STATISTICS, OPTIMIZATION AND INFORMATION COMPUTING
Stat., Optim. Inf. Comput., Vol. **x**, Month 202x, pp 0–8.
Published online in International Academic Press (www.IAPress.org)

Image Recovery from Presence of Salt-and-Pepper Noise using Robust Conjugate Gradient Method

Ayad Abdulaziz Mahmood [1], Saif A. Hussein [2], Dr. Basim A. Hassan [3,*]

[1] *Department of Statistics and Information Techniques Technical, College of Management, Mosul, Northern Technical University*
[2] *Department of Mathematic, College of Computers Sciences and Mathematics, University of Mosul, IRAQ*
[3] *Salladdin in Education Directorate, Tikrit, IRAQ*

Abstract Conjugate gradient techniques concentrate on the conjugate of the coefficient. In this research, we provide a novel coefficient conjugate gradient method to remove impulsive noise from images using the Taylor series. We have presented an intriguing conjugate gradient approach and a search strategy based on the new coefficient conjugate. Under certain conditions, we show that the proposed method converges globally. Our numerical findings show that this method works well for picture restoration.

Keywords Optimization,Robust conjugate gradient, Reducing impulse noise

AMS 2010 subject classifications 65K05, 65F10, 90C06

DOI: 10.19139/soic-2310-5070-3213

1. Introduction

The noisy pixel cleaning model is one of many complex optimization issues in engineering and management that can really be reduced to optimization problems and solved:

$$f_\alpha(\Gamma) = \sum_{(\check{\mathrm{i}},j)\in} \left[\left| \Gamma_{\check{\mathrm{i}},j} - \Gamma_{\check{\mathrm{i}},j} \right| + \frac{\beta}{2}(2 \times Z^1_{\check{\mathrm{i}},j} + Z^2_{\check{\mathrm{i}},j}) \right]. \quad (1)$$

where $Z^1_{\check{\mathrm{i}},j} = 2\sum_{(\Gamma,\Gamma)\in_{\check{\mathrm{i}},j}\cap^c} \phi_\alpha(\Gamma_{\check{\mathrm{i}},j} - \Gamma_{\Gamma,\Gamma})$, $Z^2_{\check{\mathrm{i}},j} = \sum_{(\Gamma,\Gamma)\in_{\check{\mathrm{i}},j}\cap} \phi_\alpha(\Gamma_{\check{\mathrm{i}},j} - \Gamma_{\Gamma,\Gamma})$, see [1, 2, 22]. Image restoration uses, $\Gamma_{\check{\mathrm{i}},j} = \left[\Gamma_{\check{\mathrm{i}},j}\right]_{(\check{\mathrm{i}},j)\in}$ is the picture's pixel value at position $(\check{\mathrm{i}}, j)$, $\Gamma_{\check{\mathrm{i}},j}$ is a column direction of length |N| ordered lexicographically, $\phi_\alpha = \sqrt{\alpha + \Gamma^2}$ is the edge-preserving potential function with parameter $\alpha > 0$, and is the parameter under the right circumstances [2, 21]. The work plan is depicted in the flowchart that follows:

*Correspondence to: Dr. Basim A. Hassan (Email: basimah@uomosul.edu.iq). Department of Mathematic, College of Computers Sciences and Mathematics, University of Mosul, IRAQ.

ISSN 2310-5070 (online) ISSN 2311-004X (print)

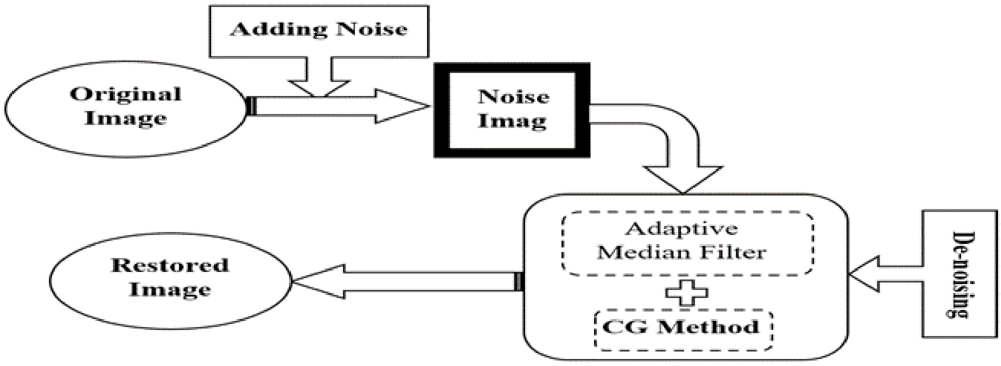

Figure 1. Our experiment's block diagram

Conjugate gradient methods are a good match for optimization issues since they can be readily repeated and need little memory:

