STATISTICS, OPTIMIZATION AND INFORMATION COMPUTING
Stat., Optim. Inf. Comput., Vol. **x**, Month 201x, pp 0–18.
Published online in International Academic Press (www.IAPress.org)

Optimality of internal time moments in 1-bullet silent duels with generalized exponentially-convex rewards

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Abstract The finite 1-bullet silent duel is a 0-value timing game, in which the duel time span is equidistantly quantized, and each of the two duelists has a generalized exponentially-convex reward function. The duel is a symmetric matrix game, and each of the duelists has the same set of optimal strategies. Such a set can consist of either optimal time moments or of probabilistic mixtures over the finite set of successive time moments of possible acting. A particular interest is in optimality of internal time moments, wherein no duel starting and duel final moments are considered. However, the starting moment is never optimal in duels with generalized exponentially-convex rewards, whichever the factor of reward steepness is. Next, there is no optimal internal moment in 3×3 duels. In 4×4 duels, the second time moment is never optimal, and the third time moment is optimal only if the reward steepness factor does not exceed a unique positive root of an algebraic equation with a sum of two exponential functions. The conditions of when an internal time moment is optimal in bigger duels are specified as well, where another algebraic equation with a sum of four exponential functions is included. In general, if the number of all time moments is even, then no optimal time moments exist in the first half of the duel span. This is also true if the number of all time moments is odd, where the time moment being right in the duel middle is non-optimal as well.

Keywords 1-Bullet silent duel, Exponentially-convex reward, Matrix game, Optimal time moment, Internal time moment

AMS 2010 subject classifications 91A05, 91A55

DOI: 10.19139/soic-2310-5070-3251

1. Exponentially-convex rewards

In many competitive timing problems involving the introduction of technologies, information releases, or protocol actions, participants face a single irreversible decision taken over a discretized planning horizon, without observing competitors' actions until the process is complete [12, 15, 13, 20]. When all participants are identical in capabilities and information, such situations are naturally modeled as symmetric silent duels with zero expected advantage ex ante [15, 5]. Empirical and technological considerations suggest, however, that the value of acting increases progressively with time due to accumulated readiness, technological maturity, or informational completeness, while remaining bounded because of saturation effects [14, 25, 6]. Linear or standard convex reward specifications inadequately reflect this combination of progressive growth and bounded valuation, and often fail to generate robust interior optimal timing behavior in symmetric settings [9, 16, 19, 10].

The generalized exponentially-convex reward model shall therefore be introduced to capture symmetric, progressive, and bounded growth of readiness over equidistantly quantized time, while preserving the silent and zero-sum nature of the duel. This framework is believed to allow one to study how the steepness of readiness accumulation influences equilibrium (optimal) timing without introducing player asymmetry, exogenous

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ISSN 2310-5070 (online) ISSN 2311-004X (print)

advantages, or informational bias, and thus provides a phenomenologically grounded yet analytically tractable generalization of classical silent duel models [12, 15, 21, 24].

A silent duel is a model of decision-making through a given time span among a group of participants (duelists) under informational uncertainty and reward limitation [9, 12, 6, 14]. There are two important specificities: the participant does not learn about actions of the other participants until the duel end, and the participant benefits from acting as late as possible but only by acting first [15, 17, 18, 20]. A finite 1-bullet silent duel, where the bullet is a metaphor of the possibility to make a definite decision or an action, is a symmetric matrix game [1, 5, 7, 15, 19]

$$\langle X_N,\ Y_N,\ \mathbf{U}_N \rangle = \left\langle \{x_i\}_{i=1}^{N},\ \{y_j\}_{j=1}^{N},\ \mathbf{U}_N \right\rangle \tag{1}$$

that models one-decision-making competition between two identical duelists through quantized time span [13, 16, 21, 20]

$$T_N = \{t_q\}_{q=1}^{N} = \left\{ \frac{q-1}{N-1} \right\}_{q=1}^{N} \subset [0;\ 1] \ \text{ for }\ N \in \mathbb{N} \setminus \{1,\ 2\} \tag{2}$$

by the first and second duelists' pure strategy sets

$$X_N = \{x_i\}_{i=1}^{N} = \left\{ \frac{i-1}{N-1} \right\}_{i=1}^{N} = T_N,\ \ Y_N = \{y_j\}_{j=1}^{N} = \left\{ \frac{j-1}{N-1} \right\}_{j=1}^{N} = T_N \tag{3}$$

and a skew-symmetric reward matrix [7, 15, 16, 10]

$$\mathbf{U}_N = [u_{ij}]_{N\times N} = [-u_{ji}]_{N\times N} = -\mathbf{U}_N^{\mathrm{T}}. \tag{4}$$

Due to (4), the optimal value of game (1) by (2) — (4) is 0, and each of the duelists has the same set of optimal strategies. Such a set can consist of either optimal time moments or of probabilistic mixtures over set (2). The structure of the optimal strategy depends on how reward matrix (4) is built by N successive time moments (2) of possible acting [3, 8, 24, 11, 22].

Commonly, the reward matrix entry [13, 15, 26, 27]

$$u_{ij} = g(x_i) - g(y_j) + g(x_i)\, g(y_j)\, \mathrm{sign}(y_j - x_i) \ \text{ for }\ i = \overline{1,\ N} \ \text{ and }\ j = \overline{1,\ N} \tag{5}$$

by some discrete exponentially-convex reward functions $g(x_i)$ and $g(y_j)$ of the first and second duelists, respectively, where [15, 1, 5, 16, 25]

$$g(x_1) = g(y_1) = g(0) = 0 \ \text{ and }\ g(x_N) = g(y_N) = g(1) = 1. \tag{6}$$

So, instead of (5), entry u_{ij} of reward matrix (4) is calculated as

$$u_{ij} = g\left(e^{\mu x_i}\right) - g\left(e^{\mu y_j}\right) + g\left(e^{\mu x_i}\right) g\left(e^{\mu y_j}\right) \mathrm{sign}(y_j - x_i) \ \text{ for }\ i = \overline{1,\ N} \ \text{ and }\ j = \overline{1,\ N} \tag{7}$$

with some factor of steepness $\mu > 0$ and by still obeying requirements similar to (6):

$$g\left(e^{\mu x_1}\right) = g\left(e^{\mu y_1}\right) = g\left(e^{0}\right) = g(1) = 0 \ \text{ and }\ g\left(e^{\mu x_N}\right) = g\left(e^{\mu y_N}\right) = g\left(e^{\mu}\right) = 1. \tag{8}$$

This reward structure models competitive environments where rewards depend jointly on absolute readiness and relative timing, incorporate asymmetric first-versus-second duelist effects, and exhibit realistic saturation — features that cannot be simultaneously captured by linear or standard convex reward functions. Parameter μ governs the steepness of readiness growth, and thus controls how sensitive rewards are to timing differences. When μ is low, function $g\left(e^{\mu z}\right)$ of variable z must grow slowly and almost linearly, differences between early and late actions are muted, and the penalty for acting slightly too early or too late is mild. In this case of gradual readiness and low aggressiveness the duelists are risk-tolerant, timing precision is less critical, and the duel resembles a soft competition where mixed or dispersed strategies are plausible. The corresponding real-world analogies by a low reward steepness factor are incremental technological improvements, markets with slow adoption curves, technologies with weak network effects, etc. A high μ is the case of explosive readiness and high aggressiveness, where readiness increases sharply near the end of the time horizon, small timing deviations cause large reward swings, the cost of being late becomes severer. In this case the duelists are highly timing-sensitive, optimal behavior concentrates sharply around a specific moment, the duel becomes knife-edge due to mistiming is heavily punished. The corresponding real-world analogies by a high reward steepness factor are incremental winner-takes-most technology races, protocol upgrades with strong network effects, blockchain systems where delayed action loses consensus relevance, etc. Hence, parameter μ reflects the aggressiveness of readiness accumulation or the strength of exponential improvement effects. Higher values of μ correspond to environments where delaying action rapidly amplifies potential reward but simultaneously increases the risk of being overtaken, thereby enforcing sharper and more decisive timing behavior.

