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Representation of Solutions of Partial Differential Equations by Contour Integrals in Two-Dimensional Complex Space

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Abstract The study aimed to develop an efficient method for obtaining solutions to partial differential equations (PDEs), particularly equations with polynomial coefficients and Helmholtz-type equations. The methodology employed the representation of solutions as series whose terms are contour integrals in a two-dimensional complex space, allowing, in some cases, reduction to special functions of two variables. This two-dimensional complex spatial approach overcomes the limitations of single-variable methods, such as the Fokas method, by explicitly capturing interactions between variables and fully accounting for the analytic and geometric structure of the equations. The results demonstrate that the proposed scheme provides stable and convergent solutions, ensures accurate representations of Helmholtz-type equations in terms of Bessel functions, and significantly simplifies the analysis and practical application of such PDEs. Moreover, the method allows for explicit construction of boundary value problem solutions in half-plane, strip, and circular domains, leveraging variable substitution and analytic function techniques, and offers potential for extension to more complex systems. The study confirms that the two-dimensional integral approach enhances both the universality and efficiency of PDE solutions, providing a robust framework beyond the constraints of single-variable integral methods.

Keywords Differential equations with polynomial coefficients, Helmholtz-type equations, special functions, contour integrals, analytical solutions

AMS 2010 subject classifications 35C15, 32A25, 35A22

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1. Introduction

In mathematical and physical research, partial differential equations (PDEs) play a central role as models for describing a wide range of physical and engineering processes, including wave propagation, diffraction, heat transfer, and other phenomena with complex geometric and physical characteristics. Solving such equations is particularly challenging in the context of complex spatial domains, where classical single-variable methods often prove inefficient or inadequate. The two-dimensional complex spatial approach overcomes the limitations of single-variable methods, such as the Fokas method, by explicitly incorporating interactions between two independent variables, which allows the construction of contour integral representations that fully capture the analytic and geometric structure of the problem. This approach enables more general and accurate solutions, effectively addresses PDEs with polynomial coefficients or Helmholtz-type equations, and simplifies the analysis of complex differential equations. By leveraging series expansions whose terms are expressed as contour integrals in two-dimensional complex space, the method provides a unified framework for obtaining exact or highly precise approximate solutions, extending the practical applicability of integral methods beyond the restrictions inherent in one-dimensional approaches.

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This work extends beyond classical contour integral methods, such as those proposed by Mackie [1] and the Fokas method [2], by introducing a framework that systematically combines series representations with multi-dimensional contour integrals in two-dimensional complex space, allowing the kernel (K sub n) and functions (phi sub n) to interact in a more flexible and analytically tractable way. Unlike earlier approaches, which often relied on one-dimensional integrals or highly specific boundary conditions, the present formulation accommodates more general domains and variable coefficient partial differential equations, providing explicit control over convergence through the analytic properties of (phi sub n) and the structure of the contour (S). This development opens new mathematical possibilities for constructing singular or fundamental solutions that are directly linked to the geometry of the problem, and offers practical advantages in numerical implementation by enabling error estimation and adaptive truncation strategies.

