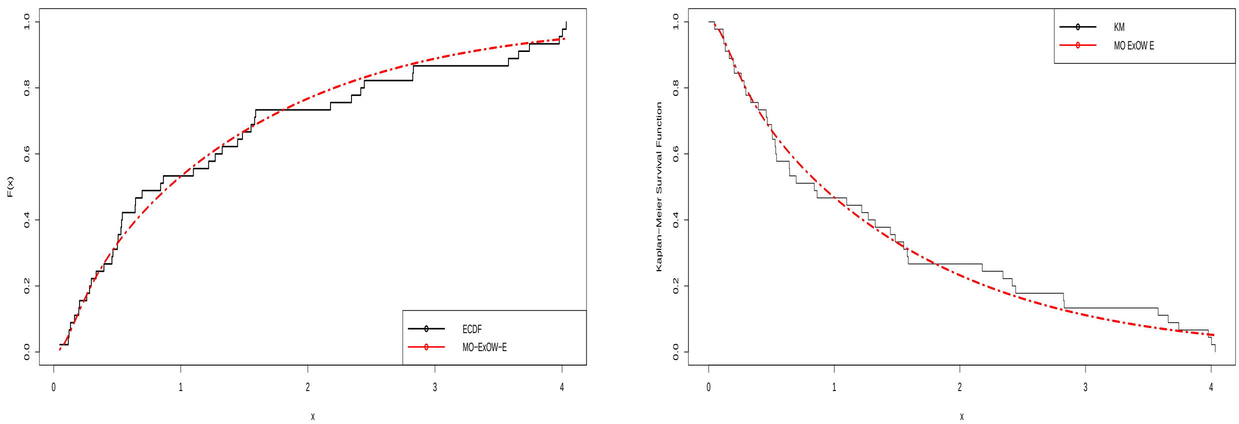
(b) Scaled TTT plot

Figure 15. Hazard rate and scaled TTT plots for the MO-ExOW-E distribution fitted to chemotherapy data.

The Likelihood Ratio Tests were conducted to compare the MO-ExOW-Exponential model with its nested models, particularly the Extended Odd Weibull-Exponential (ExOW-E) and Marshall-Olkin-Exponential (MO-E) distributions for the chemotherapy data.

Table 11. Likelihood ratio tests for the Chemotherapy dataset

Full model	**Reduced model**	**LR statistic**	**df**	**p-value**
MO-ExOW-E	ExOW-E	5.5770	1	0.0182
MO-ExOW-E	MO-E	7.1583	2	0.0279



(a) Empirical cdf plot

(b) Kaplan–Meier survival plot

Figure 16. ECDF and K–M survival plots for the MO-ExOW-E distribution fitted to chemotherapy data.

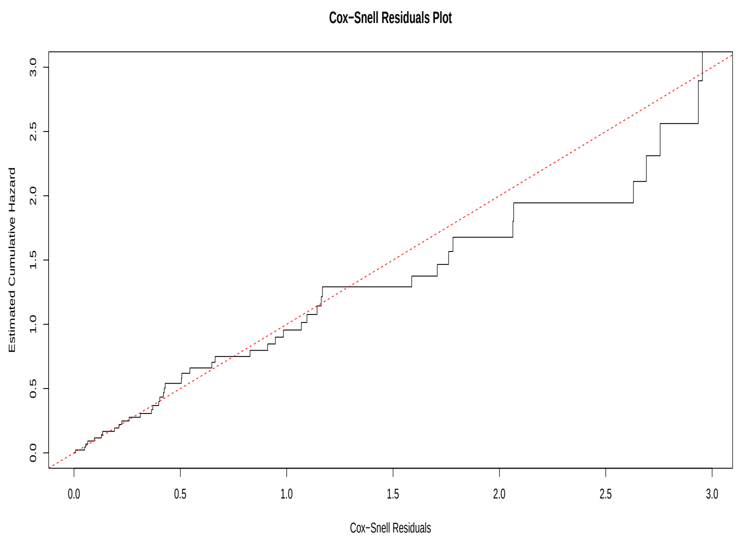


Figure 17. Cox-Snell residual plot for the chemotherapy data.

The results in Table 11 show that the MO-ExOW-E fits better than the nested models for the chemotherapy data.

6.2. *Silicon Nitride Data*

The silicon nitride dataset was analysed by Chipepa and Oluyede [9]. The data represent the fracture toughness of 119 samples of silicon nitride measured in MPam12.

The data are as follows: 5.50, 5.00, 4.90, 6.40, 5.10, 5.20, 5.20, 5.00, 4.70, 4.00, 4.50, 4.20, 4.10, 4.56, 5.01, 4.70, 3.13, 3.12, 2.68, 2.77, 2.70, 2.36, 4.38, 5.73, 4.35, 6.81, 1.91, 2.66, 2.61, 1.68, 2.04, 2.08, 2.13, 3.80, 3.73, 3.71, 3.28, 3.90, 4.00, 3.80, 4.10, 3.90, 4.05, 4.00, 3.95, 4.00, 4.50, 4.50, 4.20, 4.55, 4.65, 4.10, 4.25, 4.30, 4.50, 4.70, 5.15, 4.30, 4.50, 4.90, 5.00, 5.35, 5.15, 5.25, 5.80, 5.85, 5.90, 5.75, 6.25, 6.05, 5.90, 3.60, 4.10, 4.50, 5.30, 4.85, 5.30, 5.45, 5.10, 5.30, 5.20, 5.30, 5.25, 4.75, 4.50, 4.20, 4.00, 4.15, 4.25, 4.30, 3.75, 3.95, 3.51, 4.13, 5.40, 5.00, 2.10, 4.60, 3.20, 2.50, 4.10, 3.50, 3.20, 3.30, 4.60, 4.30, 4.30, 4.50, 5.50, 4.60, 4.90, 4.30, 3.00, 3.40, 3.70, 4.40, 4.90, 4.90, 5.00.

The silicon nitride dataset has the following descriptive statistics : mean = 4.3254, variance = 1.0373, coefficient of skewness = -0.4167 and coefficient of kurtosis = 3.0934. These data are left-skewed and leptokurtic.

The MO-ExOW-E distribution was fitted and compared with other known equi-parameter distributions, namely; Kumaraswamy-Marshall-Olkin Exponential (KMOE) by George and Thobias [15], Exponentiated Kumaraswamy-Exponential (EKwE) by Rodrigues and Silva [26], Exponentiated Generalized Exponential (EGEE) by Bukoye and Oyeyemi [8], Kumaraswamy-Extended Exponential (KwEE) by Hassan et al. [18] and the Marshall-Olkin-Kumaraswamy-Exponential (MOKwE) by Handique et al. [17].

Table 12. Parameter Estimates and GoF Statistics for Silicon Nitrite Data

					GoF Statistics							
Distribution	Estimates and SEs				-2log(L)	AIC	CAIC	BIC	W^*	A^*	K-S	p-value
	α	β	σ	λ								
MO-ExOW-E	2.9510	0.1874	4.2317	0.1837	335.5498	343.5498	343.9007	354.6663	0.0459	0.2898	0.0491	0.9368
	(1.13329)	(0.1536)	(0.4669)	(0.0427)								
	α	β	σ	λ								
ExOW-E	4.9503	2.6531	-	0.1739	395.6995	401.6995	401.9082	410.0369	0.7152	0.2482	0.1994	0.0002
	(0.3090)	(0.2122)	-	(0.0036)								
	α	β	σ	λ								
MO-E	-	-	2.5865×10^{3}	1.7572	341.9567	345.9567	346.0602	351.5150	0.0815	0.5764	0.0676	0.6485
	-	-	(3.3519×10^{-6})	(3.4666×10^{-2})								
	a	b	α	λ								
KMOE	5.3961	1.3421×10^{1}	1.0256×10^{5}	1.6220×10^{-2}	337.9650	345.9650	346.3158	357.0815	0.1005	0.6274	0.0719	0.5696
	(4.3753×10^{-1})	(9.3738×10^{-3})	(1.9273×10^{-7})	(3.5109×10^{-3})								
	α	θ	β	λ								
EKwE	107.5352	12.5668	0.5424	0.2278	337.6125	345.6125	345.9634	356.729	0.1022	0.6204	0.0775	0.4727
	(8.0345×10^{-1})	(2.0793×10^{-2})	(2.4861×10^{-4})	(3.7298×10^{-1})								
	α	β	k	θ								
EGEE	149.1216	0.6360	10.0617	5.36341	337.6890	345.6890	346.0399	356.8055	0.1029	0.6289	0.0807	0.4213
	(117.6088)	(0.3189)	(5.7762)	(2.9523)								
	ν	δ	ϕ	β								
KwEE	2.6843	1.7528×10^{4}	5.2807×10^{-2}	9.4002×10^{5}	337.7262	345.7261	346.0770	356.8426	0.0994	0.6121	0.0731	0.5480
	(0.6449×10^{-4})	(0.0199×10^{-4})	(0.0578×10^{-4})	(5.7088×10^{-4})								
	a	b	α	λ								
MOKwE	1.2588×10^{2}	4.6907×10^{-1}	2.0600×10^{2}	3.7039	340.1832	348.1832	348.5341	359.2997	0.1101	0.7449	0.0560	0.8498
	(4.1544×10^{-3})	(1.1818×10^{-1})	(4.5990×10^{-3})	(6.7794×10^{-1})								

Table 12 shows the parameter estimates, standard errors of estimates in brackets for various distributions that were fitted to the silicon nitride data, alongside with goodness-of-fit statistics. The proposed MO-ExOW-E distribution fits the silicon nitride data better than other models as it has the lowest goodness-of-fit statistics and the highest K-S p-value.

The **density and probability plots** of the MO-ExOW-E distribution against other models are presented in Figure 18 and Figure 19. These plots indicate that the MO-ExOW-E distribution fits the silicon nitride data well.

The **profile log-likelihood** plots for silicon nitride data under the MO-ExOW-E distribution are given in Figure 20. Each parameter was identifiable.

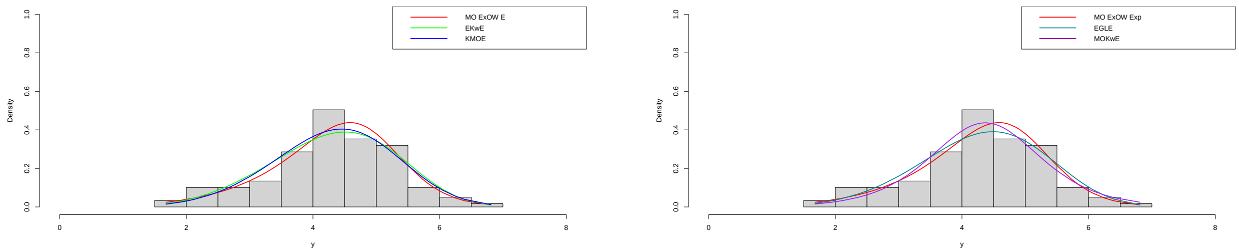


Figure 18. Density plots for the MO-ExOW-E distribution and competing models for the silicon nitride data.