An explicit exponentially-convex (which is exponentially-increasing) reward function of the duelist is

$$g\left(e^{\mu z}\right) = a e^{\mu z} + b \text{ for } z \in T_N \text{ and } a \in \mathbb{R} \setminus \{0\}, \ b \in \mathbb{R}. \tag{9}$$

As function (9) of variable z must obey requirements (8), then

$$g\left(e^{0}\right) = g(1) = a + b = 0,$$

$$g\left(e^{\mu}\right) = a e^{\mu} + b = 1,$$

whence

$$a = \frac{1}{e^{\mu} - 1} > 0, \quad b = \frac{1}{1 - e^{\mu}} < 0$$

and function (9) is

$$g\left(e^{\mu z}\right) = \frac{e^{\mu z}}{e^{\mu} - 1} - \frac{1}{e^{\mu} - 1} = \frac{e^{\mu z} - 1}{e^{\mu} - 1}. \tag{10}$$

Then, upon plugging (10) into (7), entry u_{ij} of reward matrix (4) is calculated as

$$u_{ij} = \frac{e^{\mu x_i} - 1}{e^{\mu} - 1} - \frac{e^{\mu y_j} - 1}{e^{\mu} - 1} + \frac{e^{\mu x_i} - 1}{e^{\mu} - 1} \cdot \frac{e^{\mu y_j} - 1}{e^{\mu} - 1} \cdot \operatorname{sign}(y_j - x_i) =$$

$$= \frac{e^{\mu x_i} - e^{\mu y_j}}{e^{\mu} - 1} + \frac{\left(e^{\mu x_i} - 1\right)\left(e^{\mu y_j} - 1\right)}{\left(e^{\mu} - 1\right)^2} \cdot \operatorname{sign}(y_j - x_i) \text{ for } i = \overline{1, N} \text{ and } j = \overline{1, N}. \tag{11}$$

Owing to $e^{\mu} > 1$ and $e^{\mu y_j} > 1 \ \forall j = \overline{2, N}$, it is easy to see that

$$u_{1j} = \frac{1 - e^{\mu y_j}}{e^{\mu} - 1} + \frac{(1 - 1) \cdot \left(e^{\mu y_j} - 1\right)}{\left(e^{\mu} - 1\right)^2} = \frac{1 - e^{\mu y_j}}{e^{\mu} - 1} < 0 \ \forall j = \overline{2, N}$$

and thus the starting moment $t_1 = 0$ is never optimal in duel (1) by (2) — (4) with generalized exponentially-convex rewards (11). Besides, some set-ups are such that the duel end (final) time moment $t_N = 1$ is not acceptable for acting. Henceforward, our main goal is to ascertain optimality of internal time moments (between $t_2 = \frac{1}{N-1}$ and $t_{N-1} = \frac{N-2}{N-1}$) in such 1-bullet silent duels with generalized exponentially-convex rewards at any given $\mu > 0$.

2. Reward decreases

Figure 1 shows a montage of reward matrix (4) visualization, calculated as (11) for $N = 26$, by a subset of the reward steepness factor μ values ranging from 0.05 to 10. The visualization allows understanding the impact of the reward steepness factor on the rewards. At values of μ not exceeding 1, reward matrix (4) meshed into surface (remaining still skew-symmetric) has less exposed nonlinearities. Thus, as the pure strategy of the first duelist

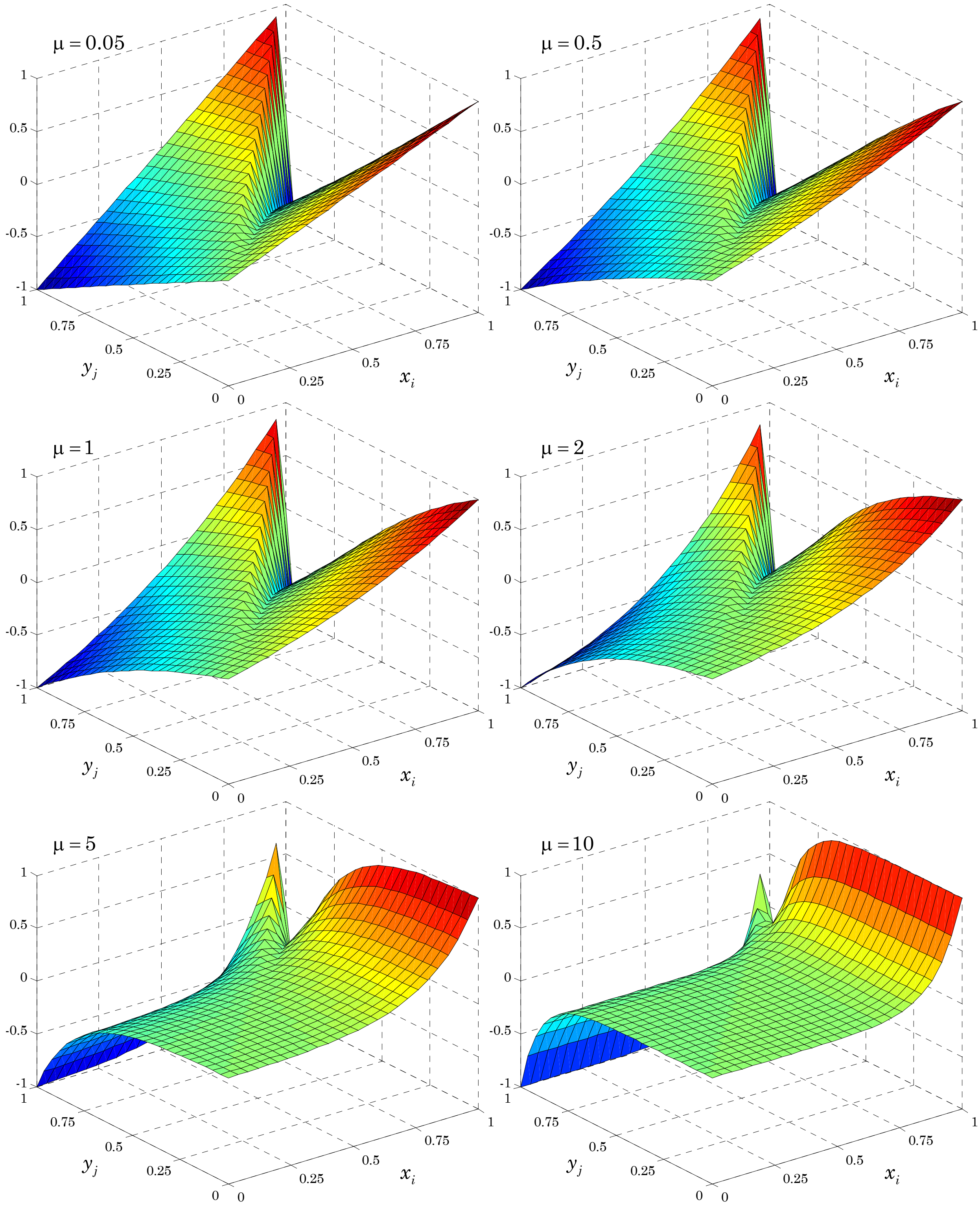

Figure 1. Reward matrix (4) for $N = 26$ as a surface defined on a 26×26 lattice by a subset of the reward steepness factor

increases, the first duelist reward grows almost linearly by an earlier pure strategy (earlier shot or earlier fired bullet) of the second duelist (it is distinctly seen by $\mu = 0.05$ and $\mu = 0.5$). As the reward steepness factor goes beyond 1, the reward surface acquires substantial nonlinearities, which are distinctly seen, e. g., by $\mu = 5$ and $\mu = 10$. In this case, the rewards of the duelists are hardly distinguishable, if each of them shoots (acts) in approximately the first third or half of the duel span (that is clearly seen in the bottom row of Figure 1). However, a subtler yet stricter property of generalized exponentially-convex rewards exists and it is covered in the following assertion.

Theorem 1
Entry u_{nj} by (11), considered as a discrete function of index $j = \overline{1,\ n-1}$ by $n \in \left\{\overline{2,\ N}\right\}$, strictly decreases as index j is increased. Entry u_{nj} by (11), considered as a discrete function of index $j = \overline{n+1,\ N}$ by $n \in \left\{\overline{2,\ N-1}\right\}$, strictly decreases as index j is increased.

Proof
Plugging $i = n$ into (11) for $n \in \left\{\overline{2,\ N}\right\}$, entry

$$u_{nj} = \frac{e^{\mu x_n} - e^{\mu y_j}}{e^{\mu} - 1} - \frac{\left(e^{\mu x_n} - 1\right)\left(e^{\mu y_j} - 1\right)}{\left(e^{\mu} - 1\right)^2} =$$
$$= \frac{e^{\mu x_n} - 1}{e^{\mu} - 1} - \left(1 + \frac{e^{\mu x_n} - 1}{e^{\mu} - 1}\right) \cdot \frac{e^{\mu y_j} - 1}{e^{\mu} - 1} \quad \text{for} \quad j = \overline{1,\ n-1} \quad \text{at} \quad n \in \left\{\overline{2,\ N}\right\}. \tag{12}$$

Due to $e^{\mu x_n} > 1$ and $e^{\mu y_j} \geqslant 1$ by $x_n > 0$ and $y_j \geqslant 0$, respectively, entry (12) is a negatively-sloped line with respect to exponent $e^{\mu y_j}$. Therefore, entry (12) strictly decreases as index j is increased off 1 up to $n-1$.