The problem of solving differential equations with polynomial coefficients has been widely investigated through analytical, numerical, and semi-analytical approaches; however, many existing techniques remain restricted in their ability to capture multi-dimensional and complex-valued structures. Ahmed [3] introduced a novel numerical framework for ordinary differential equations with polynomial coefficients that systematically handles nonlinear products of unknown functions and their derivatives, thereby improving convergence and computational robustness beyond traditional schemes. Nadeem et al. [4] extended the variational iteration method by incorporating Laplace transforms, resulting in the ML-VIM approach, which enables efficient and accurate solutions of high-order parabolic equations with variable coefficients while reducing computational cost. Within the domain of fractional calculus, Saad et al. [5] developed reliable numerical techniques for fractional biological models, with particular emphasis on equations governed by the Atangana–Baleanu derivative, allowing for a more realistic description of memory and hereditary effects. From a theoretical perspective, Bryenton et al. [6] derived necessary and sufficient conditions for the existence of polynomial solutions to linear differential equations with polynomial coefficients of orders n, n-1, and n-2, presenting a constructive approach that is both mathematically transparent and directly applicable to Hoyne and Dirac equations in theoretical physics. Sarhan et al. [7] advanced polynomial-based methodologies by introducing two-dimensional Pell polynomials, establishing their core properties, and employing them within a direct spectral method for partial differential equations, thereby enhancing accuracy in multidimensional problems. Further computational developments were proposed by Singh [8] and Srivastava [9], who formulated a flexible framework for approximating nonlinear fractional Riccati differential equations of arbitrary order through the integration of operational matrices and Jacobi polynomial collocation techniques. Samadyar and Mirzaee [10] presented a matrix-based algorithm relying on orthogonal Bernoulli polynomials to approximate solutions of fractional integro-differential equations, demonstrating the broad applicability of polynomial expansions across scientific disciplines. At a structural level, Duff et al. [11] investigated general polynomial systems with parametric coefficients and characterised their solution spaces using monodromy theory and decorated graphs, providing deeper insight into the global behaviour of such systems. Aldoghaither and Laleg-Kirati [12] proposed a reduction technique that transforms two-dimensional fractional differential equations into equivalent algebraic systems through the use of modulating functions, thereby simplifying numerical treatment. Finally, Huang et al. [13] developed an implicit alternating direction scheme for the efficient numerical solution of two-dimensional nonlinear diffusion-wave equations with fractional derivatives in both time and space. Despite these advances, the existing literature predominantly concentrates on real-valued numerical or algebraic formulations and offers limited consideration of complex geometric features. Consequently, the representation of partial differential equation solutions as contour integrals in two-dimensional complex spaces remains insufficiently addressed, highlighting an important gap for future research.

Comparatively, Fokas' method and other integral transform techniques offer systematic ways to solve certain boundary value problems, yet they typically rely on one-dimensional constructs or predefined transform kernels and are less generalizable to multi-variable, multi-dimensional complex domains. In contrast, the proposed approach explicitly constructs solutions as series of contour integrals in a two-dimensional complex space, leveraging special functions of two variables and providing a framework that accommodates Helmholtz-type equations and polynomial coefficients. This enables the method to capture geometric intricacies and boundary effects more naturally, offering precise analytical representations where classical methods may only yield approximate or numerical solutions. Consequently, the present work extends beyond the limitations of prior approaches, providing both a more general theoretical framework and practical computational advantages, while opening new avenues for applications in mathematical physics, engineering, and multi-dimensional PDE analysis.

2. Materials and Methods

In this study, the representation of partial differential equation (PDE) solutions by integrals in a two-dimensional complex space was based on the properties of differential forms, providing a consistent framework for handling equations with multiple interacting variables. Classical and numerical methods were combined with the theory of functions of a complex variable and techniques of two-dimensional contour integration, providing a unified notation for both series terms and integral kernels to simplify presentation and improve readability.

The Helmholtz and Bessel equations served as primary examples to illustrate the method, as these equations naturally describe wave propagation and other phenomena where complex spatial interactions are significant, and their solutions demonstrate the advantages of the two-dimensional approach. Dirichlet boundary conditions were employed to define constraints for the PDEs, and explanatory remarks were inserted between key equations to clarify the derivation of series representations and the role of contour integrals. A schematic or block diagram of the solution procedure can further enhance understanding, showing the stepwise transition from the original PDE to the series representation, the evaluation of integrals, and the assembly of the final solution. This structured approach not only unifies the notation and simplifies derivations but also facilitates practical implementation and extension to more complex PDE systems.

All mathematical operations were performed within the framework of the classical theory of differential equations and the theory of integrals. The sample for the study included functions describing physical and mathematical processes associated with two-dimensional complex spaces, in particular, in the fields of mechanics, electrodynamics and quantum field theory. The choice of such functions was focused on the general problem of mathematical modelling in the context of partial differential equations, which generalised the results for various application situations.

The theoretical basis of the work was the contour integrals, which describe the behaviour of functions in a two-dimensional complex space, in particular for hyperbolic equations. Various methods were used to construct solutions, among which special attention was paid to the method of contour integrals in terms of variables ξ_1 and ξ_2, which provided solutions in the form of series whose terms are integrals.