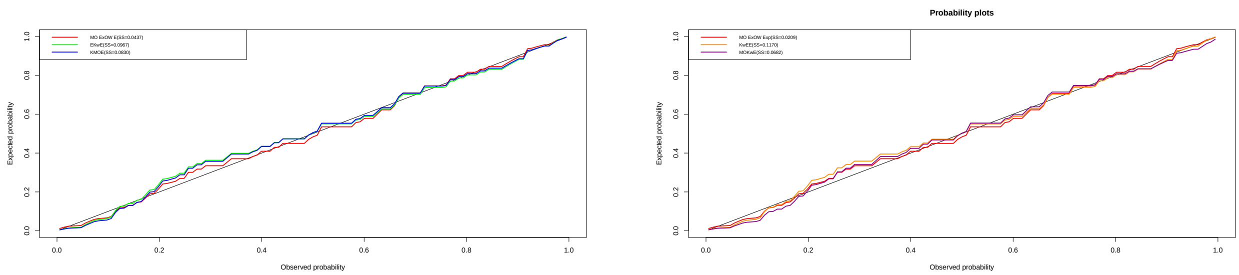


Figure 19. Probability plots for the MO-ExOW-E distribution and competing models for the silicon nitride data.

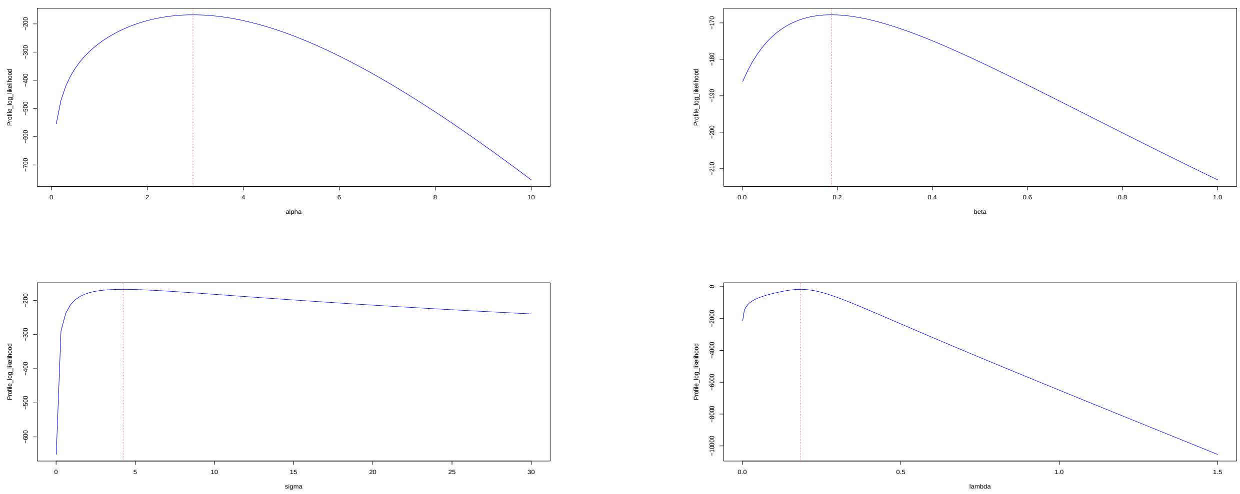
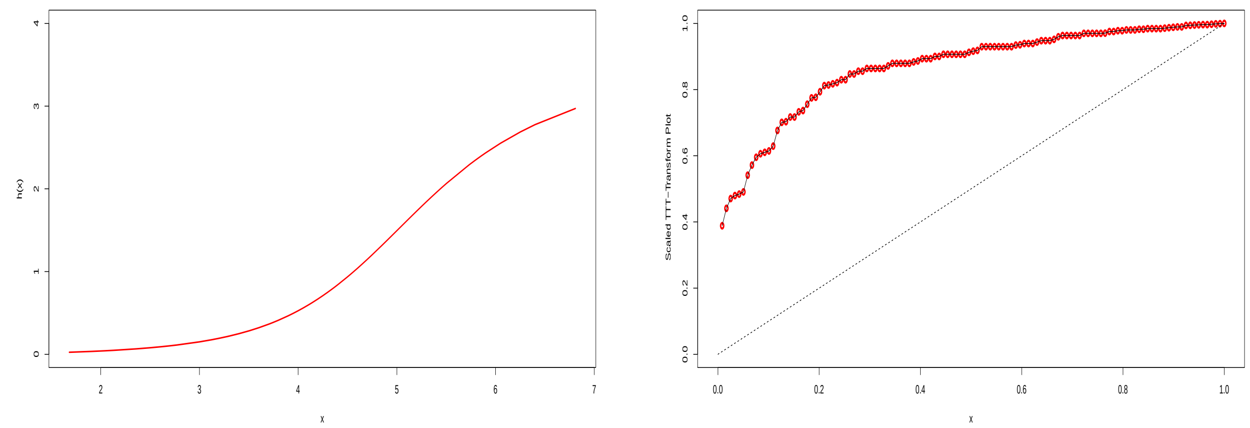


Figure 20. Profile log-likelihood plots for the parameters $(\alpha, \sigma, \beta, \lambda)$ of the MO-ExOW-E distribution fitted to the silicon nitride data.

The **hazard rate** and **scaled (TTT)** plots for silicon nitride data under the assumption of MO-ExOW-E distribution are given in Figure 21. The plots indicate an increasing failure rate.

The **K–M survival** and **ECDF** plots for silicon nitride data are given in Figure 22. The ECDF plot is in agreement with the corresponding theoretical cdf of the MO-ExOW-E distribution. The Kaplan–Meier survival curve closely aligns with the fitted survival function of the MO-ExOW-E distribution. This supports the adequacy

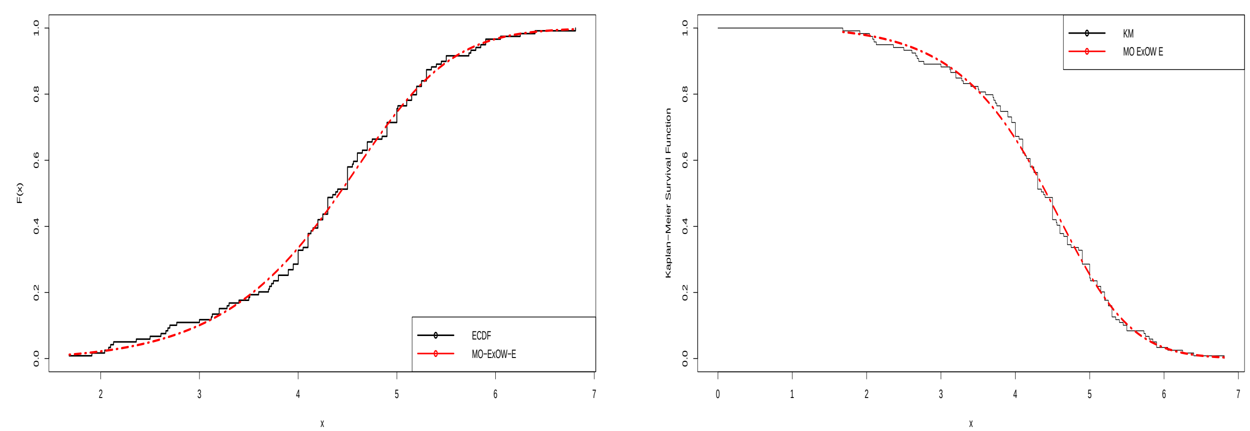


(a) Hazard rate function plot

(b) Scaled TTT plot

Figure 21. Hazard rate and scaled TTT plots of the MO-ExOW-E distribution for the silicon nitride data.

of the proposed MO-ExOW-E distribution for the silicon nitride data.



(a) Empirical cdf plot

(b) Kaplan–Meier survival plot

Figure 22. ECDF and K–M survival plots for the MO-ExOW-E distribution fitted to silicon nitride data.

Figure 23 shows the Cox–Snell residual plot for the silicon nitride data, where the estimated cumulative hazard closely follows the 45° reference line, indicating an adequate overall fit of the MO-ExOW-E distribution.

The Likelihood Ratio Tests were conducted to compare the MO-ExOW-Exponential model with its nested models, particularly the Extended Odd Weibull-Exponential (ExOW-E) and Marshall-Olkin-Exponential (MO-E) distributions for the silicon nitrite data.

The results in Table 13 show that the MO-ExOW-E fits better than the nested models for the silicon nitrite data.

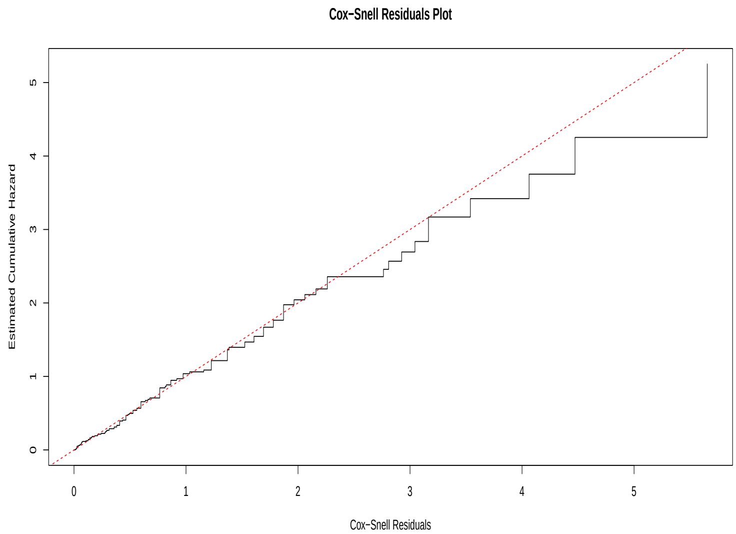


Figure 23. Cox-Snell residual plot for the silicon nitride data.