Plugging $i = n$ into (11) for $n \in \left\{\overline{2,\ N-1}\right\}$, entry

$$u_{nj} = \frac{e^{\mu x_n} - e^{\mu y_j}}{e^{\mu} - 1} + \frac{\left(e^{\mu x_n} - 1\right)\left(e^{\mu y_j} - 1\right)}{\left(e^{\mu} - 1\right)^2} =$$
$$= \frac{e^{\mu x_n} - 1}{e^{\mu} - 1} - \left(1 - \frac{e^{\mu x_n} - 1}{e^{\mu} - 1}\right) \cdot \frac{e^{\mu y_j} - 1}{e^{\mu} - 1} \quad \text{for} \quad j = \overline{n+1,\ N} \quad \text{at} \quad n \in \left\{\overline{2,\ N-1}\right\}. \tag{13}$$

Inasmuch as

$$\mu > \mu x_n \quad \forall\, \mu > 0 \quad \text{and} \quad \forall\, x_n \in \left\{\frac{i-1}{N-1}\right\}_{i=2}^{N-1} \subset \left[\frac{1}{N-1};\ \frac{N-2}{N-1}\right],$$

then $e^{\mu x_n} < e^{\mu}$ and, consequently,

$$1 > \frac{e^{\mu x_n} - 1}{e^{\mu} - 1},$$

whence entry (13) is a negatively-sloped line with respect to exponent $e^{\mu y_j}$. Therefore, entry (13) strictly decreases as index j is increased off $n+1$ up to N. □

In fact, Theorem 1 means that the first duelist reward always decreases at a fixed pure strategy of the first duelist, if the second duelist moves one's pure strategy either toward the fixed strategy or farther away from the fixed strategy, but the latter is never equal to the second duelist's pure strategy. This subtle yet remarkable property (e. g., it is hardly observable in the bottom row of Figure 1) will be very useful in proving optimality of internal time moments in the duel.

3. Optimality of internal time moments

Theorem 2
In 1-bullet silent duel (1) by (2)—(4) for (11) time moment $t_n = \dfrac{n-1}{N-1}$ for $N \in \mathbb{N} \setminus \{1,\ 2\}$ is never optimal by

$$n \leqslant \frac{N-1}{2} + 1. \tag{14}$$

So, 3×3 duels do not have their internal moment $t_2=\frac{1}{2}$, which would be optimal. On the other side, internal moment $t_3=\frac{2}{3}$ is optimal in 4×4 duels with $\mu\in(0;\ \mu_{00}]$, where μ_{00} is the positive root of equation

$$2e^{\frac{2\mu}{3}}-e^{\mu}-1=0, \tag{15}$$

and this root satisfies inequality

$$1.4436354<\mu_{00}<1.4436355. \tag{16}$$

In bigger duels, at $N\in\mathbb{N}\setminus\{1,\ 2,\ 3,\ 4\}$, internal time moment $t_n=\frac{n-1}{N-1}$ is optimal if

$$n\in\left(\frac{3+\sqrt{4N-3}}{2};\ N-1\right]\cap\mathbb{N}\ \text{ for }\ N\in\{5,\ 6\} \tag{17}$$

and

$$n\in\left(\frac{N-1}{2}+1;\ N-1\right]\cap\mathbb{N}\ \text{ for }\ N\in\mathbb{N}\setminus\left\{\overline{1,\ 6}\right\} \tag{18}$$

and $\mu\geqslant\mu_0$ along with $\mu\in(0;\ \mu_{00}]$ by $\mu_0\leqslant\mu_{00}$, where μ_0 is the unique positive root of equation

$$e^{\mu\cdot\frac{N+n-2}{N-1}}-e^{\mu\cdot\frac{N+n-3}{N-1}}+2e^{\mu\cdot\frac{n-2}{N-1}}-e^{\mu\cdot\frac{2n-3}{N-1}}-1=0 \tag{19}$$

with respect to μ and μ_{00} is the unique positive root of equation

$$2e^{\mu\cdot\frac{n-1}{N-1}}-e^{\mu}-1=0 \tag{20}$$

with respect to μ. In particular, the 5×5 duel has only one optimal internal time moment $t_4=\frac{3}{4}$ if $\mu\in[\mu_0;\ \mu_{00}]$ by the respective equation (19)

$$e^{\frac{7\mu}{4}}-e^{\frac{3\mu}{2}}+2e^{\frac{\mu}{2}}-e^{\frac{5\mu}{4}}-1=0 \tag{21}$$

and the respective equation (20)

$$2e^{\frac{3\mu}{4}}-e^{\mu}-1=0, \tag{22}$$

whose unique positive roots satisfy inequalities

$$0.79784231<\mu_0<0.79784232 \tag{23}$$

and

$$2.43751145<\mu_{00}<2.43751146, \tag{24}$$

respectively. The 6×6 duel can have two optimal internal time moments $t_4=\frac{3}{5}$ and $t_5=\frac{4}{5}$, but they are not simultaneously optimal: time moment $t_4=\frac{3}{5}$ is optimal if $\mu\in[\mu_0;\ \mu_{00}]$ by the respective equation (19)

$$e^{\frac{8\mu}{5}}-e^{\frac{7\mu}{5}}+2e^{\frac{2\mu}{5}}-e^{\mu}-1=0 \tag{25}$$

and the respective equation (20)

$$2e^{\frac{3\mu}{5}}-e^{\mu}-1=0, \tag{26}$$

whose unique positive roots satisfy inequalities

$$0.35906596<\mu_0<0.35906597 \tag{27}$$

and

$$0.82216323<\mu_{00}<0.82216324, \tag{28}$$

respectively; time moment $t_5 = \dfrac{4}{5}$ is optimal if $\mu \in [\mu_0;\ \mu_{00}]$ by the respective equation (19)

$$e^{\frac{9\mu}{5}} - e^{\frac{8\mu}{5}} + 2e^{\frac{3\mu}{5}} - e^{\frac{7\mu}{5}} - 1 = 0 \tag{29}$$

and the respective equation (20)

$$2e^{\frac{4\mu}{5}} - e^{\mu} - 1 = 0, \tag{30}$$

whose unique positive roots satisfy inequalities

$$1.743571 < \mu_0 < 1.7435711 \tag{31}$$

and

$$3.2812798 < \mu_{00} < 3.2812799, \tag{32}$$

respectively.

Proof

Internal time moment $t_n = \dfrac{n-1}{N-1}$ by $n \in \left\{\overline{2,\ N-1}\right\}$ is optimal if inequalities

$$u_{nj} = \frac{e^{\mu x_n} - e^{\mu y_j}}{e^{\mu} - 1} - \frac{\left(e^{\mu x_n} - 1\right)\left(e^{\mu y_j} - 1\right)}{\left(e^{\mu} - 1\right)^2} > 0 \ \ \forall\, y_j < x_n \ \text{ by } \ j = \overline{1,\ n-1} \tag{33}$$

and

$$u_{nj} = \frac{e^{\mu x_n} - e^{\mu y_j}}{e^{\mu} - 1} + \frac{\left(e^{\mu x_n} - 1\right)\left(e^{\mu y_j} - 1\right)}{\left(e^{\mu} - 1\right)^2} > 0 \ \ \forall\, y_j > x_n \ \text{ by } \ j = \overline{n+1,\ N} \tag{34}$$

hold. Owing to Theorem 1, function u_{nj} is strictly decreasing with respect to index $j = \overline{1,\ n-1}$, and hence inequality (33) is equivalent to inequality

$$\begin{aligned} u_{n,n-1} &= \frac{e^{\mu x_n} - e^{\mu y_{n-1}}}{e^{\mu} - 1} - \frac{\left(e^{\mu x_n} - 1\right)\left(e^{\mu y_{n-1}} - 1\right)}{\left(e^{\mu} - 1\right)^2} = \\ &= \frac{e^{\mu x_n} e^{\mu} - e^{\mu x_n} - e^{\mu y_{n-1}} e^{\mu} + e^{\mu y_{n-1}} - e^{\mu x_n} e^{\mu y_{n-1}} + e^{\mu y_{n-1}} + e^{\mu x_n} - 1}{\left(e^{\mu} - 1\right)^2} = \\ &= \frac{e^{\mu x_n} e^{\mu} - e^{\mu y_{n-1}} e^{\mu} + 2e^{\mu y_{n-1}} - e^{\mu x_n} e^{\mu y_{n-1}} - 1}{\left(e^{\mu} - 1\right)^2} > 0. \end{aligned} \tag{35}$$

Inequality (35) is simplified to inequality

$$e^{\mu x_n} e^{\mu} - e^{\mu y_{n-1}} e^{\mu} + 2e^{\mu y_{n-1}} - e^{\mu x_n} e^{\mu y_{n-1}} - 1 > 0 \tag{36}$$

written with respect to variable μ.

