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# Convergence Analysis and Numerical Approximation of the Fractional Fornberg–Whitham Equation via the Yasser–Jassim Transform

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**Abstract** This study develops a robust analytical–iterative framework for solving the fractional Fornberg–Whitham equation by combining the Yasser–Jassim integral transform with the Variational Iteration Method under the Atangana–Baleanu fractional derivative in the Caputo sense. An explicit series solution is constructed, and a rigorous convergence analysis is established, yielding sufficient conditions for existence and uniqueness of the solution. A computable bound for the truncation error is derived, providing a quantitative measure of the approximation accuracy. Numerical simulations confirm the theoretical findings, showing rapid convergence and excellent agreement with the exact solution. These results demonstrate the effectiveness, stability, and reliability of the proposed approach, indicating its potential applicability to a broad class of nonlinear fractional partial differential equations.

**Keywords** Yasser–Jassim transform, Variational Iteration Method, Fornberg–Witham equation, Atangana–Baleanu fractional derivative.

**AMS 2010 subject classifications** 26A33, 35R11, 35Q35, 60G22, 65H10.

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## 1. Introduction

*Fractional calculus* (FC) has, in recent years, solidified its role as a critical mathematical paradigm for the precise modeling of nonlocal and history-dependent behavior in diverse areas such as control theory, bioengineering, and materials science, natural sciences, engineering, fluid dynamics, life sciences, and other applied areas. By employing fractional-order tools, FC provides an effective framework for representing systems characterized by memory and hereditary properties. In particular, fractional derivatives (FDs) have shown significant capability in capturing such effects within different physical and engineering contexts. Their applications span a wide range of domains, including diffusion-reaction dynamics, frequency-dependent signal analysis, system identification, material damping behavior, and the modeling of viscoelastic responses like relaxation and creep [1, 2, 3].

The investigation of nonlinear wave equations and their associated solutions represents a cornerstone in numerous scientific disciplines. Among these, fractional nonlinear partial differential equations (FPDEs)

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are of particular interest due to their ability to capture complex dynamics through traveling wave solutions. These solutions are essential for interpreting the behavior of intricate physical and mechanical systems. Notably, several FPDEs—such as the Korteweg–de Vries and Camassa–Holm equations—exhibit a variety of traveling wave forms that effectively describe nonlinear dispersive wave processes, especially in shallow water contexts [4]. In a similar vein, the Fornberg–Whitham equation (FWE) has attracted increasing attention in mathematical physics, owing to its broad applicability across multiple scientific fields.

The fractional Fornberg–Whitham equation (FFWE) [9, 10] is defined as:

$$ {}^{AB}D_t^{\vartheta}\varpi - \varpi_{xxt} + \varpi_x + \varpi\varpi_x = 3\varpi_x\varpi_{xx} + \varpi\varpi_{xxx}, \tag{1} $$

where $0 < \vartheta \leq 1$.

This equation describes the qualitative dynamics of wave breaking in nonlinear dispersive waves. The FWE is known for its peaked solutions (peakons), which provide a mathematical framework for analyzing wave height limitations and the occurrence of wave breaking. In 1978, Fornberg and Whitham introduced a peaked solution expressed as:

$$ \varpi(x,t) = Ce^{x/2-2t/3}, \tag{2} $$

where $C$ is a constant.

Throughout the past decades, the Fornberg–Whitham equations (FWEs) have been the subject of intensive analytical and numerical investigation. Diverse methodologies have been employed to analyze their behavior, such as the Laplace decomposition method [5], Lie symmetry analysis [6], the variational iteration technique [8], differential transformation method [7], as well as iterative strategies including the new iterative method [11], the homotopy perturbation approach [12], and the homotopy analysis transform method [13]. Moreover, a variety of sophisticated tools have been utilized to tackle both linear and nonlinear fractional partial differential equations (FPDEs) [28, 29, 30, 31, 32, 33, 34, 35, 36, 37].

In light of the increasing complexity introduced by fractional operators, the time-fractional variant of the Fornberg–Whitham equation has attracted notable scholarly attention. Recent research trends have focused on constructing precise analytical frameworks capable of addressing the unique characteristics imposed by the fractional-order terms. These efforts have resulted in the development of several innovative techniques aimed at deriving meaningful and accurate solutions.

This paper proposes a hybrid analytical–iterative approach for solving the fractional Fornberg–Whitham equation by combining the Variational Iteration Method (VIM) with the Yasser–Jassim Transform (YJ) under the Atangana–Baleanu fractional derivative. The YJ is shown to be more suitable than classical transforms such as Laplace and Sumudu, as it is naturally compatible with the nonsingular kernel of the Atangana–Baleanu operator and avoids the algebraic complexity and auxiliary assumptions commonly encountered in Laplace-based methods. Unlike homotopy-based techniques, the proposed YJ–VIM framework does not require embedding parameters, leading to a more direct and efficient iterative scheme.

In contrast to most existing studies that focus mainly on numerical approximations, this work establishes a rigorous convergence analysis. Sufficient conditions for convergence are derived, and a well-defined convergence theorem is proved using appropriate norms. Moreover, a uniqueness theorem is formulated under strict and verifiable conditions, strengthening the theoretical foundation of the proposed method. Numerical results are compared with previously published methods, demonstrating the accuracy and efficiency of the YJ–VIM approach.

Specifically, the study utilizes the recently developed Yasser–Jassim integral transform in conjunction with the Variational Iteration Method (VIM) to handle the fractional Fornberg–Whitham equation formulated with Atangana–Baleanu fractional derivatives. The procedure yields semi-analytical solutions represented in the form of rapidly converging series.