Table 13. Likelihood ratio tests for the Silicon Nitrite dataset

Full model	**Reduced model**	**LR statistic**	**df**	**p-value**
MO-ExOW-E	ExOW-E	60.1497	1	< 0.0001
MO-ExOW-E	MO-E	6.4069	2	0.0406

6.3. Annual Maximum Antecedent Rainfall Measurements Data

In this section we analysed the annual maximum antecedent rainfall measurements data. This data set was analysed by Moakofi et al. [24].
The data set consists of 52 ordered annual maximum antecedent rainfall measurements in mm from Maple Ridge in British Columbia, Canada. The data are as follows: 264.9, 314.1, 364.6, 379.8, 419.3, 457.4, 459.4, 460, 490.3, 490.6, 502.2, 525.2, 526.8, 528.6, 528.6, 537.7, 539.6, 540.8, 551.0, 573.5, 579.2, 588.2, 588.7, 589.7, 592.1, 592.8, 600.8, 604.4, 608.4, 609.8, 619.2, 626.4, 629.4, 636.4, 645.2, 657.6, 663.5, 664.9, 671.7, 673.0, 682.6, 689.8, 698, 698.6, 698.8, 703.2, 755.9, 786, 787.2, 798.6, 850.4, 895.1.

The rain dataset has the following descriptive statistics : mean = 595, variance = 16158.3, coefficient of skewness = -0.1895 and coefficient of kurtosis = 3.3454. These data are left-skewed and leptokurtic.

The MO-ExOW-W distribution was fitted and compared with other known equi-parameter distributions, namely; Kumaraswamy Weibull (KW) by Cordeiro et al. [12], Marshall-Olkin-Gompertz-Weibull (MO-GOM-W) by Chipepa and Oluyede [9], Marshall-Olkin Extended Weibull (MOEW) by Cordeiro and Lemonte [11], Alpha power Topp-Leone Weibull (APTLW) by Benkhelifa [7], Marshall–Olkin log–logistic Weibull (MOLLW) by Lepetu et al. [21] and Type II exponentiated half logistic Weibull (TIIEHLW) by Al-Mofleh et al. [2]. The parameter estimates and goodness-of-fit statistics for the rainfall data are given in Table 14. The proposed MO-ExOW-W distribution fits the dataset better as indicated by the lowest values of goodness-of-fit statistics and the highest K-S p-value.

The **fitted density** and **probability** plots for the MO-ExOW-W distribution and competing models are presented in Figure 24 and Figure 25. The plots indicate that the MO-ExOW-W distribution fits the rainfall measurement

Table 14. Parameter Estimates and GoF statistics for Rainfall Measurement Data

Distribution	Estimates and SEs				GoF Statistics							
					-2log(L)	AIC	CAIC	BIC	W^*	A^*	K-S	p-value
	α	β	σ	λ								
MO-ExOW-W	3.9718×10^{-2}	4.0915×10^{-2}	1.4349×10^{3}	6.2304×10^{-1}	649.9070	657.9070	658.7582	665.7121	0.0250	0.1761	0.0633	0.9853
	(1.3296×10^{-2})	(3.1467×10^{-2})	(5.9077×10^{-6})	(6.7523×10^{-2})								
	α	β	σ	λ								
ExOW-W	0.0087	4.1006	-	0.9659	808.3742	814.3742	814.8742	820.2279	0.0810	0.5028	0.5869	< 0.0001
	(1.8802×10^{-6})	(7.7213×10^{-6})	-	(1.6480×10^{-1})								
	α	β	σ	λ								
MO-W	-	-	3.2459	0.0957	974.8362	978.8362	979.0811	982.7387	0.1459	0.8971	0.5812	< 0.0001
	-	-	(1.7377×10^{-13})	(2.0005×10^{-8})								
	a	b	α	β								
KW	2.2438	0.1438	0.0026	4.1122	652.3072	660.3078	661.1589	668.1128	0.0757	0.4666	0.1163	0.4825
	(2.2042×10^{-2})	(9.9246×10^{-5})	(5.2765×10^{-5})	(1.3313×10^{-1})								
	δ	θ	λ	γ								
MO-GOM-W	4.8130	0.0012	0.4427	0.3733	656.8790	664.8790	665.7300	672.6840	0.0604	0.3995	0.1484	0.2021
	(0.0004)	(0.0009)	(0.0376)	(0.1105)								
	α	β	λ	k								
MOEW	8.5237×10^{2}	2.9274×10^{-7}	1.1515×10^{-2}	1.9266	653.0402	661.0407	661.8917	668.8456	0.0334	0.2163	0.1311	0.3333
	(2.7697×10^{-5})	(3.8427×10^{-1})	(2.0690×10^{-3})	(2.2807×10^{-2})								
	θ	α	β	λ								
APTLWW	5.0578	9.9936×10^{3}	1.2678	6.4283×10^{-4}	656.1637	664.1637	665.0128	671.9667	0.1238	0.7664	0.1029	0.6406
	(8.1499×10^{-4})	(8.8743×10^{-8})	(7.4719×10^{-2})	(3.0650×10^{-4})								
	c	α	β	δ								
MOLLW	0.0000	4.7911×10^{-5}	1.7617	8.4445×10^{1}	654.4467	662.4499	663.3009	670.2548	0.0302	0.2094	0.1124	0.5274
	(8.6357×10^{-11})	(2.5864×10^{-4})	(7.7226×10^{-10})	(3.1656×10^{-13})								
	a	λ	δ	γ								
TIIEHLW	1.2563	8.6915	3.7502×10^{-5}	1.7054	656.8876	664.8869	665.7379	672.6919	0.1398	0.8603	0.1072	0.5882
	(2.23645×10^{-11})	(4.3483×10^{-12})	(2.1033×10^{-6})	(5.0124×10^{-10})								

data well.

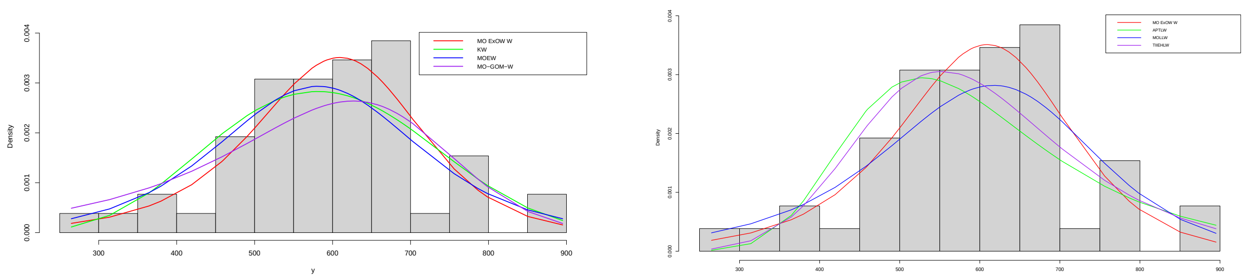


Figure 24. Density plots of the MO-ExOW-W distribution and competing models for the rainfall measurement data.

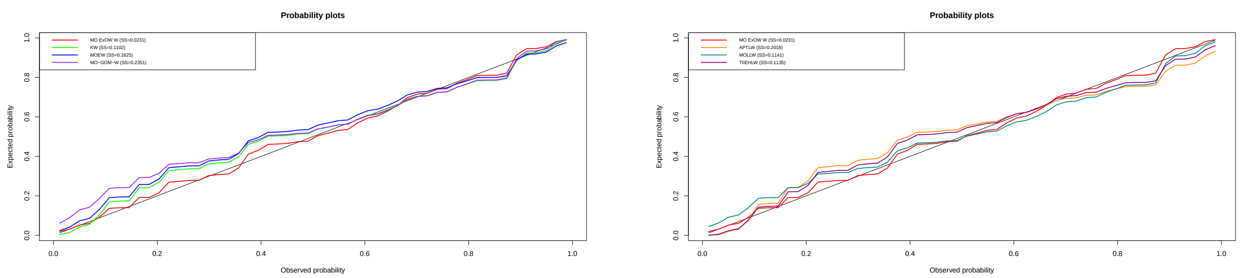


Figure 25. Probability plots of the MO-ExOW-W distribution and competing models for the rainfall measurement data.

The **profile log-likelihood** plots for the parameters $(\alpha, \sigma, \beta, \lambda)$ are shown in Figure 26. The plots show that the parameters are identifiable.

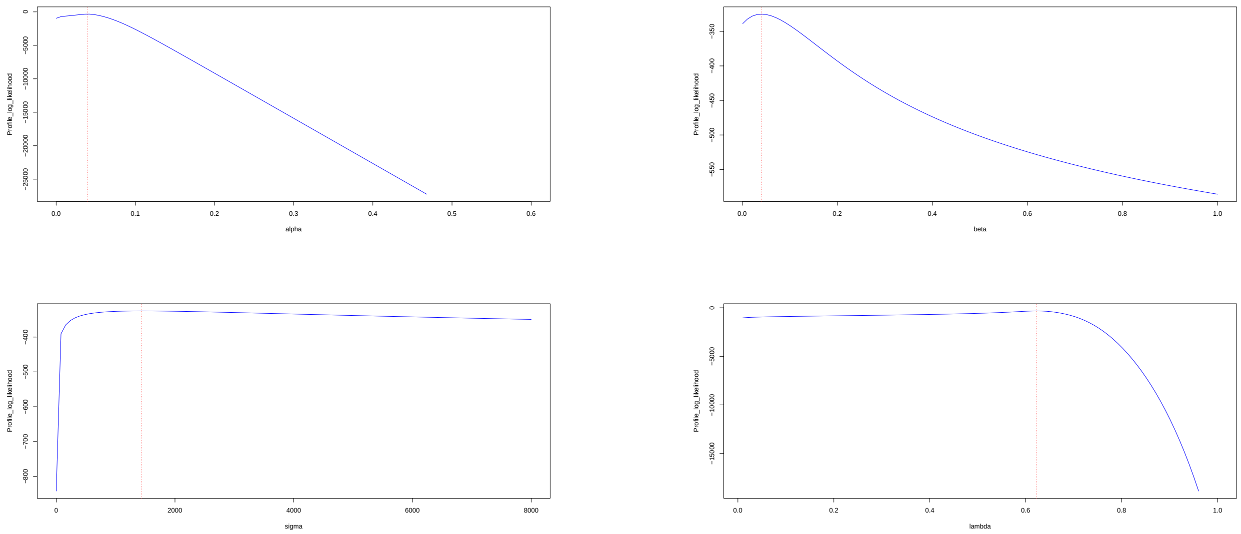
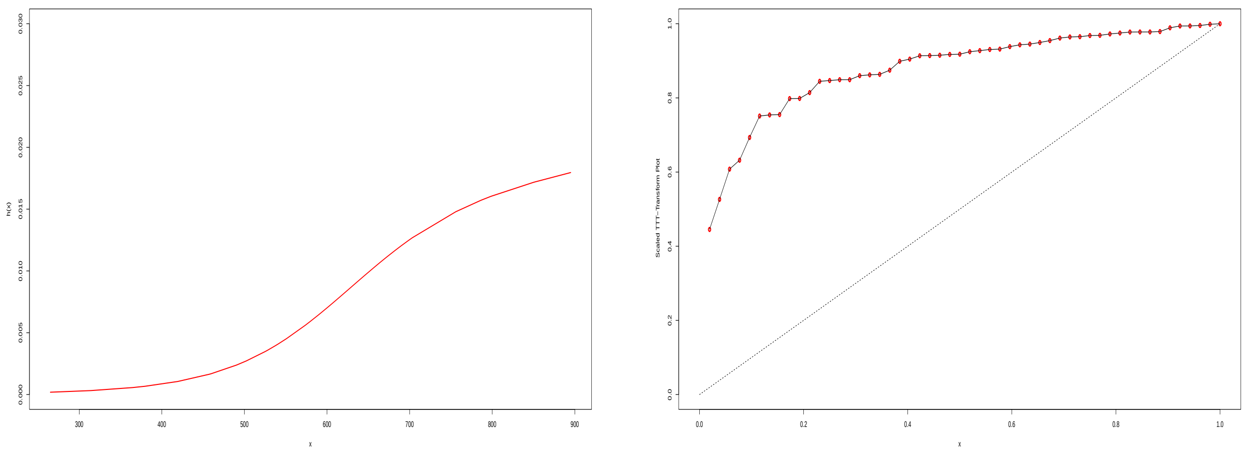