To solve inequality (36), denote

$$\begin{aligned} h\left(\mu,\ N,\ n\right) &= e^{\mu x_n} e^{\mu} - e^{\mu y_{n-1}} e^{\mu} + 2e^{\mu y_{n-1}} - e^{\mu x_n} e^{\mu y_{n-1}} - 1 = \\ &= e^{\mu \cdot \frac{n-1}{N-1}} e^{\mu} - e^{\mu \cdot \frac{n-2}{N-1}} e^{\mu} + 2e^{\mu \cdot \frac{n-2}{N-1}} - e^{\mu \cdot \frac{n-1}{N-1}} e^{\mu \cdot \frac{n-2}{N-1}} - 1 = \\ &= e^{\mu \cdot \frac{N+n-2}{N-1}} - e^{\mu \cdot \frac{N+n-3}{N-1}} + 2e^{\mu \cdot \frac{n-2}{N-1}} - e^{\mu \cdot \frac{2n-3}{N-1}} - 1. \end{aligned} \tag{37}$$

Generally speaking, function (37), considered as a function of μ at fixed integers N and n, is a sum of four exponential functions and, therefore, according to the theorem of summing exponential functions [2, 4], it can have at most three extrema. For estimating possible extrema, the first partial derivative of function (37) with respect to variable μ is:

$$\frac{\partial h}{\partial \mu} = \frac{N+n-2}{N-1} e^{\mu \cdot \frac{N+n-2}{N-1}} - \frac{N+n-3}{N-1} e^{\mu \cdot \frac{N+n-3}{N-1}} + \frac{2n-4}{N-1} e^{\mu \cdot \frac{n-2}{N-1}} - \frac{2n-3}{N-1} e^{\mu \cdot \frac{2n-3}{N-1}}. \tag{38}$$

First partial derivative (38) at point $\mu = 0$ is

$$\left.\frac{\partial h}{\partial \mu}\right|_{\mu=0} = \frac{N+n-2}{N-1} - \frac{N+n-3}{N-1} + \frac{2n-4}{N-1} - \frac{2n-3}{N-1} = 0,$$

and thus $\mu = 0$ is a critical point of function (37) as a function of variable μ. The second partial derivative of function (37) with respect to variable μ is:

$$\frac{\partial^2 h}{\partial \mu^2} = \left(\frac{N+n-2}{N-1}\right)^2 e^{\mu \cdot \frac{N+n-2}{N-1}} - \left(\frac{N+n-3}{N-1}\right)^2 e^{\mu \cdot \frac{N+n-3}{N-1}} + \\ +2 \cdot \left(\frac{n-2}{N-1}\right)^2 e^{\mu \cdot \frac{n-2}{N-1}} - \left(\frac{2n-3}{N-1}\right)^2 e^{\mu \cdot \frac{2n-3}{N-1}}. \tag{39}$$

Second partial derivative (39) at point $\mu = 0$ is

$$\left.\frac{\partial^2 h}{\partial \mu^2}\right|_{\mu=0} = \left(\frac{N+n-2}{N-1}\right)^2 - \left(\frac{N+n-3}{N-1}\right)^2 + 2 \cdot \left(\frac{n-2}{N-1}\right)^2 - \left(\frac{2n-3}{N-1}\right)^2 = \\ = \frac{(2N+2n-5) + 2n^2 - 8n + 8 - 4n^2 + 12n - 9}{(N-1)^2} = \\ = \frac{2N + 6n - 6 - 2n^2}{(N-1)^2} = 2 \cdot \frac{N + 3n - 3 - n^2}{(N-1)^2}. \tag{40}$$

As parabola

$$n^2 - 3n + 3 - N < 0 \;\; \forall n \in \left(\frac{3-\sqrt{4N-3}}{2};\; \frac{3+\sqrt{4N-3}}{2}\right), \tag{41}$$

then inequality (41) determines the sign of (40) so that

$$\left.\frac{\partial^2 h}{\partial \mu^2}\right|_{\mu=0} = 2 \cdot \frac{N + 3n - 3 - n^2}{(N-1)^2} > 0 \;\; \forall n \in \left[2;\; \frac{3+\sqrt{4N-3}}{2}\right) \cap \mathbb{N} \;\text{ and }\; N > n \tag{42}$$

and

$$\left.\frac{\partial^2 h}{\partial \mu^2}\right|_{\mu=0} = 2 \cdot \frac{N + 3n - 3 - n^2}{(N-1)^2} < 0 \;\; \forall n \in \left(\frac{3+\sqrt{4N-3}}{2};\; N-1\right] \cap \mathbb{N} \;\text{ and }\; N > n, \tag{43}$$

where

$$\left.\frac{\partial^2 h}{\partial \mu^2}\right|_{\mu=0} = 0 \;\text{ if }\; n = \frac{3+\sqrt{4N-3}}{2} < N \;\text{ and }\; \frac{3+\sqrt{4N-3}}{2} \in \mathbb{N}. \tag{44}$$

Inequality (42) means that $\mu = 0$ is a minimum point of function (37) as a function of variable μ. Inequality (43) means that $\mu = 0$ is a maximum point of function (37) as a function of variable μ, while (44) means that $\mu = 0$ may be an inflection point of such a function. Besides,

$$h(0,\; N,\; n) = 1 - 1 + 2 - 1 - 1 = 0, \tag{45}$$

whereas

$$\lim_{\mu \to -\infty} h(\mu,\; N,\; n) = \lim_{\mu \to -\infty} \left(e^{\mu \cdot \frac{N+n-2}{N-1}} - e^{\mu \cdot \frac{N+n-3}{N-1}} + 2e^{\mu \cdot \frac{n-2}{N-1}} - e^{\mu \cdot \frac{2n-3}{N-1}} - 1\right) = -1 \\ \forall n \in [3;\; N-1] \cap \mathbb{N} \tag{46}$$

and

$$\lim_{\mu \to \infty} h(\mu,\; N,\; n) = \lim_{\mu \to \infty} \left(e^{\mu \cdot \frac{N+n-2}{N-1}} - e^{\mu \cdot \frac{N+n-3}{N-1}} + 2e^{\mu \cdot \frac{n-2}{N-1}} - e^{\mu \cdot \frac{2n-3}{N-1}} - 1\right) = \infty. \tag{47}$$

At $n=2$ function (37) becomes

$$h\left(\mu,\ N,\ 2\right)=e^{\mu\cdot\frac{N}{N-1}}-e^{\mu}-e^{\mu\cdot\frac{1}{N-1}}+1, \tag{48}$$

where

$$\lim_{\mu\to-\infty}h\left(\mu,\ N,\ 2\right)=\lim_{\mu\to-\infty}\left(e^{\mu\cdot\frac{N}{N-1}}-e^{\mu}-e^{\mu\cdot\frac{1}{N-1}}+1\right)=1. \tag{49}$$

Function (48), considered as a function of μ at a fixed integer N, is a sum of three exponential functions and, therefore, according to the theorem of summing exponential functions [2, 4], it can have at most two extrema. One of those extrema, owing to (42) being true at $n=2$, is the minimum point $\mu=0$, regardless of (49). But suppose that this function has the second local extremum by $\mu>0$, which must be a local maximum. But due to (47), this local maximum must be followed by a local minimum, which is impossible. Therefore,

$$h\left(\mu,\ N,\ 2\right)>0\ \ \forall\,\mu>0\ \text{ and }\ N\in\mathbb{N}\setminus\{1,\ 2\}. \tag{50}$$

Inequality (50) means that $u_{21}>0$.

Function

$$h\left(\mu,\ N,\ 3\right)=e^{\mu\cdot\frac{N+1}{N-1}}-e^{\mu\cdot\frac{N}{N-1}}+2e^{\mu\cdot\frac{1}{N-1}}-e^{\mu\cdot\frac{3}{N-1}}-1, \tag{51}$$

considered as a function of μ at a fixed integer N, is a sum of four exponential functions and, therefore, according to the theorem of summing exponential functions [2, 4], it can have at most three extrema. One of those extrema, owing to (42) being true at $n=3$, is the minimum point $\mu=0$. But suppose that this function has a local extremum by $\mu>0$, which must be a local maximum. Due to (47), this local maximum must be followed by a local minimum. On the other side, due to (46), function (51) of μ has a local maximum by $\mu<0$. Hence, the extrema by $\mu>0$ are impossible and

$$h\left(\mu,\ N,\ 3\right)>0\ \ \forall\,\mu>0\ \text{ and }\ N\in\mathbb{N}\setminus\{1,\ 2,\ 3\}. \tag{52}$$

Inequality (52) means that $u_{31}>0$, $u_{32}>0$.