## 2. Preliminaries

**Definition 2.1.** For a function $\varpi(x)$ defined on $x>0$, the Caputo fractional derivative of order $\vartheta$ is given (see [21]) by

$$ {}^{c}D_{x}^{\vartheta}\varpi(x)=\begin{cases}\dfrac{1}{\Gamma(n-\vartheta)}\displaystyle\int_{0}^{x}(x-t)^{n-\vartheta-1}\varpi^{(n)}(t)\,dt, & n-1<\vartheta\le n,\ n\in\mathbb{N},\\ \dfrac{d^{n}}{dx^{n}}\varpi(x), & \vartheta=n,\ n\in\mathbb{N}.\end{cases} \tag{3}$$

**Remark 2.1.** By Definition 2.1, the Caputo derivative of the power function $t^{\beta}$ satisfies

$$ {}^{c}D_{t}^{\vartheta}t^{\beta}=\begin{cases}\dfrac{\Gamma(\beta+1)}{\Gamma(\beta-\vartheta+1)}t^{\beta-\vartheta}, & n-1<\vartheta\le n,\ \beta>n-1,\ \beta\in\mathbb{R},\\ 0, & n-1<\vartheta\le n,\ \beta\in\mathbb{N}.\end{cases} \tag{4}$$

**Definition 2.2.** Following [22, 23], the Atangana–Baleanu fractional derivative of order $\vartheta$ (in the Caputo sense) for a function $\varpi(t)$ on $[a,t]$ is defined by

$$ {}^{AB}D_{t}^{\vartheta}\varpi(t)=\frac{M(\vartheta)}{1-\vartheta}\int_{a}^{t}E_{\vartheta}\Big(-\tfrac{\vartheta\,(t-x)^{\vartheta}}{1-\vartheta}\Big)\varpi'(x)\,dx,\quad 0<\vartheta<1, \tag{5}$$

where the normalization function $M(\vartheta)$ obeys $M(0)=M(1)=1$.

**Definition 2.3.** The Atangana–Baleanu fractional integral of order $\vartheta$ is given in [23] by

$$ {}^{AB}I_{t}^{\vartheta}\varpi(t)=\frac{1-\vartheta}{M(\vartheta)}\varpi(t)+\frac{\vartheta}{M(\vartheta)\Gamma(\vartheta)}\int_{a}^{t}(t-x)^{\vartheta-1}\varpi(x)\,dx,\quad 0<\vartheta<1, \tag{6}$$

with $M(\vartheta)$ as above.

**Definition 2.4** (Yasser-Jassim Transform)**.** The **Yasser–Jassim transform** of a function $\varpi(t)$ is introduced in [38] as:

$$\mathscr{H}\{\varpi(t)\}=\mathscr{A}\int_{0}^{\infty}e^{-\frac{1}{\sqrt{\mathscr{A}}}t}\varpi(t)\,dt \tag{7}$$

This transformation moves the function from the time domain to a new spectral domain determined by the parameter $\mathscr{A}$.

**Some fundamental properties:**

1. $$\mathscr{H}\{1\}=\mathscr{A}\sqrt{\mathscr{A}} \tag{8}$$

2. $$\mathscr{H}\{e^{bt}\}=\mathscr{A}\sqrt{\mathscr{A}}\frac{1}{1-b\sqrt{\mathscr{A}}} \tag{9}$$

3. $$\mathscr{H}\{E_{\vartheta}(bt)\}=\mathscr{A}\sqrt{\mathscr{A}}\frac{1}{1-b\sqrt{\mathscr{A}}^{\vartheta}} \tag{10}$$

4. $$\mathscr{H}\{t^{\vartheta}\}=\mathscr{A}(\sqrt{\mathscr{A}})^{\vartheta+1}\Gamma(\vartheta+1) \tag{11}$$

**Theorem 2.1.** 1. For the Caputo fractional derivative:

$$\mathscr{H}\{{}^{c}D_t^{\vartheta}\,\varpi(t)\} = \frac{1}{(\sqrt{\mathscr{A}})^{\vartheta}}\mathscr{H}\{\varpi(t)\} - \sum_{k=0}^{n-1}\frac{\mathscr{A}}{(\sqrt{\mathscr{A}})^{\vartheta-k-1}}\varpi^{(k)}(0), \quad n-1<\vartheta\leq n. \tag{12}$$

2. For the Atangana–Baleanu derivative:

$$\mathscr{H}\{{}^{AB}D_t^{\vartheta}\,\varpi(t)\} = \frac{M(\vartheta)}{1-\vartheta+\vartheta\sqrt{\mathscr{A}}^{\vartheta}}\left[\mathscr{H}\{\varpi(t)\} - \mathscr{A}\sqrt{\mathscr{A}}\varpi(0)\right] \tag{13}$$

*Proof*

1. For the Caputo derivative:

$$\mathscr{H}\{{}^{c}D_t^{\vartheta}\,\varpi(t)\} = \mathscr{H}\left\{\frac{1}{\Gamma(n-\vartheta)}\int_0^t (t-\tau)^{n-\vartheta-1}\varpi^{(n)}(\tau)\,d\tau\right\}$$

Applying the convolution property of the YJ transform:

$$= \frac{1}{\mathscr{A}}\mathscr{H}\{t^{n-\vartheta-1}\}\cdot\mathscr{H}\{\varpi^{(n)}(t)\}$$

Substituting the transformation of the $n$th derivative yields:

$$= \frac{1}{(\sqrt{\mathscr{A}})^{\vartheta}}\mathscr{H}\{\varpi(t)\} - \sum_{k=0}^{n-1}\frac{\mathscr{A}}{(\sqrt{\mathscr{A}})^{\vartheta-k-1}}\varpi^{(k)}(0)$$

2. For the AB derivative:

$$\mathscr{H}\{{}^{AB}D_t^{\vartheta}\,\varpi(t)\} = \mathscr{H}\left\{\frac{M(\vartheta)}{1-\vartheta}\int_0^t E_{\vartheta}\left(-\frac{\vartheta(t-\tau)^{\vartheta}}{1-\vartheta}\right)\varpi'(\tau)\,d\tau\right\}$$

Using the convolution theorem:

$$= \frac{M(\vartheta)}{1-\vartheta}\cdot\frac{1}{\mathscr{A}}\cdot\mathscr{H}\left\{E_{\vartheta}\left(-\frac{\vartheta t^{\vartheta}}{1-\vartheta}\right)\right\}\cdot\mathscr{H}\{\varpi'(t)\}$$

Upon simplification, we obtain:

$$= \frac{M(\vartheta)}{1-\vartheta+\vartheta\sqrt{\mathscr{A}}^{\vartheta}}\left[\mathscr{H}\{\varpi(t)\} - \mathscr{A}\sqrt{\mathscr{A}}\varpi(0)\right]$$

□

**Definition 2.5.** The two-parameter Mittag-Leffler function is described as [19, 20]:

$$E_{\vartheta,p}(z) = \sum_{n=0}^{\infty}\frac{z^n}{\Gamma(n\vartheta+p)}, \quad \vartheta,p,z\in\mathbb{C},\ \mathrm{Re}(\vartheta)>0,\ \mathrm{Re}(p)>0. \tag{14}$$