The contour integral method for representing solutions of partial differential equations extends the classical approaches proposed by Mackie [1], while offering new opportunities for constructing solutions in two-dimensional complex space. Compared to Mackie's classical method, which relied on individual kernels and straightforward integration contours, the modern approach allows for multi-contour integrals and accommodates more complex domain geometries, providing greater flexibility and accuracy in solution representation. Fokas's method, focused on fundamental solutions and integral representations for specific cases, was limited to particular types of equations and singular points; the contemporary development generalizes these representations to a broader class of problems, including complex boundary conditions and interactions between multiple system components. Thus, the modern method not only retains the advantages of classical contour approaches but also opens new mathematical and practical possibilities, particularly for precise modeling of complex physical and engineering processes where interactions of multiple variables and singular effects in two-dimensional complex space must be taken into account.

The algorithmic representation of the proposed method involved a step-by-step construction of solutions using contour integrals in two-dimensional complex space, ensuring systematic application of the integration kernel and factor functions. Initially, the domain of definition was analyzed and the analyticity of the initial data was verified, allowing an assessment of the method's applicability and the selection of appropriate integration contours. Next, a sequence of factor functions was determined to expand the solution into a series, and the integration kernels were chosen to ensure convergence and compliance with the boundary conditions. Subsequently, the contour integrals for each series element were computed, while monitoring error magnitudes and the stability of results. In the final step, all computed elements were integrated to form the complete solution, which was then verified against the initial and boundary conditions, and the choice of contours and numerical parameters was optimized to enhance accuracy and efficiency. Presenting this procedure in a numbered format or pseudocode improves reproducibility and facilitates practical implementation of the method for various types of partial differential equation problems.

The study successfully constructed analytical solutions and determined the main characteristics of the behaviour of such solutions in a two-dimensional complex space. The use of contour integrals provided new results that have significant potential for application in various scientific and engineering fields. To demonstrate the practical applicability of the proposed method, a specific boundary value problem was considered, namely the Helmholtz-type equation in a unit disk with a simple Dirichlet boundary condition. The solution obtained in the form of a series of contour integrals was applied to this problem, showing how the abstract integral representation can be concretely evaluated to satisfy boundary constraints. In particular, the series solution, expressed through functions denoted as U_k and V_k, was substituted into the Helmholtz equation with polynomial coefficients, allowing verification that the series converges and satisfies both the differential equation and the Dirichlet condition on the boundary. This example highlights the utility of the method for practical PDE problems, demonstrating that the theoretical constructions can be effectively implemented to obtain explicit, accurate solutions in a geometrically simple domain while preserving the advantages of the two-dimensional complex space approach. The procedure confirms that the contour integral representation not only provides a systematic framework for constructing solutions but also ensures convergence and stability when applied to real boundary value problems, thus bridging the gap between abstract theory and practical computation.

3. Results

Algorithmic flowchart. For clarity, the stepwise procedure of the proposed method is summarized in Fig. 1 (Appendix A). The flowchart outlines the logical sequence from problem formulation to solution verification, emphasizing the iterative nature of contour and kernel selection. The application of the methods of the theory of functions of a complex variable to the solution of boundary value problems and, accordingly, to the representation of solutions by contour integrals makes it possible to obtain solutions in an explicit form or in an integral form. The use of differential forms allows us to reformulate boundary value problems in the form of integral equations [14, 15], which often simplifies the analysis and enables the use of various methods of integral calculus to find solutions. Let us consider the differential form of order k (1):

$$Q_z(u) = \sum_{i=0}^{k} \sum_{l=0}^{k-i} f_i^l(z_1, z_2) \frac{\partial^{i+l} u}{\partial z_1^i \partial z_2^l}, \tag{1}$$

where $f_i^l(z_1, z_2)$ are continuous functions of two variables (real or complex) with continuous partial derivatives up to order k in the domain D.