Figure 26. Profile log-likelihood plots for the parameters of the MO-ExOW-W distribution fitted to the rainfall measurement data.

The **hazard rate** and **scaled TTT plots** for rainfall measurement data are displayed in Figure 27. The plots depict an increasing-decreasing hazard rate function.



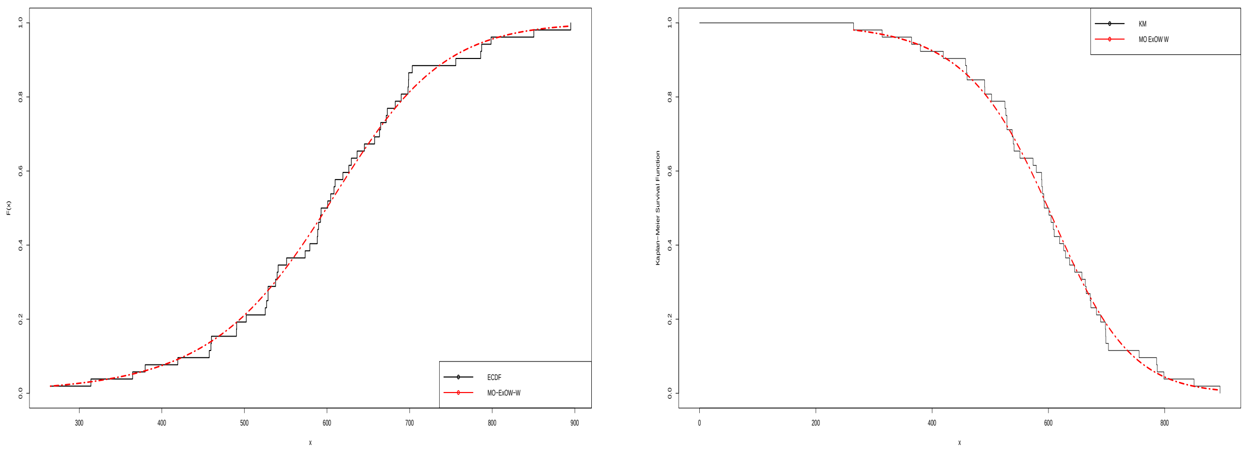
(a) Hazard rate function plot

(b) Scaled TTT plot

Figure 27. Hazard rate and scaled TTT plots of the MO-ExOW-W distribution for the rainfall measurement data.

The **empirical cdf and Kaplan–Meier survival plots** for the rainfall measurement data are presented in Figure 28.

The ECDF plot is in agreement with the corresponding theoretical cdf of the MO-ExOW-W distribution and the Kaplan–Meier survival curve closely aligns with the fitted survival function of the MO-ExOW-W distribution. This supports the adequacy of the proposed MO-ExOW-W distribution for the rainfall measurement data.



(a) Empirical cdf plot

(b) Kaplan–Meier survival plot

Figure 28. Empirical cdf and Kaplan–Meier survival plots for the MO-ExOW-W distribution fitted to the rainfall measurement data.

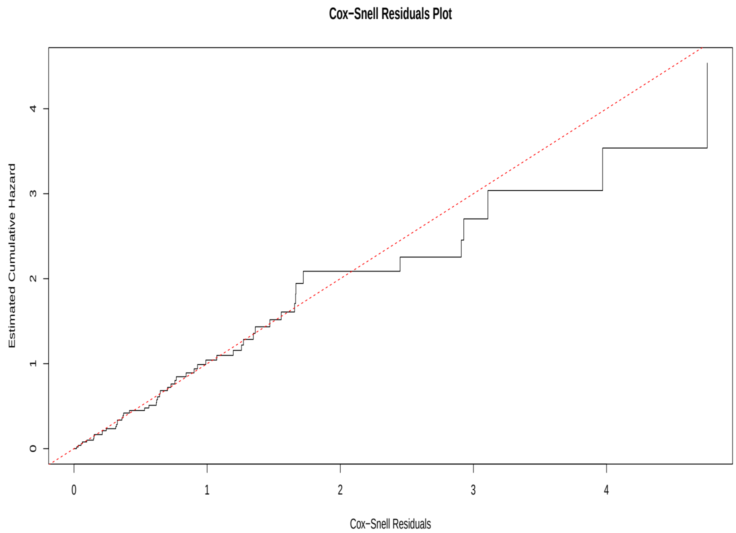


Figure 29. Cox-Snell residual plot for the rainfall data.

Figure 29 shows the Cox–Snell residual plot for the rainfall data, where the estimated cumulative hazard closely follows the 45° reference line, indicating an adequate overall fit of the MO-ExOW-W distribution.

The Likelihood Ratio Tests were conducted to compare the MO-ExOW-Weibull model with its nested models, particularly the Extended Odd Weibull-Weibull (ExOW-W) and Marshall-Olkin-Weibull (MO-W) distributions for the rainfall data.

The results in Table 15 show that the MO-ExOW-W fits better than the nested models for the rainfall data.

Table 15. Likelihood ratio tests for the Rainfall Measurement Data

Full model	Reduced model	LR statistic	df	p-value
MO-ExOW-W	ExOW-W	158.4672	1	< 0.0001
MO-ExOW-W	MO-W	324.9292	2	< 0.0001

7. Conclusion and Future Research

We have introduced the new Marshall–Olkin Extended Odd Weibull-G family of distributions as a versatile and robust framework for modelling lifetime data. This family can accommodate a wide variety of hazard rate shapes, including increasing, decreasing, decreasing-increasing-decreasing, increasing-decreasing and bathtub. The probability density functions of its special cases exhibit diverse shapes such as bimodal, nearly symmetric, right-skewed and left-skewed shapes. Statistical properties of the proposed family have been derived, and simulation studies were conducted to evaluate parameter estimation using various techniques. Moreover, we applied two special cases being the MO-ExOW-E and MO-ExOW-W distributions to real life datasets. The MO-ExOW-E distribution was applied to the chemotherapy data and silicon nitride data whereas the MO-ExOW-W was applied to the annual maximum antecedent rainfall measurement data. In all applications, the proposed models demonstrated superior performance compared to several competing distributions.

Despite its flexibility and strong empirical performance, the MO-ExOW-G family is not without limitations. The inclusion of multiple shape parameters, in addition to the baseline distribution parameters, increases model complexity and may introduce challenges related to parameter identifiability, computational burden, and potential overfitting, particularly for small or moderate sample sizes. Consequently, the use of the MO-ExOW-G model is most appropriate when the data exhibit complex features such as non-monotone hazard rates, pronounced skewness, or heavy-tailed behavior that cannot be adequately captured by simpler nested models. In practice, it is advisable for practitioners to begin with reduced sub-models, such as the ExOW-G or Marshall–Olkin-G distributions, and formally assess the need for additional flexibility using likelihood-based criteria or hypothesis testing procedures.

Another limitation of the present work is that the characterization of hazard rate shapes is primarily explored through graphical and numerical analysis. While this approach effectively illustrates the wide range of behaviors supported by the proposed family, deriving explicit analytical conditions for specific hazard shapes may be analytically intractable and is left for future investigation. In addition, although several estimation methods were considered, simulation studies were conducted for selected special cases of the MO-ExOW-G family, and broader investigations may further enhance understanding of estimator performance under different baseline distributions and sampling schemes.

Overall, the MO-ExOW-G family successfully integrates flexibility, theoretical rigor, and practical relevance. It overcomes the limitations of traditional lifetime models and provides valuable new tools for research in reliability analysis and clinical studies. Future research will include using other baseline distributions, inference under censoring schemes, Bayesian inference and the application of the new family in different fields of study, such as cure rate modelling.