**Remark 2.2.** Based on **Definition 14**, the following identities can be established:

$$E_{2,1}(x^2) = \cosh(x), \tag{15}$$

$$E_{2,2}(x^2) = \frac{\sinh(x)}{x}, \tag{16}$$

$$E_{2,3}(x^2) = \frac{1}{x^2}\left[-1+\cosh(x)\right] \tag{17}$$

Fornberg–Whitham equation with Atangana-Baleanu fractional derivative

## 3. Methodology

Consider the FFWE :

$$^{AB}D_t^{\vartheta}\varpi - \varpi_{xxt} + \varpi_x + \varpi\varpi_x = 3\varpi_x\varpi_{xx} + \varpi\varpi_{xxx}, \quad 0 < \vartheta \leq 1, \tag{18}$$

Imposing the initial condition:

$$\varpi(x,0) = g(x). \tag{19}$$

Using the VIM method, Equation (18) can be reformulated as:

$$\varpi_{n+1} = \varpi_n + \int_0^t \lambda(x,t-\xi)\Big[{}^{AB}D_\xi^{\vartheta}\varpi_n - \varpi_{n,xx\xi} + \varpi_{n,x} + \varpi_n\varpi_{n,x} - 3\varpi_{n,x}\varpi_{n,xx} - \varpi_n\varpi_{n,xxx}\Big]d\xi \tag{20}$$

Applying the YJ transform to Equation (20), we get:

$$\mathscr{H}\{\varpi_{n+1}\} = \mathscr{H}\{\varpi_n\} + \mathscr{H}\left\{\int_0^t \lambda(x,t-\xi)\Big[{}^{AB}D_\xi^{\vartheta}\varpi_n - \varpi_{n,xx\xi} + \varpi_{n,x} + \varpi_n\varpi_{n,x} - 3\varpi_{n,x}\varpi_{n,xx} - \varpi_n\varpi_{n,xxx}\Big]d\xi\right\} \tag{21}$$

Since the Lagrange multiplier depends on $t-\xi$ and the equation is written in terms of $\xi$, we apply the convolution theorem [38]

$$\mathscr{H}\{\varpi_{n+1}\} = \mathscr{H}\{\varpi_n\} + \frac{1}{\mathscr{A}}\mathscr{H}\{\lambda(x,t)\}\Big[\mathscr{H}\{{}^{AB}D_t^{\vartheta}\varpi_n - \varpi_{n,xxt} + \varpi_{n,x} + \varpi_n\varpi_{n,x} - 3\varpi_{n,x}\varpi_{n,xx} - \varpi_n\varpi_{n,xxx}\}\Big]. \tag{22}$$

Introducing $\frac{\delta}{\delta\varpi_n}$ to both sides of Equation (22), we obtain:

$$\frac{\delta}{\delta\varpi_n}\mathscr{H}\{\varpi_{n+1}\} = \frac{\delta}{\delta\varpi_n}\mathscr{H}\{\varpi_n\} + \frac{1}{\mathscr{A}}\frac{\delta}{\delta\varpi_n}\mathscr{H}\{\lambda(x,t)\}\Big[\frac{\mathscr{H}\{\varpi_n\}}{(1-\vartheta+\vartheta\sqrt{\mathscr{A}}^{\vartheta})} - \mathscr{A}\sqrt{\mathscr{A}}\varpi(x,0) - \mathscr{H}\{-\varpi_{n,xxt} + \varpi_{n,x} + \varpi_n\varpi_{n,x} - 3\varpi_{n,x}\varpi_{n,xx} - \varpi_n\varpi_{n,xxx}\}\Big] \tag{23}$$

Considering the nonlinear terms as restricted variations, i.e.,

$$\delta[\varpi_n\varpi_{n,x}-3\varpi_{n,x}\varpi_{n,xx}-\varpi_n\varpi_{n,xxx}]=0,$$

we have:

$$\mathscr{H}\{\delta\varpi_{n+1}\}=\mathscr{H}\{\delta\varpi_n\}+\mathscr{H}\{\lambda(x,t)\}\frac{\mathscr{H}\{\delta\varpi_n\}}{(1-\vartheta+\vartheta\sqrt{\mathscr{A}}^{\vartheta})} \tag{24}$$

The optimality condition for $\varpi_{n+1}$ requires that $\mathscr{H}\{\delta\varpi_{n+1}\}=0$, leading to:

$$0=\left[1+\mathscr{H}\{\lambda(x,t)\}\frac{1}{(1-\vartheta+\vartheta\sqrt{\mathscr{A}}^{\vartheta})}\right]\mathscr{H}\{\delta\varpi_n\}, \tag{25}$$

and thus:

$$\mathscr{H}\{\lambda(x,t)\}=-(1-\vartheta+\vartheta\sqrt{\mathscr{A}}^{\vartheta}) \tag{26}$$

By substituting Equation (26) into (22), we get:

$$\begin{aligned}\mathscr{H}\{\varpi_{n+1}\}=\mathscr{H}\{\varpi_n\}-(1-\vartheta+\vartheta\sqrt{\mathscr{A}}^{\vartheta})\Big[\mathscr{H}\{{}^{AB}D_t^{\vartheta}\varpi_n-\varpi_{n,xxt}+\varpi_{n,x}\\ +\varpi_n\varpi_{n,x}-3\varpi_{n,x}\varpi_{n,xx}-\varpi_n\varpi_{n,xxx}\}\Big]\end{aligned} \tag{27}$$

Utilizing the inverse YJ transform, we derive the recurrence relation:

$$\begin{aligned}\varpi_{n+1}=\varpi(x,0)-\mathscr{H}^{-1}\Big\{(1-\vartheta+\vartheta\sqrt{\mathscr{A}}^{\vartheta})\Big[\mathscr{H}\{-\varpi_{n,xxt}+\varpi_{n,x}\\ +\varpi_n\varpi_{n,x}-3\varpi_{n,x}\varpi_{n,xx}-\varpi_n\varpi_{n,xxx}\}\Big]\Big\}.\end{aligned} \tag{28}$$

Finally, the solution is expressed as:

$$\varpi(x,t)=\lim_{n\to\infty}\varpi_n. \tag{29}$$

## 4. Convergence of YJ-VIN

In this part, we analyze the convergence of the newly proposed YJ-VIM method applied to the Fornberg-Whitham equation discussed earlier. The necessary conditions ensuring the method's convergence, as well as the associated error estimates, are outlined through the upcoming theorems.

