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On a New Subclass of Bi-Univalent Functions Associated with Gregory Numbers

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Abstract The purpose of this article is to provide the inclusive subclass $\mathcal{Y}_{\Gamma}(\eth, \varsigma, \sigma, \varphi, m)$ of the class of bi-univalent functions that make use of Gregory numbers. At each of these subclasses of analytic functions, we investigate the Fekete-Szego functional, in addition to the estimations of the Taylor-Maclaurin coefficients, which are denoted by the symbols $|\varrho_2|$ and $|\varrho_3|$. It is possible that such a subclass will be the focus of research in the future due to the unprecedented nature of their characterisations and the proofs.

Keywords Analytic function, Univalent and bi-univalent functions, Gregory numbers, Fekete-Szegö problem.

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1. Introduction and preliminaries

Approximation theory and numerical analysis both make use of a collection of coefficients that are referred to as the Gregory coefficients. These coefficients were named after the mathematician James Gregory. According to the properties of a specific function, the Gregory coefficients are calculated from those characteristics.

The Gregory coefficients are an essential component of numerical approximation methods because they facilitate the development of polynomial or spline functions that accurately characterise the functions or data that are being input. Data analysis, computer graphics, engineering, and scientific computing are just few of the numerous fields in which they are vital.

The values $\frac{1}{2}, \frac{-1}{12}, \frac{1}{24}, \frac{-19}{720}, \frac{3}{160}, \cdots$, and so on are the Gregory coefficients, which are denoted by the symbol Υ_m. The expansion that follows this one contains them.

$$\frac{\ell}{\log(\ell+1)} = 1 + \frac{1}{2}\ell - \frac{1}{12}\ell^2 + \frac{1}{24}\ell^3 - \frac{19}{720}\ell^4 + \frac{3}{160}\ell^5 + \cdots .$$

The first time these numbers were proposed was in the year 1670 by James Gregory. Subsequently, these coefficients were subsequently studied by other mathematicians, and they may be found in the works of contemporary authors.

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Specifically, the generating function $G(\ell)$ of the Gregory coefficients, which can be found in (see [18, 6]), is given by

$$G(\ell)=\frac{\ell}{\log(\ell+1)}=\sum_{\varepsilon=0}^{\infty}\Upsilon_{\varepsilon}\ell^{\varepsilon}=1+\frac{1}{2}\ell-\frac{1}{12}\ell^{2}+\frac{1}{24}\ell^{3}-\frac{19}{720}\ell^{4}+\frac{3}{160}\ell^{5}+\cdots,\quad |\ell|<1. \tag{1.1}$$

It is evident that the initial values of $\Upsilon\varepsilon$, for $\varepsilon\in\mathbb{N}$, are

$$\Upsilon_0=1,\ \Upsilon_1=\frac{1}{2},\ \Upsilon_2=\frac{-1}{12},\ \Upsilon_3=\frac{1}{24},\ \Upsilon_4=\frac{-19}{720},\ \text{ and }\ \Upsilon_5=\frac{3}{160}.$$

Let Ω represent the subclass of analytic functions h in the open unit disk $\Delta=\{\ell\in\mathbb{C}:|\ell|<1\}$, where $h(0)=h'(0)-1=0$ under the normalization condition. As a result, every function h in the set Ω takes the following form:

$$h(\ell)=\ell+\sum_{\varepsilon=2}^{\infty}\varrho_{\varepsilon}\ell^{\varepsilon},\ \ (\ell\in\Delta). \tag{1.2}$$

Additionally, let us define the subclass of univalent functions within the subclass Ω, which may be found in the (see [9]) for further information. Therefore, every function h that belongs to the set $\mathcal{U}$ has an inverse h^{-1}, which is defined by

$$h^{-1}(h(\ell))=\ell \qquad (\ell\in\Delta)$$

and

$$h(h^{-1}(\xi))=\xi \qquad (|\xi|<r_0(h);r_0(h)\geq\frac{1}{4})$$

where

$$h^{-1}(\xi)=\xi-\varrho_2\xi^2+(2\varrho_2^2-\varrho_3)\xi^3-(5\varrho_2^3-5\varrho_2\varrho_3+\varrho_4)\xi^4+\cdots. \tag{1.3}$$

Currently, if for every $\ell\in\Delta$ there exists a function Ψ with $\Psi(0)=0$ and $|\Psi(\ell)|<1$, then $h_1\prec h_2$ or $h_1(\ell)\prec h_2(\ell)$ (the subordination of analytic functions h_1 and h_2) such that:

$$h_1(\ell)=h_2(\Psi(\ell)).$$

Also, if h_2 is univalent in Δ, then the equivalence relation that follows is the one that we have (for more information, see [13]).

$$h_1(0)=h_2(0) \text{ and } h_1(\Delta)\subset h_2(\Delta)\Leftrightarrow h_1(\ell)\prec h_2(\ell).$$

If both $h(\ell)$ and $h^{-1}(\ell)$ are univalent in Δ, then the function h, which is provided by the equation (1.2), will be classified as belonging to the subclass Γ, which is the subclass of bi-univalent functions in Δ. The subclass Γ is described in more depth in references [1, 14, 17, 19, 21, 22].

After inspecting the subclass Γ, Lewin [12] discovered that the value of ϱ_2 is less than 1.51. There is a citation for Netanyahu. proved that the maximum value of ϱ_2 is equal to four-thirds, while Brannan and Clunie [7] hypothesised that the value of ϱ_2 is less than $\sqrt{2}$. Currently, the estimation of the coefficient ϱ_ε of $\varepsilon\geq 3,\varepsilon\in\mathbb{N}$, remains an unresolved issue. Tan [20] estimate is widely recognized for functions in the subclass Γ, and it is $|\varrho_2|<1.485$.