In general, function (37), considered as a function of μ at a fixed integers $N-1\geqslant n\geqslant 4$ and $N\geqslant 5$, is a sum of four exponential functions and, therefore, according to the theorem of summing exponential functions [2, 4], it can have at most three extrema. If inequality (42) holds, one of those extrema is the minimum point $\mu=0$. Due to (46), this function has a local maximum by $\mu<0$. On the other side, due to (47), there are no extrema by $\mu>0$, and so

$$h\left(\mu,\ N,\ n\right)>0\ \ \forall\,\mu>0\ \text{ and }\ \forall\,n\in\left[4;\ \frac{3+\sqrt{4N-3}}{2}\right]\cap\mathbb{N}\ \text{ and }\ \forall\,N\in\mathbb{N}\setminus\{1,\ 2,\ 3,\ 4\}. \tag{53}$$

The inclusion of value $n=\dfrac{3+\sqrt{4N-3}}{2}$ in (53) will be additionally clarified below. Meanwhile, value

$$\frac{3+\sqrt{4N-3}}{2}<4\ \text{ at }\ N\in\{5,\ 6\} \tag{54}$$

and

$$\frac{3+\sqrt{4N-3}}{2}=4\ \text{ at }\ N=7, \tag{55}$$

and thus (54) and (55) allow writing inequality (53) in a better way as

$$h\left(\mu,\ N,\ n\right)>0\ \ \forall\,\mu>0\ \text{ and }\ \forall\,n\in\left[4;\ \frac{3+\sqrt{4N-3}}{2}\right]\cap\mathbb{N}\ \text{ and }\ \forall\,N\in\mathbb{N}\setminus\left\{\overline{1,\ 6}\right\}. \tag{56}$$

If inequality (43) holds, point $\mu=0$ is the maximum point, and thus function (37) has a local minimum $\mu=\mu_*$ by $\mu>0$. But, due to (47), it cannot have more than a local extremum by $\mu>0$, because then local minimum $\mu=\mu_*$ would have to be followed by a local maximum and another local minimum, which is impossible. Therefore,

$\mu = \mu_*$ is the single local minimum by $\mu > 0$, at which

$$h\left(\mu_*,\ N,\ n\right) < 0\ \ \forall n \in \left(\frac{3+\sqrt{4N-3}}{2};\ N-1\right] \cap \mathbb{N}\ \text{ and }\ \forall N \in \mathbb{N} \setminus \{1,\ 2,\ 3,\ 4\}\,. \tag{57}$$

Inequality (57) implies that function (37) has the positive zero at some $\mu = \mu_0$ that follows the single local minimum, i. e.

$$h\left(\mu_0,\ N,\ n\right) = 0\ \ \forall n \in \left(\frac{3+\sqrt{4N-3}}{2};\ N-1\right] \cap \mathbb{N}\ \text{ and }\ \forall N \in \mathbb{N} \setminus \{1,\ 2,\ 3,\ 4\} \tag{58}$$

at some $\mu_0 > \mu_*$ and

$$h\left(\mu,\ N,\ n\right) > 0\ \ \forall \mu > \mu_0$$
$$\text{and }\ \forall n \in \left(\frac{3+\sqrt{4N-3}}{2};\ N-1\right] \cap \mathbb{N}\ \text{ and }\ \forall N \in \mathbb{N} \setminus \{1,\ 2,\ 3,\ 4\}\,. \tag{59}$$

If equality (44) holds, then function (37) is

$$h\left(\mu,\ N,\ \frac{3+\sqrt{4N-3}}{2}\right) =$$
$$= e^{\mu\cdot\frac{2N+\sqrt{4N-3}-1}{2\cdot(N-1)}} - e^{\mu\cdot\frac{2N+\sqrt{4N-3}-3}{2\cdot(N-1)}} + 2e^{\mu\cdot\frac{\sqrt{4N-3}-1}{2\cdot(N-1)}} - e^{\mu\cdot\frac{\sqrt{4N-3}}{N-1}} - 1 \tag{60}$$

and its first partial derivative (38) is

$$\frac{\partial h\left(\mu,\ N,\ \frac{3+\sqrt{4N-3}}{2}\right)}{\partial \mu} =$$
$$= \frac{2N+\sqrt{4N-3}-1}{2\cdot(N-1)} e^{\mu\cdot\frac{2N+\sqrt{4N-3}-1}{2\cdot(N-1)}} - \frac{2N+\sqrt{4N-3}-3}{2\cdot(N-1)} e^{\mu\cdot\frac{2N+\sqrt{4N-3}-3}{2\cdot(N-1)}} +$$
$$+ \frac{\sqrt{4N-3}-1}{N-1} e^{\mu\cdot\frac{\sqrt{4N-3}-1}{2\cdot(N-1)}} - \frac{\sqrt{4N-3}}{N-1} e^{\mu\cdot\frac{\sqrt{4N-3}}{N-1}}\,. \tag{61}$$

The second partial derivative of function (60) is

$$\frac{\partial^2 h\left(\mu,\ N,\ \frac{3+\sqrt{4N-3}}{2}\right)}{\partial \mu^2} =$$
$$= \left(\frac{2N+\sqrt{4N-3}-1}{2\cdot(N-1)}\right)^2 e^{\mu\cdot\frac{2N+\sqrt{4N-3}-1}{2\cdot(N-1)}} - \left(\frac{2N+\sqrt{4N-3}-3}{2\cdot(N-1)}\right)^2 e^{\mu\cdot\frac{2N+\sqrt{4N-3}-3}{2\cdot(N-1)}} +$$
$$+ \frac{1}{2}\cdot\left(\frac{\sqrt{4N-3}-1}{N-1}\right)^2 e^{\mu\cdot\frac{\sqrt{4N-3}-1}{2\cdot(N-1)}} - \left(\frac{\sqrt{4N-3}}{N-1}\right)^2 e^{\mu\cdot\frac{\sqrt{4N-3}}{N-1}}\,. \tag{62}$$

Function (62) at $\mu = 0$ is:

$$\left.\frac{\partial^2 h\left(\mu,\ N,\ \frac{3+\sqrt{4N-3}}{2}\right)}{\partial \mu^2}\right|_{\mu=0} =$$
$$= \left(\frac{2N+\sqrt{4N-3}-1}{2\cdot(N-1)}\right)^2 - \left(\frac{2N+\sqrt{4N-3}-3}{2\cdot(N-1)}\right)^2 + \frac{1}{2}\cdot\left(\frac{\sqrt{4N-3}-1}{N-1}\right)^2 - \left(\frac{\sqrt{4N-3}}{N-1}\right)^2 =$$

$$= \left(\frac{2N+\sqrt{4N-3}-1}{2\cdot(N-1)} - \frac{2N+\sqrt{4N-3}-3}{2\cdot(N-1)} \right) \left(\frac{2N+\sqrt{4N-3}-1}{2\cdot(N-1)} + \frac{2N+\sqrt{4N-3}-3}{2\cdot(N-1)} \right) + \\ + \frac{4N-3-2\sqrt{4N-3}+1-8N+6}{2\cdot(N-1)^2} = \\ = \frac{1}{N-1}\cdot\left(\frac{2N+\sqrt{4N-3}-2}{N-1} \right) - \frac{\sqrt{4N-3}+2N-2}{(N-1)^2} = 0. \tag{63}$$

So, (63) means that point $\mu = 0$ is a critical point of function (61) of variable μ. The second partial derivative of this function, i. e. the third partial derivative of function (60), is

$$\frac{\partial^3 h\left(\mu,\ N,\ \frac{3+\sqrt{4N-3}}{2}\right)}{\partial\mu^3} = \\ = \left(\frac{2N+\sqrt{4N-3}-1}{2\cdot(N-1)} \right)^3 e^{\mu\cdot\frac{2N+\sqrt{4N-3}-1}{2\cdot(N-1)}} - \left(\frac{2N+\sqrt{4N-3}-3}{2\cdot(N-1)} \right)^3 e^{\mu\cdot\frac{2N+\sqrt{4N-3}-3}{2\cdot(N-1)}} + \\ + \frac{1}{4}\cdot\left(\frac{\sqrt{4N-3}-1}{N-1} \right)^3 e^{\mu\cdot\frac{\sqrt{4N-3}-1}{2\cdot(N-1)}} - \left(\frac{\sqrt{4N-3}}{N-1} \right)^3 e^{\mu\cdot\frac{\sqrt{4N-3}}{N-1}}. \tag{64}$$