Let us multiply this form by the function $v = v(z_1, z_2)$, which is continuous and has continuous partial derivatives up to order k in the domain D and then integrate over the variables z_1 and z_2. By performing the integration, an expression is obtained that includes the products of functions and their derivatives. Using the integration formula by parts (2):

$$\begin{aligned}\iint vf_i^l \frac{\partial^{i+l}u}{\partial z_1^i \partial z_2^l} dz_1 dz_2 =&(-1)^{i+l} \iint \frac{\partial^{i+l}(vf_i^l)}{\partial z_1^i \partial z_2^l} u\, dz_1 dz_2 \\ &+ (-1)^i \int \sum_{p=0}^{l-1} (-1)^p \frac{\partial^{i+p}(vf_i^l)}{\partial z_1^i \partial z_2^p} \frac{\partial^{l-p-1}u}{\partial z_2^{l-p-1}} dz_1 \\ &+ (-1)^l \int \sum_{j=0}^{i-1} (-1)^j \frac{\partial^{j+l}(vf_i^l)}{\partial z_1^j \partial z_2^l} \frac{\partial^{i-j-1}u}{\partial z_1^{i-j-1}} dz_2 \\ &+ (-1)^{i+l} \sum_{j=0}^{i-1} \sum_{p=0}^{l-1} \frac{\partial^{j+p}(vf_i^l)}{\partial z_1^j \partial z_2^p} \frac{\partial^{i+l-j-p-2}u}{\partial z_1^{i-j-1} \partial z_2^{l-p-1}},\end{aligned} \tag{2}$$

write down the Dirichlet formula (3) [16]:

$$\iint vQ_z(u)\, dz_1 dz_2 = \iint uQ_z^*(v)\, dz_1 dz_2 + \int M_1(u,v)\, dz_1 + \int M_2(u,v)\, dz_2 + M(u,v), \tag{3}$$

where $Q_z^*(v) = \sum_{i=0}^{k} \sum_{l=0}^{k-i} (-1)^{i+l} \frac{\partial^{i+l}(vf_i^l)}{\partial z_1^i \partial z_2^l}$ is the adjoint form to $Q_z(u)$, and Expressions (4-6) are bilinear differential forms:

$$M_1(u,v) = \sum_{i=0}^{k} \sum_{l=1}^{k-i} \sum_{p=0}^{l-1} (-1)^{i+p} \frac{\partial^{i+p}(vf_i^l)}{\partial z_1^i \partial z_2^p} \frac{\partial^{l-p-1}u}{\partial z_2^{l-p-1}}, \tag{4}$$

$$M_2(u,v) = \sum_{i=1}^{k} \sum_{l=0}^{k-i} \sum_{j=0}^{i-1} (-1)^{l+j} \frac{\partial^{l+j}(vf_i^l)}{\partial z_1^j \partial z_2^l} \frac{\partial^{i-j-1}u}{\partial z_1^{i-j-1}}, \tag{5}$$

$$M(u,v) = \sum_{i=1}^{k} \sum_{l=1}^{k-i} \sum_{j=0}^{i-1} \sum_{p=0}^{l-1} (-1)^{i+l} \frac{\partial^{j+p}(vf_i^l)}{\partial z_1^j \partial z_2^p} \frac{\partial^{i+l-j-p-2}u}{\partial z_1^{i-j-1} \partial z_2^{l-p-1}}, \tag{6}$$

Differentiating equality (3) first by the variable z_1 and then by the variable z_2, the Lagrange identity is obtained. This process of double differentiation identifies the relationship between the partial derivatives of functions (7):

$$vQ_z(u) - uQ_z^*(v) = \frac{\partial M_1(u,v)}{\partial z_1} + \frac{\partial M_2(u,v)}{\partial z_2} + \frac{\partial^2 M(u,v)}{\partial z_1 \partial z_2}. \tag{7}$$

The use of identity (7) allows us to represent solutions of the PDE in the form of contour integrals. To effectively solve PDEs, it is necessary to consider the solutions in the form of series or integrals, as this approach can significantly simplify calculations and provide more accurate and generalised solutions. Switching to series or integrals helps to overcome the difficulties that arise when solving such equations directly, especially when they have complex, nonlinear or variable coefficients. The use of series expansions can approximate complex functions by transforming them into computationally friendly expressions, which greatly facilitates further analytical processing. Integral representations, in turn, can be used to apply versatile integration methods that create new ways to solve problems describing physical and mathematical processes, in mechanics, thermodynamics, electrodynamics and other fields of science.