8. Appendix

The **quantile function (Q)** of the new Marshall-Olkin Extended Odd Weibull-G family of distributions follows from inverting Equation (3), as

$$Q_{MO-ExOW-G}(x;\alpha,\beta,\sigma,\xi) = F^{-1}_{MO-ExOW-G}(x;\alpha,\beta,\sigma,\xi).$$

Let $q = 1 - \frac{\sigma\left(1+\beta\left(\frac{G(x;\xi)}{1-G(x;\xi)}\right)^{\alpha}\right)^{-\frac{1}{\beta}}}{1-(1-\sigma)\left(1+\beta\left(\frac{G(x;\xi)}{1-G(x;\xi)}\right)^{\alpha}\right)^{-\frac{1}{\beta}}}$ where $0 \leq q \leq 1$,

then $q = 1 - \frac{\sigma A}{1-(1-\sigma)A}$, where $A = \left(1+\beta\left(\frac{G(x;\xi)}{1-G(x;\xi)}\right)^{\alpha}\right)^{-\frac{1}{\beta}}$.
Now,

$$\left(1+\beta\left(\frac{G(x;\xi)}{1-G(x;\xi)}\right)^{\alpha}\right)^{-\frac{1}{\beta}} = \frac{q-1}{(q-1)(1-\sigma)-\sigma} \tag{27}$$

so that,

$$\frac{G(x;\xi)}{1-G(x;\xi)} = \left(\frac{1}{\beta}\left[\left(\frac{(q-1)(1-\sigma)-\sigma}{q-1}\right)^{\beta}-1\right]\right)^{\frac{1}{\alpha}} \tag{28}$$

and

$$G(x;\xi) = \frac{\left(\frac{1}{\beta}\left[\left(\frac{(q-1)(1-\sigma)-\sigma}{q-1}\right)^{\beta}-1\right]\right)^{\frac{1}{\alpha}}}{1+\left(\frac{1}{\beta}\left[\left(\frac{(q-1)(1-\sigma)-\sigma}{q-1}\right)^{\beta}-1\right]\right)^{\frac{1}{\alpha}}}. \tag{29}$$

Consequently, the quantile function is given by

$$Q_{\text{MO-ExOW-G}}(q;\alpha,\beta,\sigma,\xi) = G^{-1}\left(\frac{\left(\frac{1}{\beta}\left[\left(\frac{(q-1)(1-\sigma)-\sigma}{q-1}\right)^{\beta}-1\right]\right)^{\frac{1}{\alpha}}}{1+\left(\frac{1}{\beta}\left[\left(\frac{(q-1)(1-\sigma)-\sigma}{q-1}\right)^{\beta}-1\right]\right)^{\frac{1}{\alpha}}}\right). \tag{30}$$

Note that the solutions to the non-linear Equation (8) gives the quantiles of the new family of distributions, for specified baseline cdf G.

Proof of Theorem 1: linear representation of the density Function
Let $f_{MO-ExOW-G}(x;\alpha,\beta,\sigma,\xi)$ be denoted by $f(x)$, $G(x;\xi)$ by $G(x)$ and $g(x;\xi)$ by $g(x)$. We apply the generalized binomial series expansion: $(1-z)^{-q} = \sum_{i=0}^{\infty}\binom{q+i-1}{i}z^i$ for $|z|<1$, where $z = (1-\sigma)\big(1+\beta\big(\frac{G(x;\xi)}{1-G(x;\xi)}\big)^{\alpha}\big)^{-\frac{1}{\beta}}$, and $q=2$ to obtain:

$$\begin{aligned}f(x) =& \frac{\sigma\alpha g(x)(G(x))^{\alpha-1}}{[1-G(x)]^{\alpha+1}}\left[1+\beta\Big(\frac{G(x)}{1-G(x)}\Big)^{\alpha}\right]^{-\frac{1}{\beta}-1}\sum_{i=0}^{\infty}\binom{2+i-1}{i}(1-\sigma)^i\left[1+\beta\Big(\frac{G(x)}{1-G(x)}\Big)^{\alpha}\right]^{-\frac{i}{\beta}}\\ =& \frac{\sigma\alpha g(x)(G(x))^{\alpha-1}}{[1-G(x)]^{\alpha+1}}\sum_{i=0}^{\infty}(1+i)(1-\sigma)^i\left[1+\beta\Big(\frac{G(x)}{1-G(x)}\Big)^{\alpha}\right]^{-(\frac{i+1}{\beta}+1)}.\end{aligned}$$

Now applying the generalised binomial series expansion: $(1+z)^{-c} = \sum_{j=0}^{\infty}\binom{c+j-1}{j}(-1)^j z^j$ for $c>0$, $|z|<1$, where $c = (\frac{(i+1)}{\beta}+1)$ and $z = \beta\left(\frac{G(x)}{1-G(x)}\right)^{\alpha}$, we obtain

$$\begin{aligned}f(x) =& \frac{\sigma\alpha g(x)(G(x))^{\alpha-1}}{[1-G(x)]^{\alpha+1}}\sum_{i=0}^{\infty}(1+i)(1-\sigma)^i\sum_{j=0}^{\infty}\binom{(\frac{i+1}{\beta}+1)+j-1}{j}(-1)^j\beta^j\Big(\frac{G(x)}{1-G(x)}\Big)^{\alpha j}\\ =& \sigma\alpha g(x)\sum_{i,j=0}^{\infty}(1-\sigma)^i(1+i)(-1)^j\beta^j\binom{(\frac{i+1}{\beta})+j}{j}[G(x)]^{\alpha j+\alpha-1}[1-G(x)]^{-(\alpha j+\alpha+1)}.\end{aligned}$$

Now applying the generalised binomial series expansion: $(1-z)^{-q} = \sum_{k=0}^{\infty} \binom{q+k-1}{k} z^k$ for $|z| < 1$, where $z = G(x)$ and $q = \alpha j + \alpha + 1$, we obtain,

$$\begin{aligned} f(x) =& \sigma\alpha g(x) \sum_{i,j=0}^{\infty} (1-\sigma)^i (1+i)(-1)^j \beta^j \binom{(\frac{i+1}{\beta}) + j}{j} [G(x)]^{\alpha j+\alpha-1} \sum_{k=0}^{\infty} \binom{\alpha j+\alpha+k}{k} [G(x)]^k \\ =& \sigma\alpha g(x) \sum_{i,j,k=0}^{\infty} (1-\sigma)^i (1+i)(-1)^j \beta^j \binom{(\frac{i+1}{\beta}) + j}{j} \binom{\alpha j+\alpha+k}{k} [1-\bar{G}(x)]^{\alpha j+\alpha+k-1}. \end{aligned}$$

Now applying the generalised binomial series expansion: $(1-z)^q = \sum_{p=0}^{\infty} \binom{q}{p} (-1)^p z^p$ for $|z| < 1$, where $z = \bar{G}(x)$ and $q = \alpha j + \alpha + k - 1$, we obtain

$$\begin{aligned} f(x) =& \sigma\alpha g(x) \sum_{i,j,k=0}^{\infty} (1-\sigma)^i (1+i)(-1)^{j+p} \beta^j \binom{(\frac{i+1}{\beta}) + j}{j} \binom{\alpha j+\alpha+k}{k} \\ &\times \sum_{p=0}^{\infty} \binom{\alpha j+\alpha+k-1}{p} [\bar{G}(x)]^p \\ =& \sigma\alpha g(x) \sum_{i,j,k,p=0}^{\infty} (1-\sigma)^i (1+i)(-1)^{j+p} \beta^j \binom{(\frac{i+1}{\beta}) + j}{j} \binom{\alpha j+\alpha+k}{k} \binom{\alpha j+\alpha+k-1}{p} \\ &\times [1-G(x)]^p. \end{aligned}$$

Now applying the generalised binomial series expansion: $(1-z)^q = \sum_{t=0}^{\infty} \binom{q}{t} (-1)^t z^t$ for $|z| < 1$, where $z = G(x)$ and $q = p$, we obtain

$$\begin{aligned} f(x) =& \sigma\alpha g(x) \sum_{i,j,k,p=0}^{\infty} (1-\sigma)^i (1+i)(-1)^{j+p} \beta^j \binom{(\frac{i+1}{\beta}) + j}{j} \binom{\alpha j+\alpha+k}{k} \binom{\alpha j+\alpha+k-1}{p} \\ &\times \sum_{t=0}^{\infty} \binom{p}{t} (-1)^t [G(x)]^t \\ =& \sigma\alpha g(x) \sum_{i,j,k,p,t=0}^{\infty} (1-\sigma)^i (1+i)(-1)^{j+p+t} \beta^j \binom{(\frac{i+1}{\beta}) + j}{j} \binom{\alpha j+\alpha+k}{k} \binom{\alpha j+\alpha+k-1}{p} \\ &\times \binom{p}{t} [G(x)]^t. \end{aligned}$$

Thus, the pdf of the MO-ExOW-G family of distributions can be written as:

$$\begin{aligned} f(x) =& \sigma\alpha \sum_{i,j,k,p,t=0}^{\infty} (-1)^{j+p+t} \frac{\beta^j (1-\sigma)^i}{(t+1)} (1+i) \binom{(\frac{i+1}{\beta}) + j}{j} \binom{\alpha j+\alpha+k}{k} \binom{\alpha j+\alpha+k-1}{p} \binom{p}{t} \\ &\times (t+1) g(x) [G(x)]^{t+1-1} \\ =& \sum_{t=0}^{\infty} \gamma_{t+1} g_{t+1}(x;\xi), \end{aligned} \tag{31}$$

where $g_{t+1}(x;\xi) = (t+1) g(x;\xi) [G(x;\xi)]^t$ and

$$\gamma_{t+1} = \sigma\alpha \sum_{i,j,k,p=0}^{\infty} (-1)^{j+p+t} \frac{\beta^j (1-\sigma)^i}{(t+1)} (1+i) \binom{(\frac{i+1}{\beta}) + j}{j} \binom{\alpha j+\alpha+k}{k} \binom{\alpha j+\alpha+k-1}{p} \binom{p}{t}. \tag{32}$$

Consequently, $f_{MO-ExOW-G}(x;\alpha,\beta,\sigma,\xi)=\sum_{t=0}^{\infty}\gamma_{t+1}g_{t+1}(x;\xi)$, where $g_{t+1}(x;\xi)=(t+1)g(x;\xi)[G(x;\xi)]^{t+1-1}$ is the exponentiated-G (Exp-G) density with power parameter (t+1).