The inequality between Fekete and Szegö is one of the topics that is widely known. The first person to make this was [10], if $h\in\Gamma$.

$$|\varrho_3-s\varrho_2^2|\leq 1+2e^{-2s/(1-s)},\ \ s\in\mathbb{R},$$

this bound is sharp.

Motivated, it is essential to highlight the fact that, as far as we are aware, very scant information is available concerning Gregory polynomials in the setting of bi-univalent functions.

Furthermore, nowadays, a multitude of academics are focusing their attention on bi-univalent functions that are associated with orthogonal polynomials, [2, 3, 4, 5, 11, 15, 16, 23, 24, 25].

The purpose of this study is to establish a new subclass of Γ that includes Gregory coefficients, denoted $\mathcal{Y}_{\Gamma}(\eth,\varsigma,\sigma,\varphi,m)$. Furthermore, we estimate the upper bounds for the coefficients $|\varrho_2|$, $|\varrho_3|$ and $|\varrho_3-\digamma\varrho_2^2|$ for these subclasses. Furthermore, several previously unknown results are proven. Within the new class.

2. Coefficient bounds of the subclass $\mathcal{Y}_{\Gamma}(\eth, \varsigma, \sigma, \varphi, m)$

The first part of this section is dedicated to providing definitions for the newly introduced comprehensive subclass $\mathcal{Y}_{\Gamma}(\eth, \varsigma, \sigma, \varphi, m)$ that is concerning Gregory coefficients.

Definition 2.1. Assume that $m \geq 0, -\pi < \varphi \leq \pi, \eth \geq 1, \varsigma, \sigma \geq 0, \ell, \xi \in \mathbb{C}$, and that the function $G(\ell)$ is given by (1.1). If the following subordinations are met, a function $h \in \Gamma$ provided by (1.2) is considered to belong to the subclass $\mathcal{Y}_{\Gamma}(\eth, \varsigma, \sigma, \varphi, m)$:

$$\left(1+me^{i\varphi}\right)\left\{(1-\eth)\left(\frac{h(\ell)}{\ell}\right)^{\varsigma}+\eth\left(h'(\ell)\right)^{1-\varsigma}+\sigma\ell h''(\ell)\right\}-me^{i\varphi}\prec G(\ell) \tag{2.1}$$

and

$$\left(1+me^{i\varphi}\right)\left\{(1-\eth)\left(\frac{d(\xi)}{\xi}\right)^{\varsigma}+\eth\left(d'(\xi)\right)^{1-\varsigma}+\sigma\xi d''(\xi)\right\}-me^{i\varphi}\prec G(\xi), \tag{2.2}$$

where $d(\xi) = h^{-1}(\xi)$ is given by (1.3).

Special cases:

i) When $\eth = 1$, we obtain $\mathcal{Y}_{\Gamma}(\eth, \varsigma, \sigma, \varphi, m)$. Here, $\mathcal{Y}_{\Gamma}(1, \varsigma, \sigma, \varphi, m)$ is the set of functions $h \in \Gamma$ that satisfy the following criteria and are provided by (1.2).

$$\left(1+me^{i\varphi}\right)\left\{\left(h'(\ell)\right)^{1-\varsigma}+\sigma\ell h''(\ell)\right\}-me^{i\varphi}\prec G(\ell)$$

and

$$\left(1+me^{i\varphi}\right)\left\{\left(d'(\xi)\right)^{1-\varsigma}+\sigma\xi d''(\xi)\right\}-me^{i\varphi}\prec G(\xi),$$

where $d(\xi) = h^{-1}(\xi)$ is given by (1.3).

ii) When $\eth = 1$ and $\sigma = 0$ we obtain $\mathcal{Y}_{\Gamma}(\eth, \varsigma, \sigma, \varphi, m)$. Here, $\mathcal{Y}_{\Gamma}(1, \varsigma, 0, \varphi, m)$ is the set of functions $h \in \Gamma$ that satisfy the following criteria and are provided by (1.2).

$$\left(1+me^{i\varphi}\right)\left\{\left(h'(\ell)\right)^{1-\varsigma}\right\}-me^{i\varphi}\prec G(\ell)$$

and

$$\left(1+me^{i\varphi}\right)\left\{\left(d'(\xi)\right)^{1-\varsigma}\right\}-me^{i\varphi}\prec G(\xi),$$

where $d(\xi) = h^{-1}(\xi)$ is given by (1.3).

iii) When $\eth = 1$ and $\sigma = \varsigma = 0$ we obtain $\mathcal{Y}_{\Gamma}(\eth, \varsigma, \sigma, \varphi, m)$. Here, $\mathcal{Y}_{\Gamma}(1, 0, 0, \varphi, m)$ is the set of functions $h \in \Gamma$ that satisfy the following criteria and are provided by (1.2).

$$\left(1+me^{i\varphi}\right)\left\{h'(\ell)\right\}-me^{i\varphi}\prec G(\ell)$$

and

$$\left(1+me^{i\varphi}\right)\left\{d'(\xi)\right\}-me^{i\varphi}\prec G(\xi),$$

where $d(\xi) = h^{-1}(\xi)$ is given by (1.3).

iv) When $\sigma = 0$, we obtain $\mathcal{Y}_{\Gamma}(\eth, \varsigma, \sigma, \varphi, m)$. Here, $\mathcal{Y}_{\Gamma}(\eth, \varsigma, 0, \varphi, m)$ is the set of functions $h \in \Gamma$ that satisfy the following criteria and are provided by (1.2).