$$f(\Gamma^*) = \min_{x \in R^N} f(\Gamma). \tag{2}$$

see [3], where f is a smooth model. In this paper, different types of problems are solved using the nonlinear conjugate gradient technique:

$$\Gamma_{k+1} = \Gamma_k + \alpha_k d_k. \tag{3}$$

Using α_k as a step length, this yields the following search direction [12]:

$$d_{k+1} = -g_{k+1} + \beta_k d_k \tag{4}$$

where robust coefficient is denoted by β_k , (see [4, 5]. The global convergence properties of convolutional gradient algorithms are noteworthy. The best convergence results were obtained using the method suggested by Fletcher and Reeves (FR) [6]. However, one of the best-performing CG algorithms, the Hestenes-Stiefel (HS) technique, failed to satisfy the requirement for global convergence in classical line search [7]. For example, protocols:

$$\beta_k^{HS} = \frac{y_k^T g_{k+1}}{d_k^T y_k}, \ \beta_k^{FR} = \frac{\|g_{k+1}\|^2}{\|g_k\|^2} \tag{5}$$

where $y_k = g_{k+1} - g_k$. In-depth research materials describing modern computer graphics techniques with impressive results may be found in [8]. One of the elements supporting the validity of the Hestenes-Stiefel formula is its ability to satisfy the conjugacy conditions. This is one of the elements that lends the formula's validity the highest level of confidence. As a direct result, it becomes an extremely practical mathematical instrument. You may use these many methods to speed up the Newton technique. The quasi-Newton approach in (4) can be approximated using d_{k+1} in accordance with the quasi-Newton direction principle, yielding the outcome that is most comparable to the other options. A parameter β_k that consequently:

$$-Q_{k+1}^{-1} g_{k+1} = -g_{k+1} + \beta_k \S_k. \tag{6}$$

Nazareth [16] states that Q_{k+1} is a Hessian matrix. This most recent work aims to provide a globally convergent descent search approach [14]. Techniques that are full of ideas, like [9, 10, 17, 18], work well in theory and in practice. The numerical findings show that the novel strategy [1] is more effective than the optimisation method [11]. The step size in [3, 15] is determined by:

$$\alpha_k = -\frac{g_k^T d_k}{d_k^T G d_k} \tag{7}$$

In light of this, we made the decision to test a few more improvements, believing that if they worked, they may improve the numerical performance. Once the conjugate gradient parameter of robust conjugate gradient was determined, the need of the modifications was assessed using a Taylor series.

2. Deriving a new parameter

The Taylor series is a key idea in determining the new conjugate gradient parameter. By expanding the function around the current iterate, the gradient can be articulated as:

$$f(\Gamma) = f(\Gamma_{k+1}) - g_{k+1}^T \S_k + \frac{1}{2} \S_k^T Q(\Gamma_{k+1}) \S_k \tag{8}$$

and:

$$g_{k+1} = g_k + Q(\Gamma_k) \S_k \tag{9}$$

Employing the relations delineated in equations (8) and (9), the second-order curvature information can be approximated as:

$$\S_k^T Q(\Gamma_k) \S_k = 1/2 y_k^T \S_k + (f_{k+1} - f_k) - g_k^T \S_k \tag{10}$$

From this preliminary expression, a more concise and computationally advantageous form can be derived, which isolates the curvature in terms of the gradient projection along $\S_k$:

$$\mathrm{\S_k^T Q(\Gamma_k) \S_k = (g_k^T \S_k)^2 / \left(2y_k^T \S_k + 2(f_k - f_{k+1}) + 2g_k^T \S_k\right)} \tag{11}$$

Through suitable algebraic manipulation, the resultant expression culminates in a matrix formulation of the form:

$$Q(\Gamma_k) = \frac{(g_k^T \S_k)^2 / \left(1/2 y_k^T \S_k + (f_{k+1} - f_k) - g_k^T \S_k\right)}{\S_k^T \S_k} I_n \tag{12}$$

where I_n is the identity matrix. By substituting equations (6) and (12), we arrive at the following compact relation:

$$\beta_k^{BBP} = \left(1 - \frac{\S_k^T \S_k}{(g_k^T \S_k)^2 / \left(1/2 y_k^T \S_k + (f_{k+1} - f_k) - g_k^T \S_k\right)}\right) \frac{g_{k+1}^T y_k}{\S_k^T y_k} \tag{13}$$

which serves as the foundation of the proposed modification.