Using equation (28) to defined the following operator

$$\begin{aligned}\mathfrak{A}[\varpi]=\varpi_{n+1}-\varpi_n=\varpi(x,0)-\varpi_n-\mathscr{H}^{-1}\Big\{(1-\vartheta+\vartheta\sqrt{\mathscr{A}}^{\vartheta})\Big[\mathscr{H}\{-\varpi_{n,xxt}+\varpi_{n,x}\\ +\varpi_n\varpi_{n,x}-3\varpi_{n,x}\varpi_{n,xx}-\varpi_n\varpi_{n,xxx}\}\Big]\Big\}.\end{aligned} \tag{30}$$

For simplicity, let $L(\varpi)=-\varpi_{xxt}+\varpi_x+\varpi\varpi_x-3\varpi_x\varpi_{xx}-\varpi\varpi_{xxx}$

By simplifying and using the convolution property, we obtain the following equivalent formula:

$$\mathfrak{A}[\varpi]=\varpi(x,0)-\varpi_n-\left\{(1-\vartheta)L(\varpi)-\int_0^t\vartheta\frac{(t-\tau)^{\vartheta-1}}{\Gamma(\vartheta)}L(\varpi)\,d\tau\right\}. \tag{31}$$

defined the component $\mho_k, k = 0,1,2,...$ as

$$\begin{aligned}
&\mho_0 = \varpi(x,0)\\
&\mho_1 = \mathfrak{A}[\mho_0]\\
&\mho_2 = \mathfrak{A}[\mho_0 + \mho_1]\\
&.\\
&.\\
&.\\
&\mho_{k+1} = \mathfrak{A}[\mho_0 + \mho_1 + ... + \mho_k]
\end{aligned} \tag{32}$$

Thus, it follows that $\varpi(x,t) = \lim_{n\to\infty} \varpi_n = \sum_{k=0}^{\infty} \mho_k$. Accordingly, the solution to the problem (18)-(19) can be expressed as

$$\varpi(x,t) = \sum_{k=0}^{\infty} \mho_k. \tag{33}$$

Where the initial approximation $\mho_0 = \varpi(x,0)$.

**Theorem 4.1.** Let $\mathfrak{A}$ be an operator such that $\mathfrak{A}: H^1 \to H^1$. The series solution $\varpi(x,t) = \sum_{k=0}^{\infty} \mho_k$ converges provided that there exists a constant $0 < \gamma < 1$ satisfying $|\mho_{k+1}| \leq \gamma |\mho_k|$ for all $k = 0,1,2,\ldots$, where $H^1$ denotes a Hilbert space.

*Proof*
let define the sequence $\{\mathfrak{S}_n\}_{n=0}^{\infty}$ as

$$\begin{aligned}
&\mathfrak{S}_0 = \mho_0\\
&\mathfrak{S}_1 = \mho_0 + \mho_1\\
&\mathfrak{S}_2 = \mho_0 + \mho_1 + \mho_2\\
&.\\
&.\\
&.\\
&\mathfrak{S}_n = \mho_0 + \mho_1 + ... + \mho_n
\end{aligned} \tag{34}$$

to demonstrate that the sequence $\{\mathfrak{S}_n\}_{n=0}^{\infty}$ is a Cauchy sequence in the Hilbert space $H^1$, consider,

$$\|\mathfrak{S}_{n+1} - \mathfrak{S}_n\| = \|\mho_{n+1}\| \leq \gamma \|\mho_n\| \leq \gamma^2 \|\mho_{n-1}\| ... \leq \gamma^{n+1} \|\mho_0\| \tag{35}$$

For every $n, j \in N, n \geq j$ we have

$$\begin{aligned}
\left\|\mathfrak{S}_n - \mathfrak{S}_j\right\| &= \left\|\mathfrak{S}_n - \mathfrak{S}_{n-1} + \mathfrak{S}_{n-1} - \mathfrak{S}_{n-2} + ... + \mathfrak{S}_{j+1} - \mathfrak{S}_j\right\|\\
&\leq \|\mathfrak{S}_n - \mathfrak{S}_{n-1}\| + \|\mathfrak{S}_{n-1} - \mathfrak{S}_{n-2}\| + ... + \left\|\mathfrak{S}_{j+1} - \mathfrak{S}_j\right\|\\
&\leq \gamma^n \|\mho_0\| + \gamma^{n-1} \|\mho_0\| + ... + \gamma^{j+1} \|\mho_0\|\\
&= \frac{1-\gamma^{n-j}}{1-\gamma} \gamma^{j+1} \|\mho_0\|
\end{aligned} \tag{36}$$

Since $0 < \gamma < 1$, we have $\lim_{n,j\to\infty} \left\|\mathfrak{S}_n - \mathfrak{S}_j\right\| = 0$ Hence, $\{\mathfrak{S}_n\}_{n=0}^{\infty}$ constitutes a Cauchy sequence in the Hilbert space, which guarantees that the series solution $\varpi(x,t) = \sum_{k=0}^{\infty} \mho_k$ is convergent. □

In this theorem, the convergence of the solution series is rigorously established, and the sufficient conditions for convergence are precisely formulated. These conditions are explicitly employed to derive the stated results, thereby ensuring the validity of the convergence analysis and the soundness of the underlying mathematical framework. The following theorem demonstrates that the convergent series indeed represents the exact solution of the governing equation, thereby confirming that the obtained solution is not merely an approximation but coincides with the true solution of the problem.

**Theorem 4.2.** If the series $\varpi(x,t) = \sum_{k=0}^{\infty} \mho_k$ converges, then it represents the exact solution to the problem (18)-(19)

*Proof*
Assuming the series solution converges, denoted by $\phi(x,t) = \sum_{k=0}^{\infty} \mho_k$, it follows that, $\lim_{j\to\infty} \mho_j = 0$, $\sum_{j=0}^{n} [\mho_{j+1} - \mho_j] = \mho_{n+1} - \mho_0$ and so,

$$\sum_{j=0}^{\infty} [\mho_{j+1} - \mho_j] = \lim_{j\to\infty} \mho_j - \mho_0 = -\mho_0 \tag{37}$$

Utilizing the operator ${}^{AB}D_t^{\vartheta}$

$$\sum_{j=0}^{\infty} {}^{AB}D_t^{\vartheta} [\mho_{j+1} - \mho_j] = 0 \tag{38}$$