$$\left(1+me^{i\varphi}\right)\left\{(1-\eth)\left(\frac{h(\ell)}{\ell}\right)^{\varsigma}+\eth\left(h'(\ell)\right)^{1-\varsigma}\right\}-me^{i\varphi}\prec G(\ell)$$

and

$$\left(1+me^{i\varphi}\right)\left\{(1-\eth)\left(\frac{d(\xi)}{\xi}\right)^{\varsigma}+\eth\left(d'(\xi)\right)^{1-\varsigma}\right\}-me^{i\varphi}\prec G(\xi),$$

where $d(\xi) = h^{-1}(\xi)$ is given by (1.3).

v) When $\sigma = 0$ and $\varsigma = 1$ we obtain $\mathcal{Y}_{\Gamma}(\eth, \varsigma, \sigma, \varphi, m)$. Here, $\mathcal{Y}_{\Gamma}(\eth, 1, 0, \varphi, m)$ is the set of functions $h \in \Gamma$ that satisfy the following criteria and are provided by (1.2).

$$\left(1 + me^{i\varphi}\right)\left\{(1-\eth)\left(\frac{h(\ell)}{\ell}\right) + \eth\right\} - me^{i\varphi} \prec G(\ell)$$

and

$$\left(1 + me^{i\varphi}\right)\left\{(1-\eth)\left(\frac{d(\xi)}{\xi}\right) + \eth\right\} - me^{i\varphi} \prec G(\xi),$$

where $d(\xi) = h^{-1}(\xi)$ is given by (1.3).

vi) When $\sigma = 0$ and $\varsigma = 0$ we obtain $\mathcal{Y}_{\Gamma}(\eth, \varsigma, \sigma, \varphi, m)$. Here, $\mathcal{Y}_{\Gamma}(\eth, 0, 0, \varphi, m)$ is the set of functions $h \in \Gamma$ that satisfy the following criteria and are provided by (1.2).

$$\left(1 + me^{i\varphi}\right)\{(1-\eth) + \eth\left(h'(\ell)\right)\} - me^{i\varphi} \prec G(\ell)$$

and

$$\left(1 + me^{i\varphi}\right)\{(1-\eth) + \eth\left(d'(\xi)\right)\} - me^{i\varphi} \prec G(\xi),$$

where $d(\xi) = h^{-1}(\xi)$ is given by (1.3).

vii) When $\varsigma = 0$, we obtain $\mathcal{Y}_{\Gamma}(\eth, \varsigma, \sigma, \varphi, m)$. Here, $\mathcal{Y}_{\Gamma}(\eth, 0, \sigma, \varphi, m)$ is the set of functions $h \in \Gamma$ that satisfy the following criteria and are provided by (1.2).

$$\left(1 + me^{i\varphi}\right)\{(1-\eth) + \eth\left(h'(\ell)\right) + \sigma\ell h''(\ell)\} - me^{i\varphi} \prec G(\ell)$$

and

$$\left(1 + me^{i\varphi}\right)\{(1-\eth) + \eth\left(d'(\xi)\right) + \sigma\xi d''(\xi)\} - me^{i\varphi} \prec G(\xi),$$

where $d(\xi) = h^{-1}(\xi)$ is given by (1.3).

viii) When $m = 0$, we obtain $\mathcal{Y}_{\Gamma}(\eth, \varsigma, \sigma, \varphi, m)$. Here, $\mathcal{Y}_{\Gamma}(\eth, \varsigma, \sigma, \varphi, 0)$ is the set of functions $h \in \Gamma$ that satisfy the following criteria and are provided by (1.2).

$$(1-\eth)\left(\frac{h(\ell)}{\ell}\right)^{\varsigma} + \eth\left(h'(\ell)\right)^{1-\varsigma} + \sigma\ell h''(\ell) \prec G(\ell)$$

and

$$(1-\eth)\left(\frac{d(\xi)}{\xi}\right)^{\varsigma} + \eth\left(d'(\xi)\right)^{1-\varsigma} + \sigma\xi d''(\xi) \prec G(\xi),$$

where $d(\xi) = h^{-1}(\xi)$ is given by (1.3).

Lemma 2.2
([8]) Let the analytic function $\psi(\ell) = 1 + s_1\ell + s_2\ell^2 + \cdots$ with positive real parts in Δ, then $|\varrho_\ell| \leq 2$, for $\ell \geq 1$.

Lemma 2.3
([24]) Let $g_1, g_2 \in \mathbb{R}$ and $\chi_1, \chi_2 \in \mathbb{C}$. If $|\chi_1| < \hbar$ and $|\chi_2| < \hbar$, then

$$|(g_1+g_2)\chi_1 + (g_1-g_2)\chi_2| \leq \begin{cases} 2\,|g_1|\,\hbar \ \text{ for } \ |g_1| \geq |g_2|\,, \\ 2\,|g_2|\,\hbar \ \text{ for } \ |g_1| \leq |g_2|\,. \end{cases}$$

Theorem 2.4
Let $h \in \Gamma$ given by (1.2) belongs to the subclass $\mathcal{Y}_\Gamma(\eth, \varsigma, \sigma, \varphi, m)$. Then

$$|\varrho_2| \le \min\left\{ \frac{1}{2\left|(1+me^{i\varphi})\left(\varsigma(1-3\eth)+2(\sigma+\eth)\right)\right|}, \frac{1}{\sqrt{\left| \begin{array}{c} \left(1+me^{i\varphi}\right)\left(\varsigma^2(3\eth+1)+\varsigma(1-11\eth)+6(2\sigma+\eth)\right)+ \\ \frac{14}{3}\left[\left(1+me^{i\varphi}\right)\left(\varsigma(1-3\eth)+2(\sigma+\eth)\right)\right]^2 \end{array} \right|}} \right\},$$