Order statistic

We apply the generalized binomial series expansion,

$(1-z)^q=\sum_{i=0}^{\infty}\binom{q}{i}(-1)^i z^i$ for $|z|<1$, where $z=\left(\frac{\sigma\left(1+\beta\left(\frac{G(x)}{1-G(x)}\right)^{\alpha}\right)^{-\frac{1}{\beta}}}{1-(1-\sigma)\left(1+\beta\left(\frac{G(x)}{1-G(x)}\right)^{\alpha}\right)^{-\frac{1}{\beta}}}\right)$, and $q=r+m-1$, to obtain

$$
\begin{aligned}
f(x)[F(x)]^{r+m-1} &= \frac{\sigma\alpha g(x)(G(x))^{\alpha-1}\left(1+\beta\left(\frac{G(x)}{1-G(x)}\right)^{\alpha}\right)^{-\frac{1}{\beta}-1}}{[1-G(x)]^{\alpha+1}\left[1-(1-\sigma)\left(1+\beta\left(\frac{G(x)}{1-G(x)}\right)^{\alpha}\right)^{-\frac{1}{\beta}}\right]^2}\sum_{i=0}^{\infty}\binom{r+m-1}{i} \\
&\times(-1)^i\frac{\sigma^i\left(1+\beta\left(\frac{G(x)}{1-G(x)}\right)^{\alpha}\right)^{-\frac{i}{\beta}}}{\left[1-(1-\sigma)\left(1+\beta\left(\frac{G(x)}{1-G(x)}\right)^{\alpha}\right)^{-\frac{1}{\beta}}\right]^i} \\
&= \sigma\alpha g(x)(G(x))^{\alpha-1}[1-G(x)]^{-\alpha-1}\left(1+\beta\left(\frac{G(x)}{1-G(x)}\right)^{\alpha}\right)^{-\frac{1}{\beta}-1} \\
&\times\sum_{i=0}^{\infty}\binom{r+m-1}{i}(-1)^i\sigma^i\left(1+\beta\left(\frac{G(x)}{1-G(x)}\right)^{\alpha}\right)^{-\frac{i}{\beta}} \\
&\times\left[1-(1-\sigma)\left(1+\beta\left(\frac{G(x)}{1-G(x)}\right)^{\alpha}\right)^{-\frac{1}{\beta}}\right]^{-(2+i)}.
\end{aligned}
$$

Now applying the generalised binomial series expansion: $(1-z)^{-q}=\sum_{j=0}^{\infty}\binom{q+j-1}{j}z^j$ for $|z|<1$, where $z=(1-\sigma)\left(1+\beta\left(\frac{G(x)}{1-G(x)}\right)^{\alpha}\right)^{-\frac{1}{\beta}}$ and $q=2+i$, we obtain

$$
\begin{aligned}
f(x)[F(x)]^{r+m-1} &= \sigma\alpha g(x)(G(x))^{\alpha-1}[1-G(x)]^{-\alpha-1}\left(1+\beta\left(\frac{G(x)}{1-G(x)}\right)^{\alpha}\right)^{-\frac{1}{\beta}-1} \\
&\times\sum_{i=0}^{\infty}\binom{r+m-1}{i}(-1)^i\sigma^i\left(1+\beta\left(\frac{G(x)}{1-G(x)}\right)^{\alpha}\right)^{-\frac{i}{\beta}} \\
&\times\sum_{j=0}^{\infty}\binom{2+i+j-1}{j}(1-\sigma)^j\left(1+\beta\left(\frac{G(x)}{1-G(x)}\right)^{\alpha}\right)^{-\frac{j}{\beta}} \\
&= \sigma\alpha g(x)(G(x))^{\alpha-1}[1-G(x)]^{-\alpha-1}\sum_{i,j=0}^{\infty}\binom{r+m-1}{i}(-1)^i\sigma^i\binom{i+j+1}{j} \\
&\times(1-\sigma)^j\left(1+\beta\left(\frac{G(x)}{1-G(x)}\right)^{\alpha}\right)^{-\left(\frac{i+j+1}{\beta}+1\right)}.
\end{aligned}
$$

Now applying the generalised binomial series expansion: $(1+z)^{-c}=\sum_{k=0}^{\infty}\binom{c+k-1}{k}(-1)^k z^k$ for $c>0, |z|<1$, where $c=\frac{i+j+1}{\beta}+1$ and $z=\beta\left(\frac{G(x)}{1-G(x)}\right)^{\alpha}$, we obtain

$$
\begin{aligned}
f(x)[F(x)]^{r+m-1} &= \sigma\alpha g(x)(G(x))^{\alpha-1}[1-G(x)]^{-\alpha-1}\sum_{i,j=0}^{\infty}\binom{r+m-1}{i}(-1)^i\sigma^i\binom{i+j+1}{j} \\
&\times(1-\sigma)^j\sum_{k=0}^{\infty}\binom{\frac{i+j+1}{\beta}+1+k-1}{k}(-1)^k\beta^k\left(\frac{G(x)}{1-G(x)}\right)^{\alpha k} \\
&= \sigma\alpha g(x)\sum_{i,j,k=0}^{\infty}\binom{r+m-1}{i}(-1)^{i+k}\sigma^i\binom{i+j+1}{j}(1-\sigma)^j \\
&\times\binom{\frac{i+j+1}{\beta}+k}{k}\beta^k[G(x)]^{\alpha k+\alpha-1}[1-G(x)]^{-(\alpha k+\alpha+1)}.
\end{aligned}
$$

Now applying the generalised binomial series expansion: $(1-z)^{-q}=\sum_{p=0}^{\infty}\binom{q+p-1}{p}z^p$ for $|z|<1$, where $z=G(x)$ and $q=\alpha k+\alpha+1$, we obtain

$$
\begin{aligned}
f(x)[F(x)]^{r+m-1} &= \sigma\alpha g(x)\sum_{i,j,k=0}^{\infty}\binom{r+m-1}{i}(-1)^{i+k}\sigma^i\binom{i+j+1}{j}(1-\sigma)^j\binom{\frac{i+j+1}{\beta}+k}{k} \\
&\times\beta^k[G(x)]^{\alpha k+\alpha-1}\sum_{p=0}^{\infty}\binom{\alpha k+\alpha+1+p-1}{p}[G(x)]^p \\
&= \sigma\alpha g(x)\sum_{i,j,k,p=0}^{\infty}\binom{r+m-1}{i}(-1)^{i+k}\sigma^i\binom{i+j+1}{j}(1-\sigma)^j\binom{\frac{i+j+1}{\beta}+k}{k} \\
&\times\binom{\alpha k+\alpha+p}{p}\beta^k[1-\bar{G}(x)]^{\alpha k+\alpha+p-1}.
\end{aligned}
$$

Now applying the generalised binomial series expansion: $(1-z)^{q}=\sum_{s=0}^{\infty}\binom{q}{s}(-1)^s z^s$ for $|z|<1$, where $z=\bar{G}(x)$ and $q=\alpha k+\alpha+p-1$, we obtain

$$
\begin{aligned}
f(x)[F(x)]^{r+m-1} &= \sigma\alpha g(x)\sum_{i,j,k,p=0}^{\infty}\binom{r+m-1}{i}(-1)^{i+k}\sigma^i\binom{i+j+1}{j}(1-\sigma)^j\binom{\frac{i+j+1}{\beta}+k}{k} \\
&\times\binom{\alpha k+\alpha+p}{p}\beta^k\sum_{s=0}^{\infty}\binom{\alpha k+\alpha+p-1}{s}(-1)^s[\bar{G}(x)]^s \\
&= \sigma\alpha g(x)\sum_{i,j,k,p,s=0}^{\infty}\binom{r+m-1}{i}(-1)^{i+k+s}\sigma^i\binom{i+j+1}{j}(1-\sigma)^j\binom{\frac{i+j+1}{\beta}+k}{k} \\
&\times\binom{\alpha k+\alpha+p}{p}\beta^k\binom{\alpha k+\alpha+p-1}{s}[1-G(x)]^s.
\end{aligned}
$$

Now applying the generalised binomial series expansion: $(1-z)^q = \sum_{t=0}^{\infty}\binom{q}{t}(-1)^t z^t$ for $|z|<1$, where $z = G(x)$ and $q = s$, we obtain

$$\begin{aligned} f(x)[F(x)]^{r+m-1} &= \sigma\alpha g(x)\sum_{i,j,k,p,s=0}^{\infty}\binom{r+m-1}{i}(-1)^{i+k+s}\sigma^i\binom{i+j+1}{j}(1-\sigma)^j\binom{\frac{i+j+1}{\beta}+k}{k} \\ &\times\binom{\alpha k+\alpha+p}{p}\beta^k\binom{\alpha k+\alpha+p-1}{s}\sum_{t=0}^{\infty}\binom{s}{t}(-1)^t[G(x)]^t \\ &= \sigma\alpha\sum_{i,j,k,p,s,t=0}^{\infty}\binom{r+m-1}{i}(-1)^{i+k+s+t}\sigma^i\binom{i+j+1}{j}(1-\sigma)^j\frac{\beta^k}{(t+1)} \\ &\times\binom{\frac{i+j+1}{\beta}+k}{k}\binom{\alpha k+\alpha+p}{p}\binom{\alpha k+\alpha+p-1}{s}\binom{s}{t}(t+1)g(x)[G(x)]^t. \end{aligned} \tag{33}$$

Renyi entropy

Let $g(x) = g(x;\xi)$ and $G(x) = G(x;\xi)$, then consider

$$\begin{aligned} I &= (\sigma\alpha)^\theta(g(x))^\theta(G(x))^{\theta(\alpha-1)}[1-G(x)]^{-\theta(\alpha+1)}\left(1+\beta\left(\frac{G(x)}{1-G(x)}\right)^\alpha\right)^{-\theta(\frac{1}{\beta}+1)} \\ &\times\left[1-(1-\sigma)\left(1+\beta\left(\frac{G(x)}{1-G(x)}\right)^\alpha\right)^{-\frac{1}{\beta}}\right]^{-2\theta}. \end{aligned} \tag{34}$$

We apply the generalized binomial series expansion: $(1-z)^{-q} = \sum_{i=0}^{\infty}\binom{q+i-1}{i}z^i$ for $|z|<1$, where $z = (1-\sigma)\big(1+\beta\big(\frac{G(x;\xi)}{1-G(x;\xi)}\big)^\alpha\big)^{-\frac{1}{\beta}}$, and $q = 2\theta$ to Equation (34) to obtain:

$$\begin{aligned} I &= \frac{(\sigma\alpha)^\theta(g(x))^\theta(G(x))^{\theta(\alpha-1)}}{[1-G(x)]^{\theta(\alpha+1)}}\left[1+\beta\Big(\frac{G(x)}{1-G(x)}\Big)^\alpha\right]^{-\theta(\frac{1}{\beta}+1)}\sum_{i=0}^{\infty}\binom{2\theta+i-1}{i} \\ &\times(1-\sigma)^i\left[1+\beta\Big(\frac{G(x)}{1-G(x)}\Big)^\alpha\right]^{-\frac{i}{\beta}} \\ &= \frac{(\sigma\alpha)^\theta(g(x))^\theta(G(x))^{\theta(\alpha-1)}}{[1-G(x)]^{\theta(\alpha+1)}}\sum_{i=0}^{\infty}\binom{2\theta+i-1}{i}(1-\sigma)^i\left[1+\beta\Big(\frac{G(x)}{1-G(x)}\Big)^\alpha\right]^{-(\frac{\theta+i}{\beta}+\theta)}. \end{aligned}$$