Meanwhile, based on definition (32), we obtain

$${}^{AB}D_t^{\vartheta} [\mho_{j+1} - \mho_j] = {}^{AB}D_t^{\vartheta} [\mathfrak{A}[\mho_0 + \mho_1 + \ldots + \mho_j] - \mathfrak{A}[\mho_0 + \mho_1 + \ldots + \mho_{j-1}]] \tag{39}$$

provided that $j \geq 1$ it follows from definition (31) that

$$\begin{aligned} {}^{AB}D_t^{\vartheta} [\mho_{j+1} - \mho_j] = {}^{AB}D_t^{\vartheta} \Bigg[ &-[\mho_j] - (1-\vartheta) \Big\{ L(\mho_0 + \mho_1 + \ldots + \mho_j) - L(\mho_0 + \mho_1 + \ldots + \mho_{j-1}) \Big\} \\ &- \int_0^t \vartheta \frac{(t-\tau)^{\vartheta-1}}{\Gamma(\vartheta)} \Big\{ L(\mho_0 + \mho_1 + \ldots + \mho_j) - L(\mho_0 + \mho_1 + \ldots + \mho_{j-1}) \Big\} d\tau \Bigg] \end{aligned} \tag{40}$$

We observe in Equation (40) that when applying the operator ${}^{AB}D_t^{\vartheta}$, the second and third terms are the inverse of the operator, resulting in the following:

$${}^{AB}D_t^{\vartheta} [\mho_{j+1} - \mho_j] = -{}^{AB}D_t^{\vartheta} [\mho_j] - \Big\{ L(\mho_0 + \mho_1 + \ldots + \mho_j) - L(\mho_0 + \mho_1 + \ldots + \mho_{j-1}) \Big\}, j \geq 1. \tag{41}$$

Consequently, we have

$$\begin{aligned} \sum_{j=0}^{n} {}^{AB}D_t^{\vartheta} [\mho_{j+1} - \mho_j] = &-{}^{AB}D_t^{\vartheta} [\mho_0] - \Big\{ L(\mho_0) \Big\} \\ &-{}^{AB}D_t^{\vartheta} [\mho_1] - \Big\{ L(\mho_0 + \mho_1) - L(\mho_0) \Big\} \\ &-{}^{AB}D_t^{\vartheta} [\mho_2] - \Big\{ L(\mho_0 + \mho_1 + \mho_2) - L(\mho_0 + \mho_1) \Big\} \\ &\vdots \\ &-{}^{AB}D_t^{\vartheta} [\mho_n] - \Big\{ L(\mho_0 + \mho_1 + \mho_2 + \ldots + \mho_n) - L(\mho_0 + \mho_1 + \ldots + \mho_{n-1}) \Big\} \end{aligned}$$

Therefore,

$$\sum_{j=0}^{\infty} {}^{AB}D_t^{\vartheta}[\mho_{j+1}-\mho_j] = -{}^{AB}D_t^{\vartheta}\left\{\sum_{j=0}^{\infty}\mho_j\right\} - L\left\{\sum_{j=0}^{\infty}\mho_j\right\} \tag{42}$$

From Equations (38) and (42), we can observe that the solution series represents the exact solution to Problem (18)-(19). □

**Theorem 4.3.** Assume the infinite series

$$\sum_{k=0}^{\infty}\mho_k$$

converges to the exact solution $\varpi(x,t)$. If one uses the partial sum

$$\sum_{k=0}^{j}\mho_k$$

as an approximation, then the truncation error $E_j(x,t)$ can be bounded by

$$E_j(x,t) \le \frac{\gamma^{j+1}}{1-\gamma}\left\|\mho_0\right\|.$$

*Proof*
from inequality (36), we have

$$\left\|\mathfrak{S}_n - \mathfrak{S}_j\right\| \le \frac{1-\gamma^{n-j}}{1-\gamma}\gamma^{j+1}\left|\mho_0\right\| \tag{43}$$

For $n \ge j$, Now, as $n \to \infty$ then $\mathfrak{S}_n \to \varpi(x,t)$. So,

$$\left\|\varpi(x,t) - \sum_{k=0}^{j}\mho_k\right\| \le \frac{1-\gamma^{n-j}}{1-\gamma}\gamma^{j+1}\left|\mho_0\right\| \tag{44}$$

Also, since $0<\gamma<1$ we have $(1-\gamma^{n-j})<1$, Then, we conclude

$$\left\|\varpi(x,t) - \sum_{k=0}^{j}\mho_k\right\| \le \frac{1}{1-\gamma}\gamma^{j+1}\left|\mho_0\right\| \tag{45}$$

□

This theorem shows that the truncation error of the solution series can be explicitly controlled and decays geometrically with respect to the contraction constant $\gamma$. Consequently, increasing the number of terms in the partial sum leads to a rapid improvement in accuracy, highlighting the numerical efficiency of the proposed method.

## 5. Uniqueness

Assume that the analytical solution of the fractional Fornberg–Whitham equation (FWE) obtained through the YJ-VIM is unique.

Define the norm on $[0,b], b>0$ by

$$\|\varpi\| = \sup_{t\in[0,b]} |\varpi(t)|, \quad \text{for all } \varpi \in H(0,b),$$

Now, consider

$$^{AB}D_t^{\vartheta}\varpi = \mathfrak{L}(\varpi) + \mathfrak{N}(\varpi), \quad 0<\vartheta\leq 1, \tag{46}$$

with the initial condition:

$$\varpi(x,0) = \mathfrak{g}(x). \tag{47}$$

Where $\mathfrak{L}(\varpi) = \varpi_{xxt} - \varpi_x$, $\mathfrak{N}(\varpi) = 3\varpi_x\varpi_{xx} + \varpi\varpi_{xxx} - \varpi\varpi_x$ are linear and nonlinear operator respectively and $\mathfrak{L}, \mathfrak{N}$ agree with lipschitz condition. Let $\psi$ and $\varphi$ be solutions of the equation (46) - (47), where the initial conditions are the same. Using equation (28) and equation(29), the solutions can be written as follows:

$$\psi = \mathfrak{g}(x) - \left\{(1-\vartheta)L(\psi) - \int_0^t \vartheta\frac{(t-\tau)^{\vartheta-1}}{\Gamma(\vartheta)}L(\psi)\,d\tau\right\}. \tag{48}$$

$$\varphi = \mathfrak{g}(x) - \left\{(1-\vartheta)L(\varphi) - \int_0^t \vartheta\frac{(t-\tau)^{\vartheta-1}}{\Gamma(\vartheta)}L(\varphi)\,d\tau\right\}. \tag{49}$$

then,

$$|\varphi-\psi| = \left|(1-\vartheta)[L(\psi)-L(\varphi)] + \left[\int_0^t \vartheta\frac{(t-\tau)^{\vartheta-1}}{\Gamma(\vartheta)}L(\varphi)-L(\psi)\,d\tau\right]\right| \tag{50}$$