Given the above approaches, the next step is to consider specific equations, in particular homogeneous PDEs that describe physical or mathematical processes that depend on two variables. Next, we consider a homogeneous PDE of order k, which describes a certain physical or mathematical process depending on two variables (8):

$$L_x(u) \equiv \sum_{i=0}^{k} \sum_{l=0}^{k-i} f_i^l(x_1,x_2) \frac{\partial^{l+i}u}{\partial x_1^i \partial x_2^l} = 0, \tag{8}$$

where $f_i^l(x_1, x_2)$ are continuous functions with continuous partial derivatives up to order k in the domain $\Omega \subset \mathbb{R}^2$.

The solutions to equation (8) will be found in the following form (9):

$$u(x_1, x_2) = \sum_{n=0}^{\infty} \int_S K_n(x_1, x_2, z_1, z_2)\, \varphi_n(z_1, z_2)\, dz_1 dz_2, \tag{9}$$

where S denotes a surface in $\mathbb{C}^2$, $\varphi_n(z_1, z_2)$ are continuous functions with continuous partial derivatives up to order k on the surface S, $K_n(x_1, x_2, z_1, z_2)$ are continuous functions with continuous partial derivatives of order k in variables x_1 and x_2 in some domain $\Omega \subset \mathbb{R}^2$ for any fixed values $(z_1, z_2) \in S$.

For the partial sum of series (9) containing N terms, the remainder (9a):

$$\begin{aligned} &R_N(x) = u(x) - \sum_{n=0}^{N} \int_S K_n \varphi_n \, dz, \quad |K_n(x, z)\varphi_n(z)| \le Cq^n, \quad 0 < C,\ 0 < q < 1,\ \forall z \in S \\ &\Rightarrow \quad |R_N(x_1, x_2)| \le \frac{C\, A(S)\, q^{N+1}}{1 - q}, \end{aligned} \tag{9a}$$

where $A(S)$ is the area (measure) of the integration contour S. This estimate demonstrates the exponential convergence rate of the method given an appropriate choice of the contour and functions.

The analysis of convergence and error estimates for the representation of solutions of partial differential equations through contour integrals is based on the properties of the kernel K_n and the functions φ_n in the sum, as well as the contour S over which the integration is performed. Convergence of the series is ensured if the contributions of the integral terms decrease sufficiently rapidly with increasing n, and if the functions φ_n are analytic in the complex region enclosing the contour S, while the kernel K_n appropriately limits the magnitude of each term in the sum. Theoretically, under these conditions, the contour integrals yield a convergent series, and the truncation error after a finite number of terms can be estimated using the upper bound of the product of K_n and φ_n on the contour and the decay rate of φ_n with n.

The method may become ineffective or lose accuracy if the functions φ_n contain poles or discontinuities near the contour S, if the contour S is chosen improperly relative to the features of the domain, or if the kernel K_n does not provide sufficient decay for large n, leading to slow convergence and accumulation of numerical errors. Therefore, the efficiency of the method strongly depends on the coordinated choice of the contour S, the kernel K_n, and the analytic properties of φ_n, as well as on the geometry of the domain and the characteristics of the solution considered in constructing the contour integral.

The study suggested that when applying differentiation operators to functions included in integrals, it is possible to interchange these operators with the integration operator, provided that the relevant conditions for their existence are met [17]. Hence, equality (10) holds:

$$\frac{\partial^{l+j}}{\partial x_1^l \partial x_2^j} \int_S K_n(x_1, x_2, z_1, z_2)\, \varphi_n(z_1, z_2)\, dz_1 dz_2 = \int_S \frac{\partial^{l+j} K_n(x_1, x_2, z_1, z_2)}{\partial x_1^l \partial x_2^j}\, \varphi_n(z_1, z_2)\, dz_1 dz_2, \tag{10}$$

where $0 \le l + j \le k$.

To construct the solution of (9), we select the differential forms:

$$Q_{z,m}^n(v) = \sum_{r=0}^{k} \sum_{l=0}^{k-r} g_{m,l,r}^n(z_1, z_2) \frac{\partial^{r+l} v}{\partial z_1^r \partial z_2^l}, \tag{11}$$

so that equality (12) holds:

$$L_x(K_n) = \sum_{m=n}^{\infty} Q_{z,m}^n(K_m), \tag{12}$$

where $g_{m,l,r}^n$ are polynomials.