Now applying the generalised binomial series expansion: $(1+z)^{-c} = \sum_{j=0}^{\infty}\binom{c+j-1}{j}(-1)^j z^j$ for $c>0$, $|z|<1$, where $c = (\frac{\theta+i}{\beta}+\theta)$ and $z = \beta\left(\frac{G(x)}{1-G(x)}\right)^\alpha$, we obtain

$$\begin{aligned} I &= \frac{(\sigma\alpha)^\theta(g(x))^\theta(G(x))^{\theta(\alpha-1)}}{[1-G(x)]^{\theta(\alpha+1)}}\sum_{i=0}^{\infty}\binom{2\theta+i-1}{i}(1-\sigma)^i\sum_{j=0}^{\infty}\binom{(\frac{\theta+i}{\beta}+\theta)+j-1}{j}(-1)^j \\ &\times\beta^j\Big(\frac{G(x)}{1-G(x)}\Big)^{\alpha j} \\ &= (\sigma\alpha)^\theta(g(x))^\theta\sum_{i,j=0}^{\infty}(1-\sigma)^i\binom{2\theta+i-1}{i}(-1)^j\beta^j\binom{\frac{\theta+i}{\beta}+\theta+j-1}{j}[G(x)]^{\alpha j+\theta(\alpha-1)} \\ &\times[1-G(x)]^{-(\theta\alpha+\theta+\alpha j)}. \end{aligned}$$

Now applying the generalised binomial series expansion: $(1-z)^{-q}=\sum_{k=0}^{\infty}\binom{q+k-1}{k}z^k$ for $|z|<1$, where $z=G(x)$ and $q=\theta\alpha+\theta+\alpha j$, we obtain,

$$
\begin{aligned}
I =&(\sigma\alpha)^{\theta}(g(x))^{\theta}\sum_{i,j=0}^{\infty}(1-\sigma)^i\binom{2\theta+i-1}{i}(-1)^j\beta^j\binom{\frac{\theta+i}{\beta}+\theta+j-1}{j}[G(x)]^{\alpha j+\theta(\alpha-1)}\\
&\times\sum_{k=0}^{\infty}\binom{\theta\alpha+\theta+\alpha j+k-1}{k}[G(x)]^k\\
=&(\sigma\alpha)^{\theta}(g(x))^{\theta}\sum_{i,j,k=0}^{\infty}(1-\sigma)^i\binom{2\theta+i-1}{i}(-1)^j\beta^j\binom{\frac{\theta+i}{\beta}+\theta+j-1}{j}\binom{\theta\alpha+\theta+\alpha j+k-1}{k}\\
&\times[1-\bar{G}(x)]^{\alpha j+\theta(\alpha-1)+k}.
\end{aligned}
$$

Now applying the generalised binomial series expansion: $(1-z)^{q}=\sum_{p=0}^{\infty}\binom{q}{p}(-1)^pz^p$ for $|z|<1$, where $z=\bar{G}(x)$ and $q=\alpha j+\theta(\alpha-1)+k$, we obtain

$$
\begin{aligned}
I =&(\sigma\alpha)^{\theta}(g(x))^{\theta}\sum_{i,j,k=0}^{\infty}(1-\sigma)^i\binom{2\theta+i-1}{i}(-1)^{j+p}\beta^j\binom{\frac{\theta+i}{\beta}+\theta+j-1}{j}\\
&\times\binom{\theta\alpha+\theta+\alpha j+k-1}{k}\sum_{p=0}^{\infty}\binom{\alpha j+\theta(\alpha-1)+k}{p}[\bar{G}(x)]^p\\
=&(\sigma\alpha)^{\theta}(g(x))^{\theta}\sum_{i,j,k,p=0}^{\infty}(1-\sigma)^i\binom{2\theta+i-1}{i}(-1)^{j+p}\beta^j\binom{\frac{\theta+i}{\beta}+\theta+j-1}{j}\\
&\times\binom{\theta\alpha+\theta+\alpha j+k-1}{k}\binom{\alpha j+\theta(\alpha-1)+k}{p}[1-G(x)]^p.
\end{aligned}
$$

Now applying the generalised binomial series expansion: $(1-z)^{q}=\sum_{t=0}^{\infty}\binom{q}{t}(-1)^tz^t$ for $|z|<1$, where $z=G(x)$ and $q=p$, we obtain

$$
\begin{aligned}
I =&(\sigma\alpha)^{\theta}(g(x))^{\theta}\sum_{i,j,k,p=0}^{\infty}(1-\sigma)^i\binom{2\theta+i-1}{i}(-1)^{j+p}\beta^j\binom{\frac{\theta+i}{\beta}+\theta+j-1}{j}\\
&\times\binom{\theta\alpha+\theta+\alpha j+k-1}{k}\binom{\alpha j+\theta(\alpha-1)+k}{p}\sum_{t=0}^{\infty}\binom{p}{t}(-1)^t[G(x)]^t\\
=&(\sigma\alpha)^{\theta}\sum_{i,j,k,p,t=0}^{\infty}(1-\sigma)^i\binom{2\theta+i-1}{i}(-1)^{j+p+t}\beta^j\binom{\frac{\theta+i}{\beta}+\theta+j-1}{j}\\
&\times\binom{\theta\alpha+\theta+\alpha j+k-1}{k}\binom{\alpha j+\theta(\alpha-1)+k}{p}\binom{p}{t}\left(\frac{1}{\frac{t}{\theta}+1}\right)^{\theta}\left(\frac{t}{\theta}+1\right)^{\theta}(g(x))^{\theta}\left([G(x)]^{\frac{t}{\theta}}\right)^{\theta}.
\end{aligned}
$$

Consequently,

$$
\begin{aligned}
H^{(\theta)}_{MO\text{-}ExOW\text{-}G} &= \frac{1}{1-\theta}\log\left(\int_0^\infty \left[(\sigma\alpha)^\theta (g(x;\xi))^\theta (G(x;\xi))^{\theta(\alpha-1)}[1-G(x;\xi)]^{-\theta(\alpha+1)}\right.\right.\\
&\times \left(1+\beta\left(\frac{G(x;\xi)}{1-G(x;\xi)}\right)^\alpha\right)^{-\theta(\frac{1}{\beta}+1)}\\
&\times \left.\left.\left[1-(1-\sigma)\left(1+\beta\left(\frac{G(x;\xi)}{1-G(x;\xi)}\right)^\alpha\right)^{-\frac{1}{\beta}}\right]^{-2\theta} dx\right)\\
&= \frac{1}{1-\theta}\log\left((\sigma\alpha)^\theta \sum_{i,j,k,p,t=0}^{\infty}(1-\sigma)^i\binom{2\theta+i-1}{i}(-1)^{j+p+t}\beta^j\binom{\frac{\theta+i}{\beta}+\theta+j-1}{j}\right.\\
&\times \binom{\theta\alpha+\theta+\alpha j+k-1}{k}\binom{\alpha j+\theta(\alpha-1)+k}{p}\binom{p}{t}\left(\frac{1}{\frac{t}{\theta}+1}\right)^\theta\\
&\times \left.\int_0^\infty \left[\left(\frac{t}{\theta}+1\right)^\theta (g(x))^\theta\left([G(x)]^{\frac{t}{\theta}}\right)^\theta\right] dx\right)\\
&= \frac{1}{1-\theta}\log\left[\sum_{t=0}^{\infty}\varrho_t e^{(1-\theta)I_{REG}}\right],
\end{aligned}
\tag{35}
$$

where

$$
\begin{aligned}
\varrho_t = (\sigma\alpha)^\theta &\sum_{i,j,k,p=0}^{\infty}(1-\sigma)^i\binom{2\theta+i-1}{i}(-1)^{j+p+t}\beta^j\binom{\frac{\theta+i}{\beta}+\theta+j-1}{j}\\
&\times \binom{\theta\alpha+\theta+\alpha j+k-1}{k}\binom{\alpha j+\theta(\alpha-1)+k}{p}\binom{p}{t}\left(\frac{1}{\frac{t}{\theta}+1}\right)^\theta,
\end{aligned}
\tag{36}
$$

and

$$
I_{REG} = \frac{1}{1-\theta}\log\left(\int_0^\infty\left[\left(\frac{t}{\theta}+1\right)g(x)\left([G(x)]^{\frac{t}{\theta}}\right)\right]^\theta dx\right)
\tag{37}
$$

is the Rényi entropy of the exp-G with power parameter $\frac{t}{\theta}$.

Table 16. Numerical classification of hazard rate shapes across parameter regimes for MO-ExOW-E Distribution

Case	α	β	σ	λ	**Hazard Shape**
1	<1	<1	<1	>1	Bathtub
2	<1	<1	>1	>1	Bathtub
3	<1	>1	<1	<1	Decreasing
4	<1	>1	>1	>1	Bathtub
5	>1	<1	<1	<1	Unimodal
6	>1	<1	>1	>1	Unimodal
7	>1	>1	<1	>1	Unimodal
8	>1	>1	>1	>1	Unimodal

Table 16 summarizes the qualitative shapes of the hazard rate function from $h'(x)$ across representative parameter regimes of MO-ExOW-E distribution.

Table 17. Numerical classification of hazard rate shapes across parameter regimes for MO-ExOW-W Distribution

Case	α	β	σ	λ	**Hazard Shape**
1	<1	<1	<1	<1	Bathtub
2	<1	<1	>1	>1	Unimodal
3	<1	>1	<1	<1	Bathtub
4	<1	>1	>1	>1	Increasing
5	>1	<1	<1	<1	Unimodal
6	>1	<1	>1	>1	Unimodal
7	>1	>1	<1	>1	Unimodal
8	>1	>1	>1	>1	Increasing

Table 17 summarizes the qualitative shapes of the hazard rate function from $h'(x)$ across representative parameter regimes of MO-ExOW-W distribution.

Table 18. Numerical classification of hazard rate shapes across parameter regimes for MO-ExOW-Burr XII Distribution

Case	α	β	σ	c	k	**Hazard Shape**
1	<1	<1	<1	<1	<1	Decreasing
2	<1	<1	>1	>1	>1	Unimodal
3	<1	>1	<1	<1	<1	Decreasing
4	<1	>1	>1	>1	>1	Unimodal
5	>1	<1	<1	<1	<1	Unimodal
6	>1	<1	>1	>1	>1	Unimodal
7	>1	>1	<1	>1	>1	Unimodal
8	>1	>1	>1	>1	>1	Unimodal

Table 18 summarizes the qualitative shapes of the hazard rate function inferred from $h'(x)$ across representative parameter regimes for MO-ExOW-Burr XII distribution.