Using the triangle inequality to obtain the following

$$|\varphi-\psi| \leq |(1-\vartheta)[L(\psi)-L(\varphi)]| + \left|\left[\int_0^t \vartheta\frac{(t-\tau)^{\vartheta-1}}{\Gamma(\vartheta)}L(\varphi)-L(\psi)\,d\tau\right]\right| \tag{51}$$

$$|\varphi-\psi| \leq |(1-\vartheta)[L(\psi)-L(\varphi)]| + \int_0^t \left|\vartheta\frac{(t-\tau)^{\vartheta-1}}{\Gamma(\vartheta)}\right| |L(\varphi)-L(\psi)|\,d\tau \tag{52}$$

let $M = Max[\left|\vartheta\frac{(t-\tau)^{\vartheta-1}}{\Gamma(\vartheta)}\right|]$ and since $0<\vartheta\leq 1$, we get

$$|\varphi-\psi| \leq \int_0^t M\,|L(\varphi)-L(\psi)|\,d\tau \tag{53}$$

Given that $L(\varpi) = \mathfrak{L}(\varpi)+\mathfrak{N}(\varpi)$, it follows that $L(\varpi)$ satisfies the Lipschitz condition because $\mathfrak{L}(\varpi)$ and $\mathfrak{N}(\varpi)$ themselves satisfy the Lipschitz condition, then, we have

$$|\varphi-\psi| \leq \int_0^t Mk\,|\varphi-\psi|\,d\tau \quad \text{where k constant} \tag{54}$$

Now, by applying Gronwall's inequality, we obtain the following inequality

$$\|\varphi-\psi\| \leq 0 \tag{55}$$

By the properties of the norm, we conclude that the two solutions are identical; hence, the solution is unique

## 6. Application

In this part, we will present an example as an application of the above method and demonstrate the steps involved in the derivation process. This example will highlight the practicality and efficiency of the proposed method in solving complex problems. Additionally, they will provide a deeper understanding of how the iterative solutions converge to the exact solution under different scenarios.

**Example 6.1.** Consider the fractional Fornberg–Whitham equation:

$$ {}^{AB}D_t^{\vartheta}\varpi - \varpi_{xxt} + \varpi_x + \varpi\varpi_x = 3\varpi_x\varpi_{xx} + \varpi\varpi_{xxx}, \quad 0 < \vartheta \leq 1 \tag{56} $$

with the initial condition:

$$ \varpi(x,0) = \left(\cosh\frac{x}{4}\right)^2. \tag{57} $$

Using Equation (28), we obtain the following iterative solutions:

$$ \begin{aligned} \varpi_0 &= \left(\cosh\frac{x}{4}\right)^2, \\ \varpi_1 &= \frac{1}{2} + \frac{1}{2}\cosh\frac{x}{2} - \left[1-\vartheta+\frac{\vartheta t^{\vartheta}}{\Gamma(\vartheta+1)}\right]\frac{11}{32}\sinh\frac{x}{2}, \\ \varpi_2 &= \frac{1}{2} + \frac{1}{2}\cosh\frac{x}{2} - \frac{11}{32}\sinh\frac{x}{2}\left[1-\vartheta+\frac{\vartheta t^{\vartheta}}{\Gamma(\vartheta+1)}\right] \\ &\quad + \frac{121}{512}\cosh\frac{x}{2}\left[(1-\vartheta)^2 + 2(1-\vartheta)\frac{\vartheta t^{\vartheta}}{\Gamma(\vartheta+1)} + \frac{t^{2\vartheta}}{\Gamma(2\vartheta+1)}\right] \\ &\quad - \frac{11}{128}\sinh\frac{x}{2}\left[\frac{\vartheta(1-\vartheta)t^{\vartheta-1}}{\Gamma(\vartheta)} + \frac{\vartheta^2 t^{2\vartheta-1}}{\Gamma(2\vartheta)}\right]. \end{aligned} $$

$$ \begin{aligned} \varpi_3 &= \frac{1}{2} + \frac{1}{2}\cosh\frac{x}{2} - \frac{11}{32}\sinh\frac{x}{2}\left[1-\vartheta+\frac{\vartheta t^{\vartheta}}{\Gamma(\vartheta+1)}\right] \\ &\quad + \frac{121}{512}\cosh\frac{x}{2}\left[(1-\vartheta)^2 + 2(1-\vartheta)\frac{\vartheta t^{\vartheta}}{\Gamma(\vartheta+1)} + \frac{t^{2\vartheta}}{\Gamma(2\vartheta+1)}\right] \\ &\quad - \frac{11}{128}\sinh\frac{x}{2}\left[\frac{\vartheta(1-\vartheta)t^{\vartheta-1}}{\Gamma(\vartheta)} + \frac{\vartheta^2 t^{2\vartheta-1}}{\Gamma(2\vartheta)}\right] \\ &\quad - \left[(1-\vartheta)^3 + 3(1-\vartheta)^2\vartheta\frac{t^{\vartheta}}{\Gamma(\vartheta+1)} + 3(1-\vartheta)\frac{\vartheta^2 t^{2\vartheta}}{\Gamma(2\vartheta+1)} + \frac{\vartheta^3 t^{3\vartheta}}{\Gamma(3\vartheta+1)}\right]\frac{1331}{8192}\sinh\frac{x}{2} \\ &\quad + \frac{121}{2048}\cosh\frac{x}{2}\left[3(1-\vartheta)^2\vartheta\frac{t^{\vartheta-1}}{\Gamma(\vartheta)} + 5(1-\vartheta)\frac{\vartheta^2 t^{2\vartheta-1}}{\Gamma(2\vartheta)} + 2\frac{\vartheta^3 t^{3\vartheta-1}}{\Gamma(3\vartheta)}\right] \\ &\quad - \frac{11}{512}\sinh\frac{x}{2}\left[(1-\vartheta)^2\vartheta\frac{t^{\vartheta-2}}{\Gamma(\vartheta-1)} + 2(1-\vartheta)\frac{\vartheta^2 t^{2\vartheta-2}}{\Gamma(2\vartheta-1)} + \frac{\vartheta^3 t^{3\vartheta-2}}{\Gamma(3\vartheta-1)}\right]. \end{aligned} $$