Function (64) at $\mu = 0$ is:

$$\left.\frac{\partial^3 h\left(\mu,\ N,\ \frac{3+\sqrt{4N-3}}{2}\right)}{\partial\mu^3}\right|_{\mu=0} = \\ = \left(\frac{2N+\sqrt{4N-3}-1}{2\cdot(N-1)} \right)^3 - \left(\frac{2N+\sqrt{4N-3}-3}{2\cdot(N-1)} \right)^3 + \frac{1}{4}\cdot\left(\frac{\sqrt{4N-3}-1}{N-1} \right)^3 - \left(\frac{\sqrt{4N-3}}{N-1} \right)^3 = \\ = \frac{\left(2N+\sqrt{4N-3}\right)^3 - 3\cdot\left(2N+\sqrt{4N-3}\right)^2 + 3\cdot\left(2N+\sqrt{4N-3}\right) - 1}{8\cdot(N-1)^3} - \\ - \frac{\left(2N+\sqrt{4N-3}\right)^3 - 9\cdot\left(2N+\sqrt{4N-3}\right)^2 + 27\cdot\left(2N+\sqrt{4N-3}\right) - 27}{8\cdot(N-1)^3} + \\ + \frac{\left(\sqrt{4N-3}\right)^3 - 3\cdot(4N-3) + 3\sqrt{4N-3} - 1 - 4\cdot\left(\sqrt{4N-3}\right)^3}{4\cdot(N-1)^3} = \\ = \frac{3\cdot\left(2N+\sqrt{4N-3}\right)^2 - 12\cdot\left(2N+\sqrt{4N-3}\right) + 13}{4\cdot(N-1)^3} + \\ + \frac{-12N+8+3\sqrt{4N-3}-3\cdot\left(\sqrt{4N-3}\right)^3}{4\cdot(N-1)^3} = \\ = \frac{12N^2+12N\sqrt{4N-3}-12N-12\sqrt{4N-3}+4}{4\cdot(N-1)^3} + \\ + \frac{-12N+8+3\sqrt{4N-3}\left(1-\left(\sqrt{4N-3}\right)^2\right)}{4\cdot(N-1)^3} = \\ = \frac{12N^2+12N\sqrt{4N-3}-24N-12\sqrt{4N-3}+12}{4\cdot(N-1)^3} +$$

$$+\frac{3\sqrt{4N-3}\,(4-4N)}{4\cdot(N-1)^3}=$$

$$=\frac{12N^2+12N\sqrt{4N-3}-24N-12\sqrt{4N-3}+12+12\sqrt{4N-3}-12N\sqrt{4N-3}}{4\cdot(N-1)^3}=$$

$$=\frac{3N^2-6N+3}{(N-1)^3}=\frac{3\cdot(N-1)^2}{(N-1)^3}=\frac{3}{N-1}>0. \tag{65}$$

Inequality (65) means that point $\mu=0$ is a local minimum point of function (61). Meanwhile,

$$\lim_{\mu\to-\infty}\frac{\partial h\left(\mu,\ N,\ \dfrac{3+\sqrt{4N-3}}{2}\right)}{\partial\mu}=$$

$$=\lim_{\mu\to-\infty}\left(\frac{2N+\sqrt{4N-3}-1}{2\cdot(N-1)}e^{\mu\cdot\frac{2N+\sqrt{4N-3}-1}{2\cdot(N-1)}}-\frac{2N+\sqrt{4N-3}-3}{2\cdot(N-1)}e^{\mu\cdot\frac{2N+\sqrt{4N-3}-3}{2\cdot(N-1)}}+\right.$$

$$\left.+\frac{\sqrt{4N-3}-1}{N-1}e^{\mu\cdot\frac{\sqrt{4N-3}-1}{2\cdot(N-1)}}-\frac{\sqrt{4N-3}}{N-1}e^{\mu\cdot\frac{\sqrt{4N-3}}{N-1}}\right)=0 \tag{66}$$

and

$$\lim_{\mu\to\infty}\frac{\partial h\left(\mu,\ N,\ \dfrac{3+\sqrt{4N-3}}{2}\right)}{\partial\mu}=$$

$$=\lim_{\mu\to\infty}\left(\frac{2N+\sqrt{4N-3}-1}{2\cdot(N-1)}e^{\mu\cdot\frac{2N+\sqrt{4N-3}-1}{2\cdot(N-1)}}-\frac{2N+\sqrt{4N-3}-3}{2\cdot(N-1)}e^{\mu\cdot\frac{2N+\sqrt{4N-3}-3}{2\cdot(N-1)}}+\right.$$

$$\left.+\frac{\sqrt{4N-3}-1}{N-1}e^{\mu\cdot\frac{\sqrt{4N-3}-1}{2\cdot(N-1)}}-\frac{\sqrt{4N-3}}{N-1}e^{\mu\cdot\frac{\sqrt{4N-3}}{N-1}}\right)=\infty. \tag{67}$$

Due to (66) and

$$\left.\frac{\partial h\left(\mu,\ N,\ \dfrac{3+\sqrt{4N-3}}{2}\right)}{\partial\mu}\right|_{\mu=0}=$$

$$=\frac{2N+\sqrt{4N-3}-1}{2\cdot(N-1)}-\frac{2N+\sqrt{4N-3}-3}{2\cdot(N-1)}+\frac{\sqrt{4N-3}-1}{N-1}-\frac{\sqrt{4N-3}}{N-1}=0, \tag{68}$$

function (61) has a local maximum by $\mu<0$. But suppose that this function has a local extremum by $\mu>0$, which must be a local maximum. Due to (67), this local maximum must be followed by a local minimum. But this is impossible due to function (61) of variable μ, being a sum of four exponential functions, can have at most three extrema. Therefore, function (61) does not have extrema by $\mu>0$ and, owing to (68),

$$\frac{\partial h\left(\mu,\ N,\ \dfrac{3+\sqrt{4N-3}}{2}\right)}{\partial\mu}>0\ \ \forall\,\mu>0. \tag{69}$$

Inequality (69) means that function (60) is an increasing function by $\mu>0$. And, owing to (45), this function is positive by $\mu>0$:

$$h\left(\mu,\ N,\ \frac{3+\sqrt{4N-3}}{2}\right)>0\ \ \forall\,\mu>0. \tag{70}$$

Inequality (70) rigorously clarifies the inclusion of value $n = \dfrac{3+\sqrt{4N-3}}{2}$ in (56). Hence, if inequality (44) holds, then inequality (35) holds as well.

Owing to Theorem 1, function u_{nj} is strictly decreasing with respect to index $j = \overline{n+1,\ N}$, and hence inequality (34) is equivalent to inequality

$$u_{nN} = \frac{e^{\mu x_n} - e^{\mu y_N}}{e^{\mu} - 1} + \frac{\left(e^{\mu x_n} - 1\right)\left(e^{\mu y_N} - 1\right)}{\left(e^{\mu} - 1\right)^2} =$$
$$= \frac{e^{\mu x_n} - e^{\mu}}{e^{\mu} - 1} + \frac{\left(e^{\mu x_n} - 1\right)\left(e^{\mu} - 1\right)}{\left(e^{\mu} - 1\right)^2} = \frac{2e^{\mu x_n} - e^{\mu} - 1}{e^{\mu} - 1} > 0. \tag{71}$$

Inequality (71) is simplified to inequality

$$2e^{\mu \cdot \frac{n-1}{N-1}} - e^{\mu} - 1 > 0 \tag{72}$$

written with respect to variable μ. To solve inequality (72), denote

$$f(\mu,\ N,\ n) = 2e^{\mu \cdot \frac{n-1}{N-1}} - e^{\mu} - 1. \tag{73}$$

The first derivative of function (73) is

$$\frac{\partial f}{\partial \mu} = 2 \cdot \frac{n-1}{N-1} \cdot e^{\mu \cdot \frac{n-1}{N-1}} - e^{\mu}. \tag{74}$$

First derivative (74) turns into zero if

$$e^{\mu} = 2 \cdot \frac{n-1}{N-1} \cdot e^{\mu \cdot \frac{n-1}{N-1}},$$

whence

$$\mu = \ln\left(2 \cdot \frac{n-1}{N-1} \cdot e^{\mu \cdot \frac{n-1}{N-1}}\right) = \ln\left(2 \cdot \frac{n-1}{N-1}\right) + \ln\left(e^{\mu \cdot \frac{n-1}{N-1}}\right) = \ln\left(2 \cdot \frac{n-1}{N-1}\right) + \mu \cdot \frac{n-1}{N-1},$$

$$\mu \cdot \frac{N-n}{N-1} = \ln\left(2 \cdot \frac{n-1}{N-1}\right),$$

and thus

$$\mu = \mu_{**} = \frac{N-1}{N-n} \cdot \ln\left(2 \cdot \frac{n-1}{N-1}\right) \tag{75}$$

is the unique critical point of function (73). The second derivative of function (73) is

$$\frac{d^2 f}{d\mu^2} = 2 \cdot \left(\frac{n-1}{N-1}\right)^2 \cdot e^{\mu \cdot \frac{n-1}{N-1}} - e^{\mu}. \tag{76}$$