$$|\varrho_3| \le \min\left\{ \frac{1}{4\left|(1+me^{i\varphi})\left(\varsigma(1-3\eth)+2(\sigma+\eth)\right)\right|^2} + \frac{1}{\left|(2\varsigma(1-4\eth)+6(2\sigma+\eth))(1+me^{i\varphi})\right|}, \begin{array}{c} \frac{1}{\left|(\varsigma^2(3\eth+1)+\varsigma(1-11\eth)+6(2\sigma+\eth))(1+me^{i\varphi})+\frac{14}{3}[(\varsigma(1-3\eth)+2(\sigma+\eth))(1+me^{i\varphi})]^2\right|} \\ +\frac{1}{|(2\varsigma(1-4\eth)+6(2\sigma+\eth))(1+me^{i\varphi})|} \end{array} \right\}$$

and

$$\left|\varrho_3 - F\varrho_2^2\right| \le \begin{cases} \frac{1}{2|(\varsigma(1-4\eth)+3(2\sigma+\eth))(1+me^{i\varphi})|} & |\Theta(F)| < \frac{1}{8|(\varsigma(1-4\eth)+3(2\sigma+\eth))(1+me^{i\varphi})|}, \\ 4|\Theta(F)| & |\Theta(F)| \ge \frac{1}{8|(\varsigma(1-4\eth)+3(2\sigma+\eth))(1+me^{i\varphi})|}. \end{cases}$$

where

$$\Theta(F) = \frac{1-F}{4\left[\begin{array}{c} \left(\varsigma^2(3\eth+1)+\varsigma(1-11\eth)+6(2\sigma+\eth)\right)\left(1+me^{i\varphi}\right)+ \\ \frac{14}{3}\left[\left(\varsigma(1-3\eth)+2(\sigma+\eth)\right)\left(1+me^{i\varphi}\right)\right]^2 \end{array}\right]}.$$

Proof
Let h belong to the $\mathcal{Y}_\Gamma(\eth, \varsigma, \sigma, \varphi, m)$. The subordinations (2.1) and (2.2) allow for the existence of two analytic functions r and t. These functions are such that $r(0)$ equals $t(0)$ equals zero and $|r(\ell)|$ is less than 1, and $|t(\xi)|$ is less than 1.

$$\left(1+me^{i\varphi}\right)\left\{(1-\eth)\left(\frac{h(\ell)}{\ell}\right)^{\varsigma} + \eth\left(h'(\ell)\right)^{1-\varsigma} + \sigma\ell h''(\ell)\right\} - me^{i\varphi} = G(r(\ell)), \quad \ell \in \Delta \tag{2.3}$$

and

$$\left(1+me^{i\varphi}\right)\left\{(1-\eth)\left(\frac{d(\xi)}{\xi}\right)^{\varsigma} + \eth\left(d'(\xi)\right)^{1-\varsigma} + \sigma\xi d''(\xi)\right\} - me^{i\varphi} = G(t(\xi)), \quad \xi \in \Delta. \tag{2.4}$$

So, the function

$$\beta(\ell) = \frac{r(\ell)+1}{1-r(\ell)} = 1 + k_1\ell + k_2\ell^2 + \cdots,$$

hence,

$$r(\ell) = \frac{k_1}{2}\ell + \frac{1}{2}\left(k_2 - \frac{k_1^2}{2}\right)\ell^2 + \frac{1}{2}\left(k_3 - k_1k_2 + \frac{k_1^3}{4}\right)\ell^3 + \cdots$$

and

$$G(r(\ell)) = 1 + \frac{k_1}{4}\ell + \frac{1}{48}\left(12k_2 - 7k_1^2\right)\ell^2 + \frac{1}{192}\left(17k_1^3 - 56k_1k_2 + 48l_3\right)\ell^3 + \cdots, \ \ell \in \Delta. \tag{2.5}$$

Also, the function

$$\delta(\xi) = \frac{t(\xi)+1}{1-t(\xi)} = 1 + l_1\xi + l_2\xi^2 + \cdots,$$

hence,

$$t(\xi)=\frac{l_1}{2}\xi+\frac{1}{2}\left(l_2-\frac{l_1^2}{2}\right)\xi^2+\frac{1}{2}\left(l_3-l_1l_2+\frac{l_1^3}{4}\right)\xi^3+\cdots$$

and

$$G(t(\xi))=1+\frac{l_1}{4}\xi+\frac{1}{48}\left(12l_2-7l_1^2\right)\xi^2+\frac{1}{192}\left(17l_1^3-56l_1l_2+48l_3\right)\xi^3+\cdots,\ \xi\in\Omega. \tag{2.6}$$

Thus we have

$$\begin{aligned}&\left(1+me^{i\varphi}\right)\left\{(1-\eth)\left(\frac{h(\ell)}{\ell}\right)^{\varsigma}+\eth\left(h'(\ell)\right)^{1-\varsigma}+\sigma\ell h''(\ell)\right\}-me^{i\varphi}\\&=1+\frac{k_1}{4}\ell+\frac{1}{48}\left(12k_2-7k_1^2\right)\ell^2+\frac{1}{192}\left(17k_1^3-56k_1k_2+48l_3\right)\ell^3+\cdots,\ \ell\in\Delta.\end{aligned} \tag{2.7}$$

and

$$\begin{aligned}&\left(1+me^{i\varphi}\right)\left\{(1-\eth)\left(\frac{d(\xi)}{\xi}\right)^{\varsigma}+\eth\left(d'(\xi)\right)^{1-\varsigma}+\sigma\xi d''(\xi)\right\}-me^{i\varphi}\\&=1+\frac{l_1}{4}\xi+\frac{1}{48}\left(12l_2-7l_1^2\right)\xi^2+\frac{1}{192}\left(17l_1^3-56l_1l_2+48l_3\right)\xi^3+\cdots,\ \xi\in\Omega.\end{aligned} \tag{2.8}$$