Next, let us apply the identity (7) to the differential forms $Q_{z,m}^n(K_m)$. It is obtain (13):

$$\varphi_n Q^n_{z,m}(K_m) = K_m Q^{n*}_{z,m}(\varphi_n) + \frac{\partial M_1(K_m,\varphi_n)}{\partial z_1} + \frac{\partial M_2(K_m,\varphi_n)}{\partial z_2} + \frac{\partial^2 M(K_m,\varphi_n)}{\partial z_1 \partial z_2}, \tag{13}$$

Now it is possible to calculate the operator $L_x(u)$. To do this, the solutions of (9) was substituted into equation (8) and relations (12) and (13) were used.

The method of representing solutions of partial differential equations (PDEs) through contour integrals in two-dimensional complex space is based on the idea of transferring information about initial or boundary conditions along a carefully chosen contour S using the integral kernel K_n, which determines the weight of each point on the contour in forming the solution. The functions φ_n capture the local properties of the system and may include singular solutions that appear at points with concentrated effects or special features of the domain, such as corners or isolated critical points. These singular solutions are not always fundamental in the classical sense, but they allow an accurate description of local behaviors and critical points of the solution, reflecting the influence of boundary conditions and the shape of the domain on system behavior.

The geometry of the problem directly determines the location and nature of these singularities, while the contour S and kernel K_n provide the integral combination of local effects into a global solution. This approach offers both analytical precision and intuitive understanding of how local features propagate influence throughout the domain, combining the properties of boundary conditions, domain geometry, and the integral structure of the solution into a coherent framework for analyzing partial differential equations, where S indicates the integration path, K_n represents the weighting function for each point along S, and φ_n denotes the local or singular solutions contributing to the global behavior.

It is important to assume that the operations of integration, differentiation, and summation can be interchanged without loss of generality, which makes it possible to simplify the expression $L_x(u)$ and obtain a convenient form for further calculations [18]. The expression for the operator $L_x(u)$ (14) is obtained:

$$\begin{aligned}
L_x(u) &= \sum_{n=0}^{\infty} \int_S L_x(K_n)\, \varphi_n \, dz_1 dz_2 \\
&= \sum_{n=0}^{\infty} \sum_{m=n}^{\infty} \int_S Q^n_{z,m}(K_m)\, \varphi_n \, dz_1 dz_2 \\
&= \sum_{n=0}^{\infty} \sum_{m=n}^{\infty} \int_S K_m \, Q^{n*}_{z,m}(\varphi_n)\, dz_1 dz_2 \\
&\quad + \sum_{n=0}^{\infty} \sum_{m=n}^{\infty} \int_S \left[\frac{\partial M_1(K_m,\varphi_n)}{\partial z_1} + \frac{\partial M_2(K_m,\varphi_n)}{\partial z_2} + \frac{\partial^2 M(K_m,\varphi_n)}{\partial z_1 \partial z_2} \right] dz_1 dz_2 \\
&= \sum_{m=0}^{\infty} \int_S K_m \sum_{n=0}^{m} Q^{n*}_{z,m}(\varphi_n)\, dz_1 dz_2 \\
&\quad + \sum_{m=0}^{\infty} \sum_{n=0}^{m} \int_S \left[\frac{\partial M_1(K_m,\varphi_n)}{\partial z_1} + \frac{\partial M_2(K_m,\varphi_n)}{\partial z_2} + \frac{\partial^2 M(K_m,\varphi_n)}{\partial z_1 \partial z_2} \right] dz_1 dz_2 .
\end{aligned} \tag{14}$$

For further analysis, it was assumed that the set of functions $\varphi_n(z_1, z_2)$ must satisfy certain conditions, which will simplify the expression for the operator $L_x(u)$. The choice of such functions is an important step, as these functions must satisfy the equations that determine their interaction with other model components.

In particular, the functions $\varphi_n(z_1, z_2)$ are chosen so that they satisfy equation (15):

$$\sum_{n=0}^{m} Q^{n*}_{z,m}(\varphi_n) = 0, \quad n = 0, 1, \ldots \tag{15}$$

Taking into account relation (15), equation (8) will take on a simpler form (16):

$$\sum_{m=0}^{\infty}\sum_{n=0}^{m}\int_S\left[\frac{\partial M_1(K_m,\varphi_n)}{\partial z_1}+\frac{\partial M_2(K_m,\varphi_n)}{\partial z_2}+\frac{\partial^2 M(K_m,\varphi_n)}{\partial z_1\partial z_2}\right]dz_1dz_2=0. \tag{16}$$

Thus, the representation of the solutions of equation (8) in the form of (9) is reduced to the construction of identity (12), solving the system of equations (15) and establishing equality (16). At this stage, it is necessary to complete the mathematical transformations that will ensure the consistency and correctness of the equations that appear in the process of solving the problem. After completing these steps, which include checking for consistency and accuracy, the next step is to present the solutions of the partial differential equation with polynomial coefficients.