The BBP Algorithm.

For the sake of simplicity, the proposed methodology is designated as the BBP method, which integrates the newly derived parameter into the classical conjugate gradient framework:

Stage 1. suppose $\Gamma_0 \in R^n$, let $k = 0$.

Stage 2. If $\|g_k\| \le \epsilon$ stop.

Stage 3. In [19, 20]. Compute the α_k using Wolfe conditions:
$f(\Gamma_k + \alpha_{k\,k}) \le f(\Gamma_k) + \delta \alpha_k g_k^T {}_k$,
${}_k^T g(\Gamma_k + \alpha_{k\,k}) \ge \sigma \, {}_k^T g_k$.

Stage 4. Update the parameter β_k using equation (13) and compute the next iterate $\Gamma_{k+1} = \Gamma_k + \alpha_{\Gamma k}$.

Stage 5. Determine new search direction ${}_{k+1} = -g_{k+1} + \beta_{k\,k}$.

Stage 6. Let $k = k + 1$, go to 2.

3. Convergence property

The following presumptions are used to determine the BBP algorithm's global convergence:

i. The iterates remain in a bounded set $\Psi = \{\Gamma \in R^n : f(\Gamma) \leqslant f(\Gamma_0)\}$

ii. Lipschitz Gradient, there exists $L > 0$ as:

$$\|g(\Gamma) - g(\Gamma)\| \leq L\|\Gamma - \Gamma\|, \forall \Gamma, \Gamma \in R^n \tag{14}$$

In this case, the establishment of a stable connection is conditioned on the satisfaction of certain functional assumptions $\Gamma \geqslant 0$, as mentioned in $\|\nabla f(x)\| \leqslant \Gamma$, see [1, 7].

Determine the potential uses for the descent condition by analysing the theorem.

Theorem 1
he BBP algorithm produces search directions that are descent directions, i.e, ${}^T_{k+1} g_{k+1} \leqslant 0$.

Proof
Assume, by induction, we have ${}_0 = -g_0$,which yields $g_0^T {}_0 = -\|g_0\|^2 < 0$. By inference, we assume that for some iteration k, it holds that ${}^T_k g_k \leq 0$. When equation (5) is multiplied by g_{k+1}, the result is:

$$ {}^T_{k+1} g_{k+1} = -g^T_{k+1} g_{k+1} + \left(1 - \frac{\S^T_k \S_k}{\left(g^T_k \S_k\right)^2 / \left(1/2 y^T_k \S_k + (f_{k+1} - f_k) - g^T_k \S_k\right)} \right) \frac{y^T_k g_{k+1}}{\S^T_k y_k} \S^T_k g_{k+1} \tag{15}$$

We may infer from equations (9) and (11) that:

$$\mathrm{\S^T_k y_k = (g^T_k \S_k)^2 / \left(2y^T_k \S_k + 2\left(f_k - f_{k+1}\right) + 2g^T_k \S_k\right)} \tag{16}$$

We obtain $y^T_k g_{k+1} \leqslant L\S^T_k g_{k+1}$ and $\S^T_k y_k \leqslant L \ \S^T_k \S_k$ by applying the Lipschitz continuity condition to the gradient. These boundaries can be substituted into the preceding equation to obtain:

$$ {}^T_{k+1} g_{k+1} \leqslant -\|g_{k+1}\|^2 + \left(\frac{L\S^T_k \S_k - \S^T_k \S_k}{\S^T_k y_k} \right) \frac{L\left(\S^T_k g_{k+1}\right)^2}{\S^T_k y_k} \tag{17}$$

The second term is insignificant since L and α_k^2 are both small enough, so:

$$ {}^T_{k+1} g_{k+1} \leqslant 0 \tag{18}$$

Therefore, ${}_{k+1}$ is a descent direction. □

In accordance with Dai et al. [8], the convergence features of any conjugate gradient approach utilising the Wolfe-conditions may be expressed as follows.