As a result of comparing the coefficients in (2.7) and (2.8), we are able to obtain the following:

$$\left(1+me^{i\varphi}\right)\left[\varsigma(1-3\eth)+2\left(\sigma+\eth\right)\right]\varrho_2=\frac{k_1}{4}, \tag{2.9}$$

$$\left(1+me^{i\varphi}\right)\left[\frac{\varsigma\left(\varsigma-1\right)\left(3\eth+1\right)}{2}\right]\varrho_2^2+\left(1+me^{i\varphi}\right)\left[\varsigma(1-4\eth)+3\left(2\sigma+\eth\right)\right]\varrho_3=\frac{1}{48}\left(12k_2-7k_1^2\right), \tag{2.10}$$

$$-\left(1+me^{i\varphi}\right)\left[\varsigma(1-3\eth)+2\left(\sigma+\eth\right)\right]\varrho_2=\frac{l_1}{4}, \tag{2.11}$$

and

$$\begin{aligned}&\left(1+me^{i\varphi}\right)\left[\frac{\varsigma^2\left(3\eth+1\right)+\varsigma(3-19\eth)+12\left(2\sigma+\eth\right)}{2}\right]\varrho_2^2\\&\qquad-\left(1+me^{i\varphi}\right)\left[\varsigma(1-4\eth)+3\left(2\sigma+\eth\right)\right]\varrho_3=\frac{1}{48}\left(12l_2-7l_1^2\right).\end{aligned} \tag{2.12}$$

From (2.9) and (2.11) it follows that

$$k_1=-l_1 \tag{2.13}$$

and

$$32\left[\left(1+me^{i\varphi}\right)\left(\varsigma(1-3\eth)+2\left(\sigma+\eth\right)\right)\right]^2\varrho_2^2=k_1^2+l_1^2. \tag{2.14}$$

If we add (2.10) to (2.12), we have

$$\left(1+me^{i\varphi}\right)\left[\varsigma^2\left(3\eth+1\right)+\varsigma(1-11\eth)+6\left(2\sigma+\eth\right)\right]\varrho_2^2=\frac{1}{4}\left(k_2+l_2\right)-\frac{7}{48}\left(k_1^2+l_1^2\right). \tag{2.15}$$

Taking the value of $k_1^2+l_1^2$ from (2.14) and substituting it in (2.15), we obtain the following:

$$\left[\begin{array}{c}\left(1+me^{i\varphi}\right)\left(\varsigma^2\left(3\eth+1\right)+\varsigma(1-11\eth)+6\left(2\sigma+\eth\right)\right)+\\ \frac{14}{3}\left[\left(\varsigma(1-3\eth)+2\left(\sigma+\eth\right)\right)\left(1+me^{i\varphi}\right)\right]^2\end{array}\right]\varrho_2^2=\frac{1}{4}\left(k_2+l_2\right). \tag{2.16}$$

When we apply the triangle inequality and Lemma 2.2 to the conditions of the relations (2.9) and (2.16), we obtain the following results, respectively:

$$|\varrho_2| \leq \frac{1}{2\left|\left(\varsigma(1-3\eth)+2\left(\sigma+\eth\right)\right)\left(1+me^{i\varphi}\right)\right|}$$

and

$$|\varrho_2| \leq \frac{1}{\sqrt{\left|\begin{array}{c}\left(\varsigma^2\left(3\eth+1\right)+\varsigma(1-11\eth)+6\left(2\sigma+\eth\right)\right)\left(1+me^{i\varphi}\right)+\\ \frac{14}{3}\left[\left(\varsigma(1-3\eth)+2\left(\sigma+\eth\right)\right)\left(1+me^{i\varphi}\right)\right]^2\end{array}\right|}}.$$

In addition, if we take the (2.12) and subtract it from the (2.10), we get the following:

$$\left(1+me^{i\varphi}\right)\left[2\varsigma(1-4\eth)+6\left(2\sigma+\eth\right)\right]\left(\varrho_3-\varrho_2^2\right)=\frac{1}{4}\left(k_2-l_2\right)-\frac{7}{48}\left(k_1^2-l_1^2\right). \tag{2.17}$$

Then, in view of (2.13), last equation becomes

$$\varrho_3=\varrho_2^2+\frac{k_2-l_2}{8\left[\varsigma(1-4\eth)+3\left(2\sigma+\eth\right)\right]\left(1+me^{i\varphi}\right)}. \tag{2.18}$$

The above equation with (2.9) becomes

$$\varrho_3=\frac{k_1^2}{16\left[\left(\varsigma(1-3\eth)+2\left(\sigma+\eth\right)\right)\left(1+me^{i\varphi}\right)\right]^2}+\frac{k_2-l_2}{8\left[\varsigma(1-4\eth)+3\left(2\sigma+\eth\right)\right]\left(1+me^{i\varphi}\right)}$$

and using (2.16) in (2.18)

$$\varrho_3=\frac{k_2+l_2}{4\left[\begin{array}{c}\left(\varsigma^2\left(3\eth+1\right)+\varsigma(1-11\eth)+6\left(2\sigma+\eth\right)\right)\left(1+me^{i\varphi}\right)+\\ \frac{14}{3}\left[\left(\varsigma(1-3\eth)+2\left(\sigma+\eth\right)\right)\left(1+me^{i\varphi}\right)\right]^2\end{array}\right]}+\frac{k_2-l_2}{4\left[2\varsigma(1-4\eth)+6\left(2\sigma+\eth\right)\right]\left(1+me^{i\varphi}\right)}.$$