Second derivative (76) at critical point (75) is

$$\left.\frac{d^2 f}{d\mu^2}\right|_{\mu=\mu_{**}} = 2 \cdot \left(\frac{n-1}{N-1}\right)^2 \cdot e^{\frac{N-1}{N-n} \cdot \ln\left(2 \cdot \frac{n-1}{N-1}\right) \cdot \frac{n-1}{N-1}} - e^{\frac{N-1}{N-n} \cdot \ln\left(2 \cdot \frac{n-1}{N-1}\right)} =$$
$$= 2 \cdot \left(\frac{n-1}{N-1}\right)^2 \cdot e^{\frac{n-1}{N-n} \cdot \ln\left(2 \cdot \frac{n-1}{N-1}\right)} - e^{\frac{N-1}{N-n} \cdot \ln\left(2 \cdot \frac{n-1}{N-1}\right)} =$$
$$= 2 \cdot \left(\frac{n-1}{N-1}\right)^2 \cdot \left(2 \cdot \frac{n-1}{N-1}\right)^{\frac{n-1}{N-n}} - \left(2 \cdot \frac{n-1}{N-1}\right)^{\frac{N-1}{N-n}} =$$

$$= 2^{\frac{N-1}{N-n}} \cdot \left(\frac{n-1}{N-1}\right)^{\frac{2N-n-1}{N-n}} - 2^{\frac{N-1}{N-n}} \cdot \left(\frac{n-1}{N-1}\right)^{\frac{N-1}{N-n}} =$$

$$= 2^{\frac{N-1}{N-n}} \cdot \left(\frac{n-1}{N-1}\right)^{\frac{N-1}{N-n}} \left(\frac{n-1}{N-1} - 1\right) =$$

$$= \left(2 \cdot \frac{n-1}{N-1}\right)^{\frac{N-1}{N-n}} \left(\frac{n-N}{N-1}\right) < 0. \tag{77}$$

Inequality (77) means that critical point (75) is the unique maximum point of function (73) of variable μ and there are no other extrema. This function at maximum point (75) rises to value

$$f(\mu_{**},\ N,\ n) = 2e^{\frac{N-1}{N-n}\cdot\ln\left(2\cdot\frac{n-1}{N-1}\right)\cdot\frac{n-1}{N-1}} - e^{\frac{N-1}{N-n}\cdot\ln\left(2\cdot\frac{n-1}{N-1}\right)} - 1 =$$

$$= 2e^{\frac{n-1}{N-n}\cdot\ln\left(2\cdot\frac{n-1}{N-1}\right)} - e^{\frac{N-1}{N-n}\cdot\ln\left(2\cdot\frac{n-1}{N-1}\right)} - 1 =$$

$$= 2 \cdot \left(2 \cdot \frac{n-1}{N-1}\right)^{\frac{n-1}{N-n}} - \left(2 \cdot \frac{n-1}{N-1}\right)^{\frac{N-1}{N-n}} - 1 =$$

$$= 2^{\frac{N-1}{N-n}} \cdot \left(\frac{n-1}{N-1}\right)^{\frac{n-1}{N-n}} - 2^{\frac{N-1}{N-n}} \cdot \left(\frac{n-1}{N-1}\right)^{\frac{N-1}{N-n}} - 1 =$$

$$= 2^{\frac{N-1}{N-n}} \cdot \left(\frac{n-1}{N-1}\right)^{\frac{n-1}{N-n}} \cdot \left(1 - \frac{n-1}{N-1}\right) - 1$$

$$= 2^{\frac{N-1}{N-n}} \cdot \left(\frac{n-1}{N-1}\right)^{\frac{n-1}{N-n}} \cdot \left(\frac{N-n}{N-1}\right) - 1. \tag{78}$$

Meanwhile, consider (75) as a function of integer variable n at a fixed integer N, $2 \leqslant n \leqslant N-1$. The first derivative of such a function is

$$\frac{\partial \mu_{**}}{\partial n} = \frac{N-1}{(N-n)^2} \cdot \ln\left(2 \cdot \frac{n-1}{N-1}\right) + \frac{N-1}{(N-n)(n-1)} =$$

$$= \frac{N-1}{(N-n)} \cdot \left(\frac{\ln\left(2 \cdot \frac{n-1}{N-1}\right)}{N-n} + \frac{1}{n-1}\right). \tag{79}$$

First derivative (79) turns into zero if

$$\frac{\ln\left(2 \cdot \frac{n-1}{N-1}\right)}{N-n} + \frac{1}{n-1} = 0. \tag{80}$$

As $t_n = \dfrac{n-1}{N-1}$, then

$$n = t_n(N-1) + 1$$

and equation (80) can be re-written as

$$\ln(2t_n) + \frac{N - t_n(N-1) - 1}{t_n(N-1)} = \ln(2t_n) + \frac{1-t_n}{t_n} = 0. \tag{81}$$

Consider a function of variable t_n

$$\varphi(t_n) = \ln(2t_n) + \frac{1-t_n}{t_n} \tag{82}$$

in the left side of (81). The first derivative of function (82) is

$$\frac{d\varphi}{dt_n} = \frac{1}{t_n} - \frac{1}{t_n^2} = \frac{t_n - 1}{t_n^2} < 0 \;\; \forall\, t_n \in (0;\; 1)$$

and so function (82) is strictly decreasing. But

$$\varphi(1) = \ln(2) > 0$$

and this means than function (82) is positive $\forall\, t_n \in (0;\; 1)$. Therefore, equation (81) does not have roots and first derivative (79) is positive $\forall\, n = \overline{2,\; N-1}$. This means that (75) as a function of integer variable n at a fixed integer N increases as n increases off 2 up to $N-1$. Meanwhile, $\mu_{**} \leqslant 0$ if

$$\frac{N-1}{N-n} \cdot \ln\left(2 \cdot \frac{n-1}{N-1}\right) \leqslant 0,$$

$$2 \cdot \frac{n-1}{N-1} \leqslant 1,$$

whence (14) follows. In addition,

$$f(0,\; N,\; n) = 2 - 1 - 1 = 0. \tag{83}$$

So, if (14) holds then function (73) of variable μ reaches its maximum (78) at $\mu_{**} \leqslant 0$, which, according to (83), is followed by crossing the abscissa axis exactly at $\mu = 0$, whence

$$f(\mu,\; N,\; n) < 0 \;\; \forall\, \mu > 0 \;\text{ and }\; \forall\, n \in \left[2;\; \frac{N-1}{2} + 1\right] \cap \mathbb{N}. \tag{84}$$

Inequality (84) means that time moment t_n is not optimal by (14). In particular, time moment $t_2 = \frac{1}{2}$ is not optimal in any 3×3 duel, whichever μ is. Moreover, inequality (14) holds for $n = 2$ and any N, and thus time moment $t_2 = \frac{1}{N-1}$ is not optimal in any $N \times N$ duel.

If

$$n > \frac{N-1}{2} + 1 \tag{85}$$

then $\mu_{**} > 0$ and function (73) of variable μ has a zero at some $\mu = \mu_{00} > \mu_{**}$, whence

$$f(\mu,\; N,\; n) > 0 \;\; \forall\, \mu \in (0;\; \mu_{00})$$
$$\text{and } \; \forall\, n \in \left(\frac{N-1}{2} + 1;\; N-1\right] \cap \mathbb{N} \;\text{ and }\; N \in \mathbb{N} \setminus \{1,\; 2,\; 3\}, \tag{86}$$

$$f(\mu_{00},\; N,\; n) = 0 \;\; \forall\, n \in \left(\frac{N-1}{2} + 1;\; N-1\right] \cap \mathbb{N} \;\text{ and }\; N \in \mathbb{N} \setminus \{1,\; 2,\; 3\}, \tag{87}$$

$$f(\mu,\; N,\; n) < 0 \;\; \forall\, \mu > \mu_{00} \;\text{ and }\; \forall\, n \in \left(\frac{N-1}{2} + 1;\; N-1\right] \cap \mathbb{N} \;\text{ and }\; N \in \mathbb{N} \setminus \{1,\; 2,\; 3\}. \tag{88}$$

In particular, at $N = 4$ inequality (86) holds for $n = 3$ and some

$$\mu_{00} > 3 \cdot \ln\left(\frac{4}{3}\right) > 0$$

in (86), and thus, owing to (52), time moment $t_3 = \frac{2}{3}$ is optimal in any 4×4 duel with $\mu \in (0;\; \mu_{00})$, where μ_{00} is the positive zero of function

$$f(\mu,\; 4,\; 3) = 2e^{\frac{2\mu}{3}} - e^{\mu} - 1,$$

being the positive root of equation (15). With using the bisection method for approximating this zero [28, 29], value μ_{00} satisfies inequality (16). At $\mu = \mu_{00}$ time moment $t_3 = \frac{2}{3}$ is optimal as well in the respective 4×4 duel, but then time moment $t_N = 1$ is also optimal: indeed,

$$f\left(\mu_{00},\ 4,\ 3\right) = 0 = u_{34} = u_{43} = u_{33} = u_{44}$$

and

$$u_{41} = \frac{e^{\mu_{00}} - e^{\mu_{00} \cdot 0}}{e^{\mu_{00}} - 1} - \frac{\left(e^{\mu_{00}} - 1\right)\left(e^{\mu_{00} \cdot 0} - 1\right)}{\left(e^{\mu_{00}} - 1\right)^2} = 1,$$

$$u_{42} = \frac{e^{\mu_{00}} - e^{\frac{\mu_{00}}{3}}}{e^{\mu_{00}} - 1} - \frac{\left(e^{\mu_{00}} - 1\right)\left(e^{\frac{\mu_{00}}{3}} - 1\right)}{\left(e^{\mu_{00}} - 1\right)^2} = \frac{e^{\mu_{00}} - 2e^{\frac{\mu_{00}}{3}} + 1}{e^{\mu_{00}} - 1} > 0$$

in this case. By the way, the positiveness of the last row of matrix (4), except for entries $u_{43} = 0$ and $u_{44} = 0$, follows from Theorem 1.