Lemma 1
Examine the conjugate gradient approach, which is described by equation (16). Under the following circumstances, the produced search direction $d_{k+1} = -g_{k+1} + \beta_k \S_k$, is guaranteed to be a downward direction by using the strong Wolfe line search:

$$\sum\nolimits_{k>1} \frac{1}{\|{}_{k+1}\|^2} = \infty \tag{19}$$

Then:

$$\underset{k\to\infty}{lim} \left(\inf \|g_{k+1}\|\right) = 0 \tag{20}$$

The convergence property may be proved using Lemma 1, which offers a theoretical basis for the findings presented in [4, 8, 13].

Theorem 2
Assume that $\Gamma > 0$ is a constant and that $\{g_{k+1}\}$ is the gradient sequence produced by the BBP algorithm. This way,

$$\mathrm{(\nabla f(\Gamma) - \nabla f(\Gamma))^T \geqslant \Gamma\|\Gamma - \Gamma\|^2, \forall \Gamma, \Gamma \in R^n} \tag{21}$$

The sequence then fulfils:

$$\lim_{\mathrm{k}\to\infty} \left(\inf \left\|\mathrm{g_{k+1}}\right\|\right) = 0 \tag{22}$$

Proof
Using equations (5) and (13), it follows that the update formulas yield:

$$\|_{k+1}\| = \|g_{k+1}\| + \left|(1-\omega)\,\frac{g_{k+1}^T y_k}{\S_k^T y_k}\right| \|\S_k\| \tag{23}$$

where $\omega = (g_k^T \S_k)^2 / \left(1/2 y_k^T \S_k + (f_{k+1} - f_k) - g_k^T \S_k\right)$. To make use of Cauchy's inequality,

$$\|_{k+1}\| \leqslant \|g_{k+1}\| + |(1-\omega)|\,\frac{\|g_{k+1}\|\,\|y_k\|}{\|\S_k\|\,\|y_k\|}\,\|\S_k\| \leqslant (2-\omega)\,\|g_{k+1}\| \tag{24}$$

As a result, $\|\nabla f(x)\| \leqslant \Gamma$ gives us:

$$\sum\nolimits_{k\geqslant 1} \frac{1}{\|_k\|^2} \geqslant \left(\frac{1}{2-\omega}\right)\frac{1}{\Gamma}\sum\nolimits_{k\geqslant 1} 1 = \infty \tag{25}$$

Lemma 1 leads us to the conclusion that $\underset{k\to\infty}{lim} \inf \|g_k\| = 0$. □

4. Numerical Results

The effectiveness of the BBP algorithm in reducing salt-and-pepper impulse noise is assessed in this work. Table 1, which also shows the research procedure and the first test photographs, summarises several algorithm settings and methodologies. On a personal computer, MATLAB 2015a was used for all calculations. Comparisons between the FR technique and the BBP-based approaches are made. The primary goal is to identify practical methods for cutting carbon emissions, and the Signal-to-Noise Ratio (PSNR) is used to measure:

$$PSNR = 10\log_{10}\frac{255^2}{\frac{1}{MN}\sum_{\check{\imath},j}\left(\Gamma_{\check{\imath},j}^r - \Gamma_{\check{\imath},j}^*\right)^2} \tag{26}$$

where $\Gamma_{\check{\imath},j}^r$ and $\Gamma_{\check{\imath},j}^*$ represent the pixel values of the original and restored images, respectively. Both algorithms must meet certain convergence requirements in order to stop running:

$$\frac{|f(\Gamma_k) - f(\Gamma_{k-1})|}{|f(\Gamma_k)|} \leqslant 10^{-4} \text{ and } \|f(\Gamma_k)\| \leqslant 10^{-4}\left(1 + |f(\Gamma_k)|\right) \tag{27}$$

Both algorithms need the satisfaction of certain convergence conditions in order to stop running. The results of the trials are shown in Table 1, which includes the number of function evaluations (NF), iterations (NI), and Peak Signal-to-Noise Ratio (PSNR). Several research have looked at this subject from different angles [23, 24, 25, 26, 27, 28] in an effort to support the investigation's underlying theoretical framework. Recent advancements have been observed in the domain of Quasi-Newton methods, as evidenced by the citations [29, 30, 31, 32].

Table 1. Performance Results of FR and BBP Methods.