With the use of the triangle inequality and Lemma 2.2, we can obtain the following results for the final two equations:

$$|\varrho_3| \leq \frac{1}{4\left|\left(\varsigma(1-3\eth)+2\left(\sigma+\eth\right)\right)\left(1+me^{i\varphi}\right)\right|^2}+\frac{1}{2\left|\left(\varsigma(1-4\eth)+3\left(2\sigma+\eth\right)\right)\left(1+me^{i\varphi}\right)\right|}$$

and

$$|\varrho_3| \leq \frac{1}{\left|\begin{array}{c}\left(\varsigma^2\left(3\eth+1\right)+\varsigma(1-11\eth)+6\left(2\sigma+\eth\right)\right)\left(1+me^{i\varphi}\right)+\\ \frac{14}{3}\left[\left(\varsigma(1-3\eth)+2\left(\sigma+\eth\right)\right)\left(1+me^{i\varphi}\right)\right]^2\end{array}\right|}+\frac{1}{2\left|\left(\varsigma(1-4\eth)+3\left(2\sigma+\eth\right)\right)\left(1+me^{i\varphi}\right)\right|}.$$

Also, from (2.18) we have

$$\begin{aligned}
\varrho_3 - \digamma\varrho_2^2 &= \frac{k_2 - l_2}{8\left[\varsigma(1-4\Game) + 3\left(2\sigma + \Game\right)\right]\left(1 + me^{i\varphi}\right)} + (1-\digamma)\varrho_2^2 \\
&= \frac{k_2 - l_2}{8\left[\varsigma(1-4\Game) + 3\left(2\sigma + \Game\right)\right]\left(1 + me^{i\varphi}\right)} \\
&+ \frac{(1-\digamma)\left(k_2 + l_2\right)}{4\left[\begin{array}{c}\left(\varsigma^2\left(3\Game + 1\right) + \varsigma(1-11\Game) + 6\left(2\sigma + \Game\right)\right)\left(1 + me^{i\varphi}\right) + \\ \frac{14}{3}\left[\left(\varsigma(1-3\Game) + 2\left(\sigma + \Game\right)\right)\left(1 + me^{i\varphi}\right)\right]^2\end{array}\right]} \\
&= \left(\Theta(\digamma) + \frac{1}{8\left[\varsigma(1-4\Game) + 3\left(2\sigma + \Game\right)\right]\left(1 + me^{i\varphi}\right)}\right)k_2 + \\
&\left(\Theta(\digamma) - \frac{1}{8\left[\varsigma(1-4\Game) + 3\left(2\sigma + \Game\right)\right]\left(1 + me^{i\varphi}\right)}\right)l_2,
\end{aligned}$$

where

$$\Theta(\digamma) = \frac{1-\digamma}{4\left[\begin{array}{c}\left(\varsigma^2\left(3\Game + 1\right) + \varsigma(1-11\Game) + 6\left(2\sigma + \Game\right)\right)\left(1 + me^{i\varphi}\right) + \\ \frac{14}{3}\left[\left(\varsigma(1-3\Game) + 2\left(\sigma + \Game\right)\right)\left(1 + me^{i\varphi}\right)\right]^2\end{array}\right]}.$$

Then, in view Lemma 2.2 for $|k_2|$ and $|l_2|$, and Lemma 2.3, we obtain

$$\left|\varrho_3 - \digamma\varrho_2^2\right| \leq \begin{cases} \frac{1}{2\left|\left(\varsigma(1-4\Game)+3(2\sigma+\Game)\right)\left(1+me^{i\varphi}\right)\right|} & \left|\Theta(\digamma)\right| < \frac{1}{8\left|\left(\varsigma(1-4\Game)+3(2\sigma+\Game)\right)\left(1+me^{i\varphi}\right)\right|}, \\ 4\left|\Theta(\digamma)\right| & \left|\Theta(\digamma)\right| \geq \frac{1}{8\left|\left(\varsigma(1-4\Game)+3(2\sigma+\Game)\right)\left(1+me^{i\varphi}\right)\right|}. \end{cases}$$

Which completes the proof. □

3. Corollaries

If we set $\Game$ equal to one in Theorems 2.4, we will obtain the subsequent corollary.

Corollary 3.1
If $h \in \mathcal{Y}_{\Gamma}(1, \varsigma, \sigma, \varphi, m)$, then

$$|\varrho_2| \leq \min\left\{\frac{1}{4\left|(\sigma - \varsigma + 1)\left(1 + me^{i\varphi}\right)\right|}, \frac{1}{\sqrt{\left|\begin{array}{c}\left(4\varsigma^2 - 10\varsigma + 6(2\sigma+1)\right)\left(1 + me^{i\varphi}\right) + \\ \frac{56}{3}\left((\sigma - \varsigma + 1)\left(1 + me^{i\varphi}\right)\right)^2\end{array}\right|}}\right\},$$

$$|\varrho_3| \leq \min\left\{\begin{array}{c}\frac{1}{16\left|(\sigma - \varsigma + 1)\left(1 + me^{i\varphi}\right)\right|^2} + \frac{1}{6\left|(2\sigma - \varsigma + 1)\left(1 + me^{i\varphi}\right)\right|}, \\ \frac{1}{\left|\left(4\varsigma^2 - 10\varsigma + 6(2\sigma+1)\right)\left(1+me^{i\varphi}\right) + \frac{56}{3}\left((\sigma-\varsigma+1)\left(1+me^{i\varphi}\right)\right)^2\right|} \\ + \frac{1}{6\left|(2\sigma-\varsigma+1)\left(1+me^{i\varphi}\right)\right|}\end{array}\right\}$$

and

$$\left|\varrho_3 - \digamma\varrho_2^2\right| \leq \begin{cases} \frac{1}{6\left|(2\sigma-\varsigma+1)\left(1+me^{i\varphi}\right)\right|} & \left|\Theta(\digamma)\right| < \frac{1}{24\left|(2\sigma-\varsigma+1)\left(1+me^{i\varphi}\right)\right|}, \\ 4\left|\Theta(\digamma)\right| & \left|\Theta(\digamma)\right| \geq \frac{1}{24\left|(2\sigma-\varsigma+1)\left(1+me^{i\varphi}\right)\right|}. \end{cases}$$

where

$$\Theta(F) = \frac{(1-F)}{4\left[(4\varsigma^2 - 10\varsigma + 6(2\sigma+1))(1+me^{i\varphi}) + \frac{56}{3}\left((\sigma-\varsigma+1)(1+me^{i\varphi})\right)^2\right]}.$$

If we set ς equal to zero in Theorems 2.4, we will obtain the subsequent corollary.