Specifically, time moment $t_n = \frac{n-1}{N-1}$ is optimal if inequalities (56) and (86) hold, i. e. if

$$\frac{N-1}{2} + 1 < n \leqslant \frac{3 + \sqrt{4N-3}}{2},$$

whence

$$\frac{N-1}{2} + 1 < \frac{3 + \sqrt{4N-3}}{2},$$
$$N - 2 < \sqrt{4N-3},$$
$$N^2 - 4N + 4 < 4N - 3,$$
$$N^2 - 8N + 7 < 0,$$
$$(N-7)(N-1) < 0.$$

The last inequality here is impossible, so time moment $t_n = \frac{n-1}{N-1}$ is not optimal by (56). This condition is immersed in condition (14). Finally, time moment $t_n = \frac{n-1}{N-1}$ is optimal if inequalities (59) and (86) hold, or equality (58) and inequality (86) hold, or inequality (59) and equality (87) hold, or equalities (58) and (87) hold by $\mu_0 \leqslant \mu_{00}$, where the optimality remains for the nonstrict inequalities as well owing to inequalities (33) and (34) can be nonstrict. These conditions are immersed in the assertion with equations (19) and (20) by inequality (85) as condition (18) for 7×7 duels and bigger, whereas in 5×5 and 6×6 duels optimal internal time moments can be determined by straightforwardly using condition (17). As (54) holds and

$$\frac{3 + \sqrt{4N-3}}{2} > 3 \ \text{ at } \ N \in \{5,\ 6\}$$

then $t_4 = \frac{3}{4}$ is the only one optimal internal time moment in the 5×5 duel by $\mu \in [\mu_0;\ \mu_{00}]$ and equations (21), (22), whose respective unique positive roots μ_0 and μ_{00} are approximated by using the bisection method [28, 29], and these values satisfy inequalities (23), (24). Time moment $t_4 = \frac{3}{5}$ is optimal in the 6×6 duel by $\mu \in [\mu_0;\ \mu_{00}]$ and equations (25), (26), whose respective unique positive roots μ_0 and μ_{00} are approximated by using the bisection method [28, 29], and these values satisfy inequalities (27), (28). Time moment $t_5 = \frac{4}{5}$ is optimal in the 6×6 duel by $\mu \in [\mu_0;\ \mu_{00}]$ and equations (29), (30), whose respective unique positive roots μ_0 and μ_{00} are approximated by using the bisection method [28, 29], and these values satisfy inequalities (31), (32). Inequalities (28) and (31) confirm that the 6×6 duel can have only one optimal internal time moment. □

Basically, Theorem 2 states that, in duels with seven time moments or more, internal time moment $t_n = \frac{n-1}{N-1}$ is optimal if it is in the second half of the duel span and the reward steepness factor lies between the unique positive roots of equations (19) and (20), respectively. The optimal internal time moments in the smaller duels are determined explicitly: 4×4 and 5×5 duels can have only one optimal internal time moment, which is the penultimate one, whereas the penultimate time moment and the preceding one can be optimal in the 6×6 duel, although the duel with six time moments can have only one internal time moment as well.

4. Discussion

The obtained results clarify how optimal timing behavior emerges in symmetric 1-bullet silent duels with generalized exponentially-convex rewards and how it fundamentally differs from duels with linear or standard convex reward structures. Theorem 1 establishes a monotonicity property of rewards with respect to the opponent's timing choice: for any fixed pure strategy of the first duelist, the reward strictly decreases when the second duelist shifts one's action either closer to or farther from that fixed time moment, except at coincidence. This property confirms that strategic interaction in the duel is governed not by local dominance or discontinuous jumps in incentives, but by a smooth and symmetric erosion of advantage as relative timing deviates. Such behavior reinforces the interpretation of the duel as a competition over relative readiness accumulation, rather than over discrete positional advantage.

Theorem 2 provides a more structurally informative result by characterizing when an internal time moment is optimal or non-optimal. A key implication is that optimality of internal time moments is entirely determined by the second half of the duel span. In particular, no internal optimal time moment can exist in the first half of the duel, regardless of the discretization level or the specific form of the generalized exponentially-convex reward, as long as it satisfies the imposed normalization and monotonicity properties in reward matrix (4) calculated as (11). This explains why no internal optimum exists in 3×3 duels and why the second time moment is never optimal in 4×4 duels. Even in bigger duels, passing the midpoint of the duel is a necessary but not sufficient condition for internal optimality, as additional restrictions on the reward steepness parameter must also be satisfied.

The dependence of internal optimality on the reward steepness factor μ highlights a qualitative transition in strategic behavior. For those values of μ that do not drop below the critical threshold given by the unique positive root of the corresponding algebraic equation (19), internal optimal moments may arise in the second half of the duel, reflecting environments where readiness accumulates gradually and timing precision is moderate. However, once μ exceeds the critical threshold given by the unique positive root of the corresponding algebraic equation (20), internal optimality disappears altogether, and no internal time moment can be optimal. In this regime, the duel effectively collapses to a boundary-dominated timing problem, where only extreme actions remain candidates for optimal behavior. This phenomenon illustrates how increasingly aggressive or sharply accelerating reward growth destabilizes interior timing optimality even in fully symmetric and silent settings.

An important methodological implication concerns the role of mixed strategies. From a purely game-theoretic perspective, the absence of optimal pure strategies implies the existence of an optimal solution in mixed strategies [7, 15, 5]. However, in practical systems modeled by silent duels, such as, e. g., competitive technology deployment, protocol actions, or time-sensitive information release, the use of mixed strategies is largely infeasible [3, 14, 25]. Implementing a mixed strategy requires stable environmental conditions, repeatability of interaction, and reliable randomization across multiple realizations [26, 13, 15, 11]. These assumptions are typically violated in the intended application domains, where actions are irreversible, environments evolve endogenously, and competitive encounters are effectively one-shot. Consequently, the practical relevance of the results lies precisely in identifying conditions under which pure-strategy timing equilibria (i. e., optimal time moments) exist, and in understanding when such equilibria necessarily fail to arise.

When compared with silent duels under linear and convex-but-not-exponential reward functions, the present findings reinforce a general and practically meaningful principle: no internal optimal timing can occur in the first half of the duel span. In models with linear and convex-but-not-exponential rewards, early internal optima may appear or timing incentives may remain diffuse, leading to unstable or highly sensitive equilibrium behavior

[15, 17, 19, 20]. By contrast, generalized exponentially-convex rewards impose a far more disciplined structure on readiness accumulation, ensuring that premature action is always suboptimal and that any candidate for optimal timing must lie strictly beyond the midpoint of the planning span. This feature aligns well with real competitive settings, where acting before sufficient technological maturity or informational completeness has been achieved is systematically dominated.

It is worth noting that, for the case of at least two optimal time moments, the model itself does not enforce a unique selection between them. From a practical standpoint, however, such selection would be naturally guided by risk dominance, robustness to timing perturbations, and minimal regret considerations, which typically favor later optimal moments because deviations or implementation noise tend to be less costly there under progressive reward growth [24]. So, if multiple optimal time moments exist, practical selection favors the latest optimal moment due to greater robustness, lower regret under timing errors, and focal salience in one-shot silent competition [7, 15, 10].

5. Conclusion

Optimality of internal time moments in 1-bullet silent duels with generalized exponentially-convex rewards, where exponentially-convex reward function (10) exists only without influence of the other duelist, is determined by the second half of the duel span. Thus, there is no optimal internal moment in 3×3 duels, and the second time moment is never optimal in 4×4 duels. Nevertheless, the third time moment in 4×4 duels is optimal only if the reward steepness factor μ does not exceed the unique positive root of algebraic equation (15), where the root estimation is (16).

Theorem 2 specifies the conditions of when an internal time moment is optimal in bigger duels, but the optimality is only possible if the first half of the duel span is passed, given by inequality (14). Obviously, inequality (85), being complementary to inequality (14), does not mean that an optimal internal time moment exists in the second half. In particular, if the reward steepness factor exceeds the unique positive root of equation (20), then inequality (88) holds and there are no optimal internal time moments in the duel.

As the duel starting moment $t_1 = 0$ is never optimal in silent duels [23, 16, 18, 20], the only non-internal time moment, whose optimality conditions are yet to be determined for generalized exponentially-convex rewards, is the duel final moment. In addition, possibilities of a few optimal time moments in a duel with a constant reward steepness factor are to be studied also. Moreover, it would be quite interesting to find more specific conditions of internal time moment optimality in the second half of the duel.

Acknowledgement

The work was technically supported by the Faculty of Mechanical and Electrical Engineering at the Polish Naval Academy, Gdynia, Poland.

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