Image	**Noise**	**FR-Method**			**BBP-Method**		
		NI	**NF**	**PSNR (dB)**	**NI**	**NF**	**PSNR (dB)**
Le	50%	82.0	153.0	30.5529	55.0	111.0	30.6783
	70%	81.0	155.0	27.4824	52.0	104.0	27.2918
	90%	108.0	211.0	22.8583	52.0	104.0	22.8430
Ho	50%	52.0	53.0	30.6845	37.0	75.0	34.9332
	70%	63.0	116.0	31.2564	34.0	68.0	31.2440
	90%	111.0	214.0	25.2870	48.0	96.0	25.0390
El	50%	35.0	36.0	33.9129	27.0	53.0	33.8902
	70%	38.0	39.0	31.8640	30.0	59.0	31.8778
	90%	65.0	114.0	28.2019	40.0	79.0	28.2288
c512	50%	59.0	87.0	35.5359	28.0	55.0	35.1534
	70%	78.0	142.0	30.6259	31.0	61.0	30.7814
	90%	121.0	236.0	24.3962	40.0	79.0	24.8811

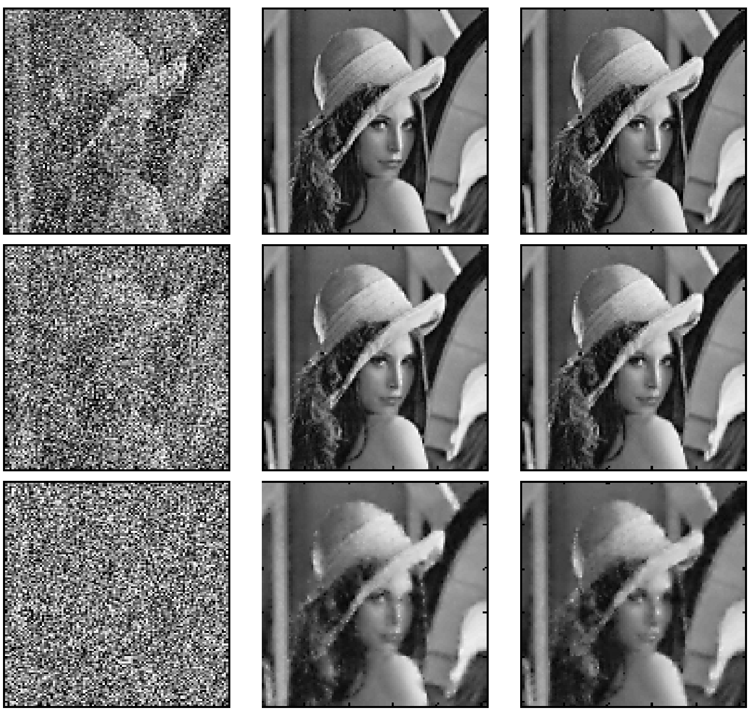

Figure 2. Fallouts of the FR and BBP algorithms for the 256×256 Lena picture are shown

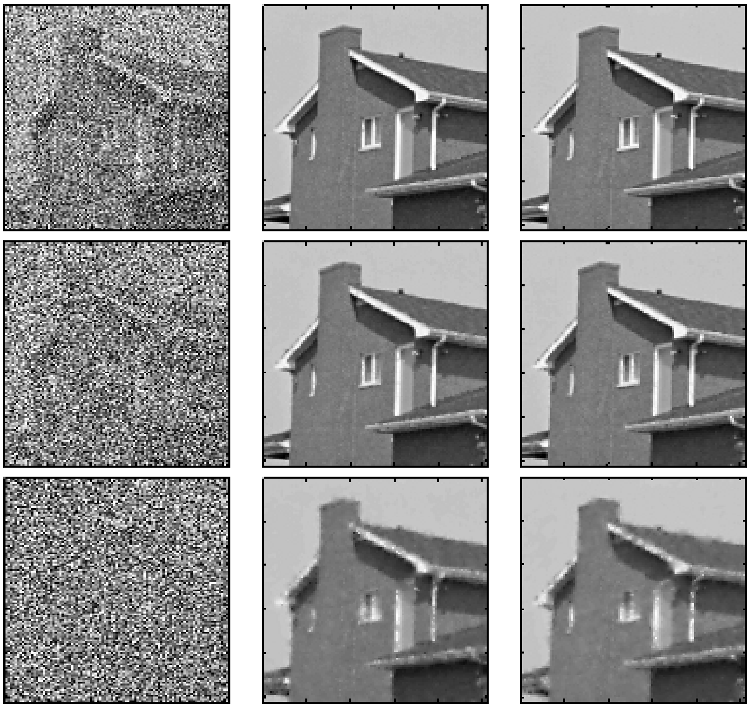

Figure 3. Fallouts of the FR and BBP algorithms for the 256×256 House picture are shown