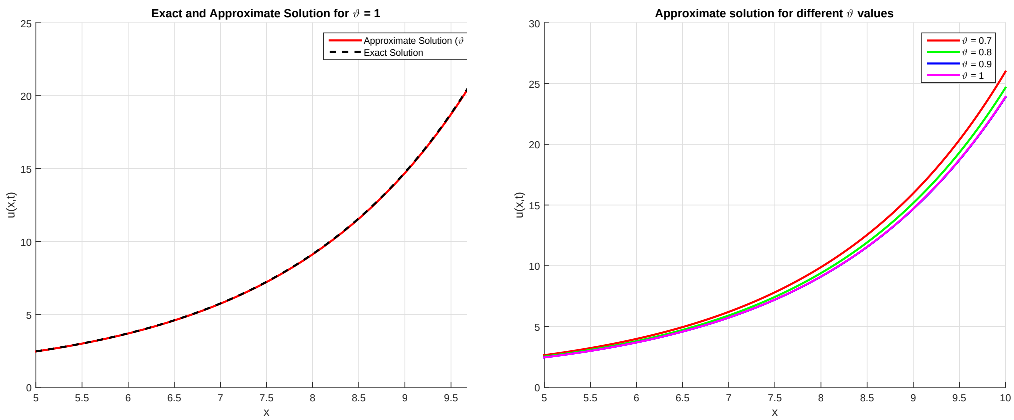
The approximate solution is:

$$\begin{aligned}
\varpi(x,t) = & \frac{1}{2} + \frac{1}{2}\cosh\frac{x}{2} - \frac{11}{32}\sinh\frac{x}{2}\left[1-\vartheta+\frac{\vartheta t^{\vartheta}}{\Gamma(\vartheta+1)}\right] \\
& + \frac{121}{512}\cosh\frac{x}{2}\left[(1-\vartheta)^2+2(1-\vartheta)\frac{\vartheta t^{\vartheta}}{\Gamma(\vartheta+1)}+\frac{t^{2\vartheta}}{\Gamma(2\vartheta+1)}\right] \\
& - \frac{11}{128}\sinh\frac{x}{2}\left[\frac{\vartheta(1-\vartheta)t^{\vartheta-1}}{\Gamma(\vartheta)}+\frac{\vartheta^2 t^{2\vartheta-1}}{\Gamma(2\vartheta)}\right] \\
& - \left[(1-\vartheta)^3+3(1-\vartheta)^2\vartheta\frac{t^{\vartheta}}{\Gamma(\vartheta+1)}+3(1-\vartheta)\frac{\vartheta^2 t^{2\vartheta}}{\Gamma(2\vartheta+1)}+\frac{\vartheta^3 t^{3\vartheta}}{\Gamma(3\vartheta+1)}\right]\frac{1331}{8192}\sinh\frac{x}{2} \\
& + \frac{121}{2048}\cosh\frac{x}{2}\left[3(1-\vartheta)^2\vartheta\frac{t^{\vartheta-1}}{\Gamma(\vartheta)}+5(1-\vartheta)\frac{\vartheta^2 t^{2\vartheta-1}}{\Gamma(2\vartheta)}+2\frac{\vartheta^3 t^{3\vartheta-1}}{\Gamma(3\vartheta)}\right] \\
& - \frac{11}{512}\sinh\frac{x}{2}\left[(1-\vartheta)^2\vartheta\frac{t^{\vartheta-2}}{\Gamma(\vartheta-1)}+2(1-\vartheta)\frac{\vartheta^2 t^{2\vartheta-2}}{\Gamma(2\vartheta-1)}+\frac{\vartheta^3 t^{3\vartheta-2}}{\Gamma(3\vartheta-1)}\right].
\end{aligned}$$

For $\vartheta = 1$, the exact solution is:

$$\varpi(x,t) = \cosh^2\left(\frac{x}{4}-\frac{11t}{24}\right). \tag{58}$$

Figure 1. In **Example 6.1**, plots (a) and (b) demonstrate that the curve progressively converges toward the exact solution as $\vartheta$ approaches 1. At $\vartheta = 0.9$, the curve aligns almost perfectly with the one corresponding to $\vartheta = 1$.

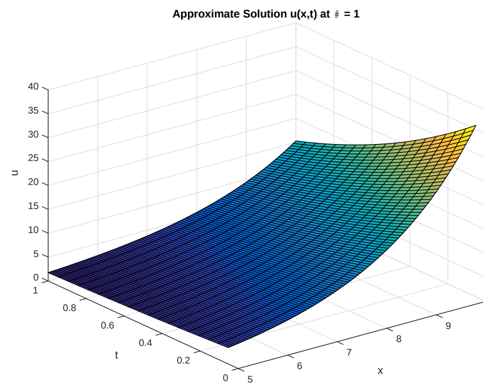



(a) (b)

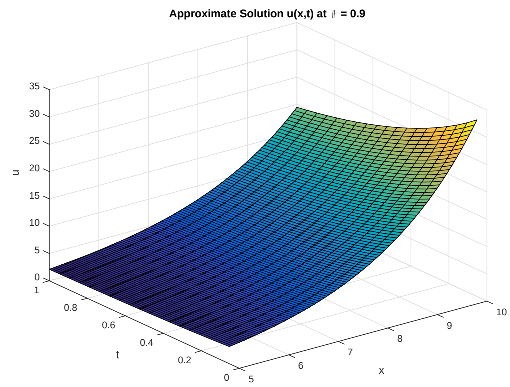
**Remark 6.1.** The parameter $\vartheta$ denotes the order of the fractional derivative used in the Atangana–Baleanu operator.

The results presented in Table 2 clearly indicate the superiority of the YJ–VIM over the mVIM, as it produces considerably smaller absolute errors for all values of $\varkappa$ and fractional order $\vartheta$. This superiority becomes more evident for smaller values of $\vartheta$, highlighting the accuracy and robustness of the proposed method in comparison with previously published approaches for Fornberg–Whitham type equations.

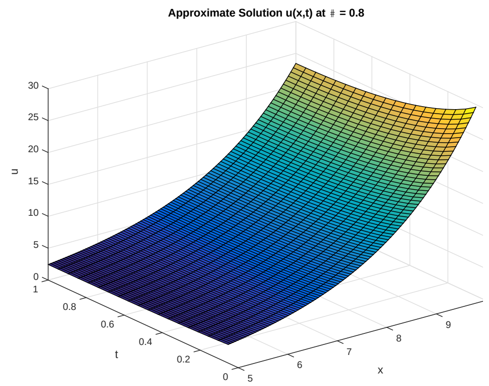
Figure 2. In **Example 6.1**, plots (a), (b), (c), and (d) provide surface representations that highlight the close correspondence between the numerical solution and the exact solution, shown in plot (e).