Maximum Likelihood Estimation

The maximum likelihood estimate (MLEs) of the parameters of the MO-ExOW-G is found by finding the maxima of the log-likelihood function $\ell_{MO-ExOW-G}(\alpha,\beta,\sigma,\xi)$ from a random sample of size n, where $\ell_{MO-ExOW-G}(\alpha,\beta,\sigma,\xi) = \ell(\mathbf{\Theta})$ is;

$$\begin{aligned}\ell(\mathbf{\Theta}) = \sum_{i=1}^{n} \Bigg[& \ln(\sigma) + \ln(\alpha) + \ln(g(x_i;\xi)) + (\alpha-1)\ln(G(x_i;\xi)) \\ & - \left(\frac{1}{\beta}+1\right)\ln\left(1+\beta\left(\frac{G(x_i;\xi)}{1-G(x_i;\xi)}\right)^{\alpha}\right) - (\alpha+1)\ln[1-G(x_i;\xi)] \\ & - 2\ln\left(1-(1-\sigma)\left(1+\beta\left(\frac{G(x_i;\xi)}{1-G(x_i;\xi)}\right)^{\alpha}\right)^{-1/\beta}\right)\Bigg].\end{aligned}$$

Numerical methods are then used to solve the nonlinear equations $(\frac{\partial \ell(\mathbf{\Theta})}{\partial \alpha}, \frac{\partial \ell(\mathbf{\Theta})}{\partial \beta}, \frac{\partial \ell(\mathbf{\Theta})}{\partial \sigma}, \frac{\partial \ell(\mathbf{\Theta})}{\partial \xi})^T = \mathbf{0}$. These solutions are the maximum likelihood estimates of the parameters α, β, σ and ξ.

Least Squares and Weighted Least Square Estimation

The method of **least-squares** (LS) involves determining the unknown parameters by minimizing the following function:

$$LS(\boldsymbol{\Theta}) = \sum_{i=1}^{n} \left(F\big(x_{(i)}; \alpha, \beta, \sigma, \xi\big) - \frac{i}{n+1} \right)^2. \tag{38}$$

Numerical methods are then used to solve the nonlinear equations $(\frac{\partial LS(\boldsymbol{\Theta})}{\partial \alpha}, \frac{\partial LS(\boldsymbol{\Theta})}{\partial \beta}, \frac{\partial LS(\boldsymbol{\Theta})}{\partial \sigma}, \frac{\partial LS(\boldsymbol{\Theta})}{\partial \xi})^T = \mathbf{0}$. These gives the least square estimates for the parameters α, β, σ and ξ. The method of **weighted least square** (WLS), applies similar principles, but the function to be minimized is weighted as:

$$WLS(\boldsymbol{\Theta}) = \sum_{i=1}^{n} \frac{(n+1)^2(n+2)}{i(n-i+1)} \left(F\big(x_{(i)}; \alpha, \beta, \sigma, \xi\big) - \frac{i}{n+1} \right)^2. \tag{39}$$

Numerical methods are then used to solve the nonlinear equations $(\frac{\partial WLS(\boldsymbol{\Theta})}{\partial \alpha}, \frac{\partial WLS(\boldsymbol{\Theta})}{\partial \beta}, \frac{\partial WLS(\boldsymbol{\Theta})}{\partial \sigma}, \frac{\partial WLS(\boldsymbol{\Theta})}{\partial \xi})^T = \mathbf{0}$. These gives the weighted least square estimates for the parameters α, β, σ and ξ.

Cramér-von Mises Estimation

The Cramér-von Mises (CVM) estimation technique, determines the unknown parameters by minimizing the following function:

$$CVM(\boldsymbol{\Theta}) = \frac{1}{12n} + \sum_{i=1}^{n} \left(F(x_{(i)}; \alpha, \beta, \sigma, \xi) - \frac{2i-1}{2n} \right)^2. \tag{40}$$

Numerical methods are then used to solve the nonlinear equations $(\frac{\partial CVM(\boldsymbol{\Theta})}{\partial \alpha}, \frac{\partial CVM(\boldsymbol{\Theta})}{\partial \beta}, \frac{\partial CVM(\boldsymbol{\Theta})}{\partial \sigma}, \frac{\partial CVM(\boldsymbol{\Theta})}{\partial \xi})^T = \mathbf{0}$. These gives the Cramér-von Mises estimates for the parameters α, β, σ and ξ.

Anderson-Darling Estimation

The Anderson-Darling (AD) technique estimates the unknown parameters by minimizing the following function:

$$AD(\boldsymbol{\Theta}) = -n - \frac{1}{n} \sum_{i=1}^{n} \left[(2i-1) \left(\log\left(F\left(x_{(i)}; \alpha, \beta, \sigma, \xi\right)\right) + \log(1 - F(x_{n+1-i}; \alpha, \beta, \sigma, \xi) \right) \right]. \tag{41}$$

Numerical methods are then used to solve the nonlinear equations $(\frac{\partial AD(\boldsymbol{\Theta})}{\partial \alpha}, \frac{\partial AD(\boldsymbol{\Theta})}{\partial \beta}, \frac{\partial AD(\boldsymbol{\Theta})}{\partial \sigma}, \frac{\partial AD(\boldsymbol{\Theta})}{\partial \xi})^T = \mathbf{0}$. These gives the Anderson-Darling estimates for the parameters α, β, σ and ξ.

The pdf of non nested models used for comparison on the applications are given below:

Kumaraswamy Marshall-Olkin Exponential (KMOE)

$$f_{\text{KMOE}}(x; a, b, \alpha, \lambda) = \frac{ab\lambda(1-\alpha)e^{-\lambda x}\left(1 - e^{-\lambda x}\right)^{a-1}}{\left(1 - \alpha e^{-\lambda x}\right)^{a+1}} \left\{ 1 - \left[\frac{1 - e^{-\lambda x}}{1 - \alpha e^{-\lambda x}} \right]^a \right\}^{b-1}.$$

Exponentiated Kumaraswamy-Exponential (EKwE)

$$f_{EKwE}(x; \alpha, \beta, \lambda, \theta) = \alpha\beta\lambda\theta \left(1 - e^{-\alpha x}\right)^{\beta-1} \left[1 - \left(1 - e^{-\alpha x}\right)^{\beta}\right]^{\lambda-1} e^{-\alpha x} \times \left\{ \left[1 - \left(1 - e^{-\alpha x}\right)^{\beta}\right]^{\lambda} \right\}^{\theta-1}.$$

Exponentiated Generalized Exponential (EGEE)

$$f_{EGEE}(x;\alpha,\beta,k,\theta)=\frac{\alpha\beta k}{\theta}e^{-\left(\frac{x}{\theta}\right)}\left[1-e^{-\left(\frac{x}{\theta}\right)}\right]^{k-1}\left\{1-\left[1-e^{-\left(\frac{x}{\theta}\right)}\right]^{k}\right\}^{\alpha-1}$$
$$\times\left[\left\{1-\left[1-e^{-\left(\frac{x}{\theta}\right)}\right]^{k}\right\}^{\alpha}\right]^{\beta-1}.$$

Kumaraswamy-Extended Exponential (KwEE)

$$f_{KwEE}(x;\upsilon,\delta,\phi,\beta)=\frac{\upsilon\delta\phi^{2}(1+\beta x)e^{-\phi x}}{\phi+\beta}\left[1-\frac{(\phi+\beta+\phi\beta x)e^{-\phi x}}{\phi+\beta}\right]^{\upsilon-1}$$
$$\times\left\{\left[1-\frac{(\phi+\beta+\phi\beta x)e^{-\phi x}}{\phi+\beta}\right]^{\upsilon}\right\}^{\delta-1}$$

Marshall-Olkin-Kumaraswamy-Exponential (MOKwE)

$$f_{MOKwE}(x;a,b,\alpha,\beta)=\frac{\alpha ab\lambda e^{-\lambda x}\left(1-e^{-\lambda x}\right)^{a-1}\left[1-\left(1-e^{-\lambda x}\right)^{a}\right]^{b-1}}{\left\{1-(1-\alpha)\left[1-(1-e^{-\lambda x})^{a}\right]^{b}\right\}^{2}}$$

Kumaraswamy Weibull (KW)

$$f_{KW}(x;a,b,\alpha,\beta)=ab\alpha\beta x^{\beta-1}e^{-(\alpha x)^{\beta}}\left(1-e^{-(\alpha x)^{\beta}}\right)^{a-1}\left[1-\left(1-e^{-(\alpha x)^{\beta}}\right)^{a}\right]^{b-1}$$

Marshall-Olkin-Gompertz-Weibull (MO-GOM-W)

$$f_{MO-GOM-W}(x;\theta,\gamma,\delta,\lambda)=\frac{\delta\theta\gamma\lambda x^{\lambda-1}e^{-x^{\lambda}}\left(e^{-x^{\lambda}}\right)^{-\gamma-1}e^{\frac{\theta}{\gamma}\left(1-\left(e^{-x^{\lambda}}\right)^{-\gamma}\right)}}{\left[1-(1-\delta)e^{\frac{\theta}{\gamma}\left(1-\left(e^{-x^{\lambda}}\right)^{-\gamma}\right)}\right]^{2}}$$

Marshall-Olkin Extended Weibull (MOEW)

$$f(x;\alpha,\beta,\lambda,k)=\frac{\alpha\left(\lambda+\beta kx^{k-1}\right)\,\exp\left[-(\lambda x+\beta x^{k})\right]}{\left[1-(1-\alpha)\,\exp\left(-(\lambda x+\beta x^{k})\right)\right]^{2}}$$

Alpha power Topp-Leone Weibull (APTLW)

$$f_{APTLW}(x;\theta,\alpha,\beta,\lambda)=\frac{2\beta\theta\lambda\log(\alpha)\,x^{\beta-1}\exp(-2\lambda x^{\beta})\alpha^{\left(1-\exp(-2\lambda x^{\beta})\right)^{\theta}}\left(1-\exp(-2\lambda x^{\beta})\right)^{\theta-1}}{(\alpha-1)}$$

Marshall–Olkin log–logistic Weibull (MOLLW)

$$f_{MOLLW}(x; c, \alpha, \beta, \delta) = \frac{\delta(1+x^c)^{-1}e^{-\alpha x^\beta}\left\{\alpha\beta x^{\beta-1} + cx^{c-1}(1+x^c)^{-1}\right\}}{\left[1-(1-\delta)(1+x^c)^{-1}e^{-\alpha x^\beta}\right]^2}$$

Type II exponentiated half logistic Weibull (TIIEHLW)

$$f_{TIIEHLW}(x; a, \delta, \lambda, \gamma) = 2a\lambda\delta\gamma\, x^{\gamma-1}e^{-\delta x^\gamma}\left[1-e^{-\delta x^\gamma}\right]\frac{\left[1-\left(1-e^{-\delta x^\gamma}\right)^\lambda\right]^{a-1}}{\left[1+\left(1-e^{-\delta x^\gamma}\right)^\lambda\right]^{a-1}}.$$

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