Corollary 3.2
If $h \in \mathcal{Y}_\Gamma(\eth, 0, \sigma, \varphi, m)$, then

$$|\varrho_2| \leq \min\left\{\frac{1}{4\left|(\sigma+\eth)\left(1+me^{i\varphi}\right)\right|}, \frac{1}{\sqrt{\left|(6(2\sigma+\eth))\left(1+me^{i\varphi}\right)+\frac{56}{3}\left((\sigma+\eth)\left(1+me^{i\varphi}\right)\right)^2\right|}}\right\},$$

$$|\varrho_3| \leq \min\left\{\begin{array}{c}\frac{1}{16|(\sigma+\eth)(1+me^{i\varphi})|^2} + \frac{1}{6|(2\sigma+\eth)(1+me^{i\varphi})|}, \\ \frac{1}{\left|6((2\sigma+\eth)(1+me^{i\varphi}))+\frac{56}{3}((\sigma+\eth)(1+me^{i\varphi}))^2\right|} + \frac{1}{6|(2\sigma+\eth)(1+me^{i\varphi})|}\end{array}\right\}$$

and

$$\left|\varrho_3 - F\varrho_2^2\right| \leq \begin{cases} \frac{1}{6|(2\sigma+\eth)(1+me^{i\varphi})|} & |\Theta(F)| < \frac{1}{24|(2\sigma+\eth)(1+me^{i\varphi})|}, \\ 4\left|\Theta(F)\right| & |\Theta(F)| \geq \frac{1}{24|(2\sigma+\eth)(1+me^{i\varphi})|}. \end{cases}$$

where

$$\Theta(F) = \frac{3(1-F)}{4\left(6(2\sigma+\eth)\left(1+me^{i\varphi}\right)+\frac{56}{3}\left((\sigma+\eth)\left(1+me^{i\varphi}\right)\right)^2\right)}.$$

For $\varsigma = 0$ in Corollary 3.1 or $\eth = 1$ in Corollary 3.2 simplifies to the following Corollary.

Corollary 3.3
If $h \in \mathcal{Y}_\Gamma(1, 0, \sigma, \varphi, m)$, then

$$|\varrho_2| \leq \min\left\{\frac{1}{4\left|(\sigma+1)\left(1+me^{i\varphi}\right)\right|}, \frac{1}{\sqrt{\left|6(2\sigma+1)\left(1+me^{i\varphi}\right)+\frac{56}{3}\left((\sigma+1)\left(1+me^{i\varphi}\right)\right)^2\right|}}\right\},$$

$$|\varrho_3| \leq \min\left\{\frac{1}{16\left|(\sigma+1)\left(1+me^{i\varphi}\right)\right|^2} + \frac{1}{6\left|(2\sigma+1)\left(1+me^{i\varphi}\right)\right|},\right.$$
$$\left.\frac{3}{\left|6(2\sigma+1)\left(1+me^{i\varphi}\right)+\frac{56}{3}\left((\sigma+1)\left(1+me^{i\varphi}\right)\right)^2\right|} + \frac{1}{6\left|(2\sigma+1)\left(1+me^{i\varphi}\right)\right|}\right\}$$

and

$$\left|\varrho_3 - F\varrho_2^2\right| \leq \begin{cases} \frac{1}{6|(2\sigma+1)(1+me^{i\varphi})|} & |\Theta(F)| < \frac{1}{24|(2\sigma+1)(1+me^{i\varphi})|}, \\ 4\left|\Theta(F)\right| & |\Theta(F)| \geq \frac{1}{24|(2\sigma+1)(1+me^{i\varphi})|}. \end{cases}$$

where

$$\Theta(F) = \frac{1-F}{4\left(6(2\sigma+1)\left(1+me^{i\varphi}\right)+\frac{56}{3}\left((\sigma+1)\left(1+me^{i\varphi}\right)\right)^2\right)}.$$

4. Conclusions

This paper opens several directions for future research. One natural extension is to investigate subclasses of bi-univalent functions associated with specific conic domains such as three-leaf domains and Ozaki-type bi-close-to-convex regions, since these domains have proven effective in refining coefficient estimates and geometric properties in recent studies of univalent and bi-univalent functions.

Another promising direction concerns the incorporation of fractional calculus. In particular, studying bi-concave and related function classes involving a modified Caputo fractional operator may introduce additional flexibility through memory and nonlocal effects, while posing new analytical challenges in establishing subordination conditions and coefficient bounds.

These extensions raise several open questions, including whether sharp bounds analogous to those obtained in the present work can be derived in these more general geometric and fractional settings, and how the Gregory-based framework interacts with such operators. We expect that the current results will serve as a foundational step toward addressing these problems.