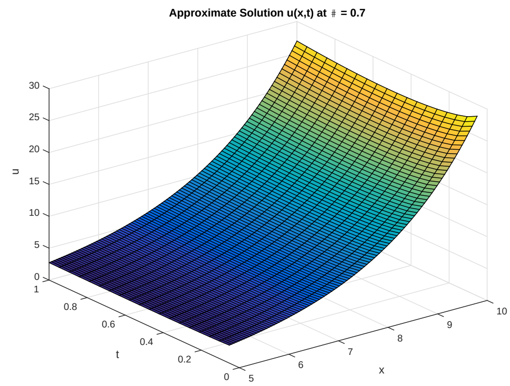

(a)



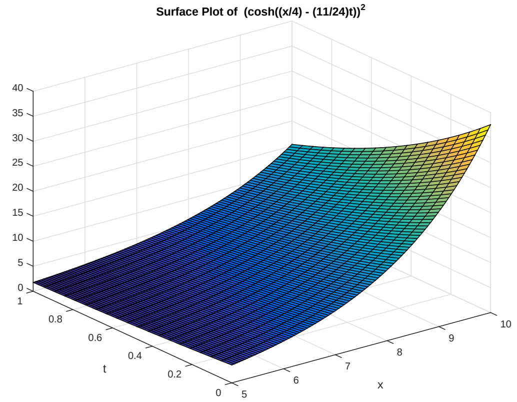

(b)




(c)




(d)




(e)

Table 1. A table displaying the absolute error for **Example 6.1**, where the Atangana-Baleanu operator is applied.

| $x$ | $AB_{\vartheta=1}$ | $AB_{\vartheta=0.9}$ | $AB_{\vartheta=0.8}$ | $AB_{\vartheta}=0.7$ |
|---|---|---|---|---|
| 5 | 0.005757 | 0.0039514 | 0.065314 | 0.17893 |
| 5.5 | 0.0070306 | 0.0070339 | 0.079532 | 0.22314 |
| 6 | 0.008746 | 0.010558 | 0.098746 | 0.28137 |
| 6.5 | 0.011011 | 0.014746 | 0.12416 | 0.35728 |
| 7 | 0.013967 | 0.01986 | 0.15738 | 0.45564 |
| 7.5 | 0.017801 | 0.026222 | 0.20049 | 0.58262 |
| 8 | 0.022754 | 0.034232 | 0.25619 | 0.74621 |
| 8.5 | 0.029136 | 0.044392 | 0.32799 | 0.95667 |
| 9 | 0.037348 | 0.057341 | 0.42039 | 1.2272 |
| 9.5 | 0.047907 | 0.073892 | 0.53921 | 1.5749 |

Table 2. Comparison of absolute errors for Example 6.1 demonstrates that the errors obtained using the proposed method are significantly smaller than those reported in the previously published study [41], where the modified variational iteration method (mVIM) was applied. These results indicate that the present approach provides improved accuracy and convergence efficiency in solving Fornberg–Whitham–type equations.

| $\varkappa$ | $AB_{\vartheta=1}$ | | $AB_{\vartheta=0.9}$ | | $AB_{\vartheta=0.8}$ | |
|---|---|---|---|---|---|---|
| | YJ–VIM | mVIM | YJ–VIM | mVIM | YJ–VIM | mVIM |
| 5 | 0.005757 | 0.005757 | 0.0039514 | 0.10332 | 0.065314 | 0.17664 |
| 5.5 | 0.0070306 | 0.0070306 | 0.0070339 | 0.13346 | 0.079532 | 0.22897 |
| 6 | 0.008746 | 0.008746 | 0.010558 | 0.17199 | 0.098746 | 0.29568 |
| 6.5 | 0.011011 | 0.011011 | 0.014746 | 0.22132 | 0.12416 | 0.38096 |
| 7 | 0.013967 | 0.013967 | 0.01986 | 0.28455 | 0.15738 | 0.49019 |
| 7.5 | 0.017801 | 0.017801 | 0.026222 | 0.36567 | 0.20049 | 0.63021 |
| 8 | 0.022754 | 0.022754 | 0.034232 | 0.46975 | 0.25619 | 0.80982 |
| 8.5 | 0.029136 | 0.029136 | 0.044392 | 0.60335 | 0.32799 | 1.0403 |
| 9 | 0.037348 | 0.037348 | 0.057341 | 0.77486 | 0.42039 | 1.3362 |
| 9.5 | 0.047907 | 0.047907 | 0.073892 | 0.99504 | 0.53921 | 1.716 |

## 7. Conclusion

In this work, a closed-form series solution for the fractional Fornberg–Whitham equation has been derived using the Yasser–Jassim Variational Iteration Method (YJ–VIM) under the Atangana–Baleanu fractional derivative. The proposed approach provides an explicit analytical representation of the solution together with a systematic procedure for computing successive approximations.

A rigorous convergence analysis has been established by proving that the associated iteration operator is contractive, which guarantees convergence of the solution series in the Hilbert space $H^1$. In addition, an explicit truncation error bound has been obtained, allowing a precise quantitative assessment of the approximation accuracy. Existence and uniqueness of the solution have also been ensured through operator-theoretic arguments.

From a comparative perspective, the proposed YJ–VIM method exhibits clear advantages over the mVIM reported in the literature. While classical and mVIM-based approaches mainly focus on constructing approximate solutions, the present method additionally provides a rigorous convergence theorem, a computable error estimate, and explicit uniqueness conditions. Moreover, the incorporation of the Yasser–Jassim transform simplifies the treatment of the Atangana–Baleanu fractional operator and leads to faster convergence with fewer iterations.

Numerical results confirm the theoretical analysis and demonstrate that the proposed approach achieves high accuracy with reduced computational effort. Overall, the YJ–VIM framework offers a robust and efficient alternative for solving nonlinear fractional partial differential equations. Future work may focus on higher-dimensional problems, more general boundary conditions, or alternative fractional derivatives.

Table 3. List of Abbreviations

| **Notation** | **Comment** |
|---|---|
| YJ | Yasser–Jassim Transform. |
| $\Gamma(\varkappa)$ | Gamma Function. |
| $E_{\vartheta}(t)$ | Mittage-Leffler Function. |
| $H^1$ | Hilbert space. |
| mVIM | modified variational iteration method |
| ABFD | Atangana-Baleanu Fractional Derivative. |
| ABFI | Atangana-Baleanu Fractional Integral. |
| FC | Fractional Calculus. |
| FWE | Fornberg-Whitham Equation. |
| FFWE | Fractional Fornberg-Whitham Equation. |
| FPDE | Fractional Partial Differential Equation. |
| YJ–VIM | Yasser–Jassim Transform-variational iteration method. |

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