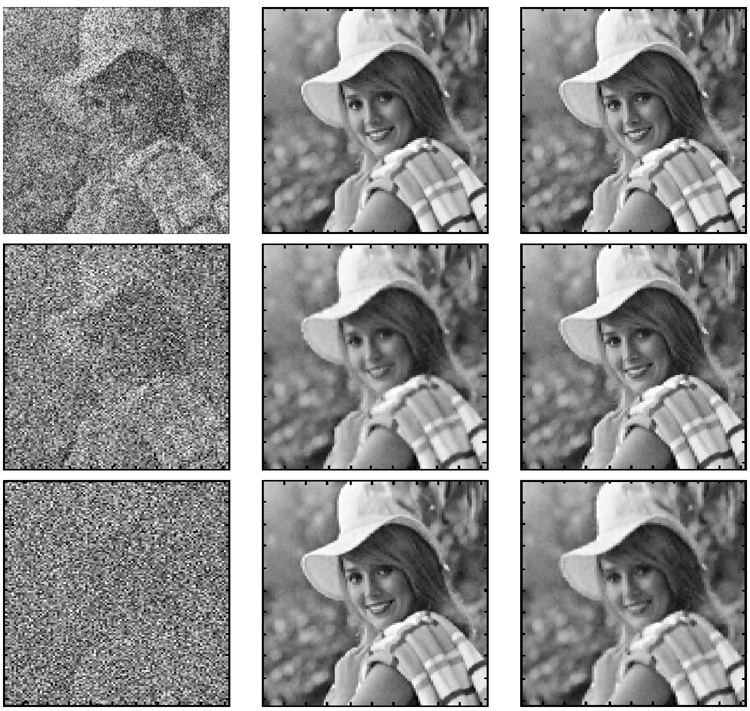

Figure 4. Fallouts of the FR and BBP algorithms for the 256×256 Elaine picture are shown

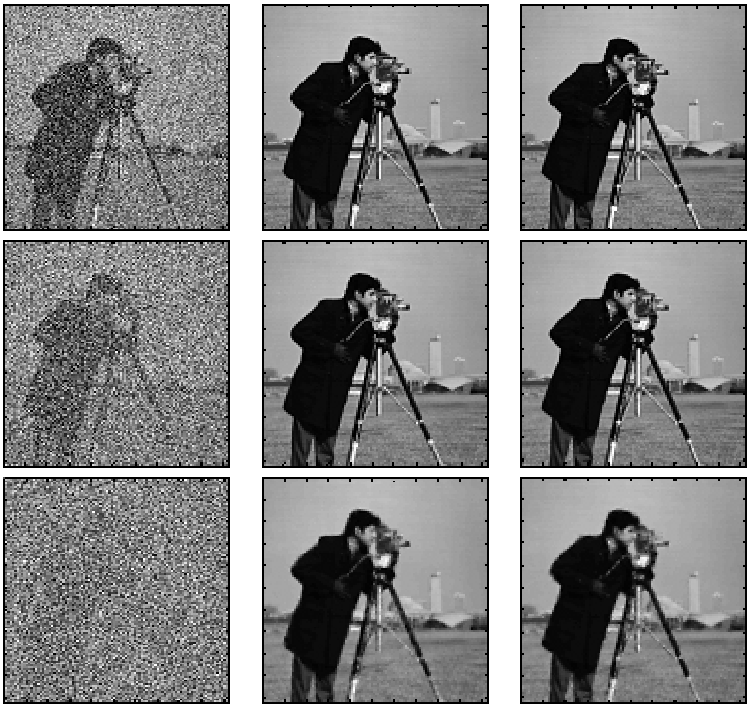

Figure 5. Fallouts of the FR and BBP algorithms for the 256×256 Cameraman picture are shown

5. Conclusions

In conclusion, we have combined the BBP approach with robust conjugate gradient equation. We were able to ascertain its worldwide convergence by employing the line search parameters. The results of the simulations have shown that BBP has the capability to drastically reduce the amount of function computations and iterations while keeping image quality.

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