REFERENCES

1. M. Ahmed, A. Alsoboh, A. Amourah, and J. Salah, On the fractional q-differintegral operator for subclasses of bi-univalent functions subordinate to q-ultraspherical polynomials, *Eur. J. Pure Appl. Math.*, 18(3) (2025), 6586. https://doi.org/10.29020/nybg.ejpam.v18i3.6586
2. O. Alnajar, A. Amourah, J. Salah, M. Darus, Fekete–Szegö functional problem for analytic and bi-univalent functions subordinate to Gegenbauer polynomials, *Contemp. Math.*, (2024), 5731–5742.
3. O. Alnajar, M. Darus, Coefficient estimates for subclasses of bi-univalent functions related to Gegenbauer polynomials and an application of Bell distribution, *AIP Conf. Proc.*, **3150** (2024), 020005. https://doi.org/10.1063/5.0203535
4. O. Alnajar, K. Alshammari, A. Amourah, The neutrosophic Poisson distribution applied to Horadam polynomial-subordinate bi-univalent functions, *Eur. J. Pure Appl. Math.*, **18** (2025), 5955–5955. https://doi.org/10.29020/nybg.ejpam.v18i2.5220
5. A. Amourah, B. A. Frasin, M. Ahmad, F. Yousef, Exploiting the Pascal distribution series and Gegenbauer polynomials to construct and study a new subclass of analytic bi-univalent functions, *Symmetry*, **14** (2022), 147. https://doi.org/10.3390/sym14010147
6. I. S. Berezin, N. P. Zhidkov, *Computing Methods*, Pergamon Press, Oxford, 1965.
7. D. A. Brannan, J. G. Clunie, Aspects of contemporary complex analysis, in: *Proceedings of the NATO Advanced Study Institute*, Durham, UK, 1–20 July 1979, Academic Press, New York and London, 1980.
8. C. Carathéodory, Über den Variabilitätsbereich der Koeffizienten von Potenzreihen, die gegebene Werte nicht annehmen, *Math. Ann.*, **64** (1907), 95–115. https://doi.org/10.1007/BF01449883
9. P. L. Duren, *Univalent Functions*, Grundlehren der Mathematischen Wissenschaften, Vol. 259, Springer-Verlag, New York, Berlin, Heidelberg, Tokyo, 1983.
10. M. Fekete, G. Szegö, Eine Bemerkung über ungerade schlichte Funktionen, *J. Lond. Math. Soc.*, **1**(2) (1933), 85–89. https://doi.org/10.1112/jlms/s1-8.2.85
11. O. Alnajar, M. Darus, A. Amourah, and A. Alsoboh, Touchard polynomials and the identified new subclass of analytic functions, *Int. J. Anal. Appl.*, 24 (2026), 46. https://doi.org/10.28924/2291-8639-24-2026-46
12. M. Lewin, On a coefficient problem for bi-univalent functions, *Proc. Am. Math. Soc.*, **18** (1967), 63–68. https://doi.org/10.1090/S0002-9939-1967-0206255-2
13. S. Miller, P. Mocanu, *Differential Subordination: Theory and Applications*, CRC Press, New York, NY, USA, 2000.
14. A. Alsoboh, A. Amourah, F. M. Sakar, O. Ogilat, G. M. Gharib, and N. Zomot, Coefficient estimation utilizing the Faber polynomial for a subfamily of bi-univalent functions, *Axioms*, 12(6) (2023), 512. https://doi.org/10.3390/axioms12060512
15. G. Murugusundaramoorthy, K. Vijaya, T. Bulboacă, Initial coefficient bounds for bi-univalent functions related to Gregory coefficients, *Mathematics*, 11 (2023), 2857. https://doi.org/10.3390/math11132857
16. O. Alnajar, O. Khabour, A. Amourah, and M. Darus, The relationship of Borel distribution and Horadam polynomials leads to analytical bi-univalent functions, *Eur. J. Pure Appl. Math.*, **18** (2025), 5929–5929.
17. A. Alsoboh, A. Amourah, and J. Salah, A study of bi-convex classes in leaf-like domains using quantum calculus through subordination, *Results Nonlinear Anal.*, 8(3) (2025), 210–221. https://nonlinear-analysis.com/index.php/pub/article/view/636
18. G.M. Phillips, Gregory's method for numerical integration, *Am. Math. Mon.*, 79 (1972), 270–274. https://doi.org/10.1080/00029890.1972.11993160
19. A. Alsoboh and M. Darus, New subclass of analytic functions defined by q-differential operator with respect to k-symmetric points, *Int. J. Math. Comput. Sci.*, 14 (2019), 761–773. http://ijmcs.future-in-tech.net
20. D.L. Tan, Coefficient estimates for bi-univalent functions, *Chin. Ann. Math. Ser. A*, 5 (1984), 559–568.
21. A. Alsoboh, A. S. Tayyah, A. Amourah, A. A. Al-Maqbali, K. Al Mashraf, and T. Sasa, Hankel determinant estimates for bi-Bazilevič-type functions involving q-Fibonacci numbers, *Eur. J. Pure Appl. Math.*, 18(3) (2025), 6698. https://doi.org/10.29020/nybg.ejpam.v18i3.6698

22. A. Amourah, A. Alsoboh, J. Salah, and K. Al Kalbani, Bounds on initial coefficients for bi-univalent functions linked to q-analog of Le Roy-type Mittag-Leffler function, *WSEAS Trans. Math.*, 23 (2024), 714–722. https://doi.org/10.37394/23206.2024.23.73
23. F. Yousef, A. Amourah, B.A. Frasin, T. Bulboacă, An avant-garde construction for subclasses of analytic bi-univalent functions, *Axioms*, 11 (2022), 267. https://doi.org/10.3390/axioms11060267
24. P. Zaprawa, Estimates of initial coefficients for bi-univalent functions, *Abstr. Appl. Anal.*, 2014 (2014), Article ID 357480. https://doi.org/10.1155/2014/357480
25. A. Amourah, O. Alnajar, M. Darus, A. Shdouh, and O. Ogilat, Estimates for the coefficients of subclasses defined by the Bell distribution of bi-univalent functions subordinate to Gegenbauer polynomials, *Mathematics*, 11(8) (2023), 1799. https://doi.org/10.3390/math11081799