The general PDE serves as a basis for modelling a wide range of phenomena. Such equations are often used to model various phenomena where the unknown functions depend on several variables, and the coefficients of the equation can be functions of these variables. In this case, a differential equation with polynomial coefficients is obtained (17):

$$L_x(u)\equiv\sum_{r=0}^{k}\sum_{q=0}^{k-r}f_{r,q}(x_1,x_2)\frac{\partial^{r+q}u}{\partial x_1^r\partial x_2^q}+f(x_1,x_2)u=0, \tag{17}$$

The presence of polynomial coefficients $f_{r,q}(x_1,x_2)$ results in the kernel K_n and the functions φ_n having to satisfy differential equations with variable coefficients (15)–(16). This complicates the construction of explicit φ_n, but also allows the geometric features of the domain and boundary conditions to be taken into account through an appropriate choice of the contour S.

Equality (18, 19) denote homogeneous polynomials:

$$f_{r,q}(x_1,x_2)=\sum_{m=0}^{r+q}a_m^{r,q}\,x_1^m x_2^{r+q-m}, \tag{18}$$

$$f(x_1,x_2)=\sum_{r=0}^{s}b_{r,s-r}\,x_1^r x_2^{s-r}, \tag{19}$$

The problem of solving a differential equation that contains complex operators and depends on several variables was considered [19]. For this type of equation, series expansion methods are often used to find solutions in the form of infinite sums of integrals. In this case, it is necessary to search for a solution to equation (17) in the form of a functional series, where each term of the series depends on integrals over certain variables. This allows to take into account the complexity of the equation and ensure its exact or approximate solution in general. This form of the solution should be as follows (20):

$$u_0(x_1,x_2)=\sum_{n=0}^{\infty}\int_S\xi^n\,\varphi_n(z_1,z_2)\,dz_1dz_2, \tag{20}$$

where $\xi=\omega+x_1z_1+x_2z_2$ ($\omega\neq0$) is a real or complex number, S is a closed surface, $\varphi_n(z_1,z_2)$ are continuous functions with continuous partial derivatives of order k on the surface S.

For further analysis of solutions to equation (17), the method of substituting solutions of the form (20) into this equation was used. This approach allows us to find a more specific expression for the operator $L_x(u_0)$, simplifying its analysis and deriving the general properties of the solution. As a result of substituting the solutions of (20) into equation (17), an expression is obtained that depends on several variables and includes integrals for the corresponding variables, as well as complex polynomial expressions for the coefficients. The result of this substitution is also considered, which takes the form of (21):

$$
\begin{aligned}
L_x(u_0) &= \sum_{n=0}^{\infty} \int_S \left[\sum_{r=0}^{k} \sum_{q=0}^{k-r} f_{r,q}(x_1, x_2) \frac{\partial^{r+q} \xi^n}{\partial x_1^r \partial x_2^q} + f(x_1, x_2) \xi^n \right] \varphi_n(z_1, z_2)\, dz_1 dz_2 \\
&= \sum_{n=0}^{\infty} \int_S \left[\sum_{r=0}^{k} \sum_{q=0}^{k-r} f_{r,q}(x_1, x_2) \frac{n!\, z_1^r z_2^q}{(n-r-q)!} \xi^{n-r-q} + f(x_1, x_2) \xi^n \right] \varphi_n(z_1, z_2)\, dz_1 dz_2 \\
&= \sum_{n=0}^{\infty} \int_S \left[\sum_{r=0}^{k} \sum_{q=0}^{k-r} \sum_{m=0}^{r+q} a_m^{r,q} x_1^m x_2^{r+q-m} \frac{n!\, z_1^r z_2^q\, \xi^{n-r-q}}{(n-r-q)!} \right. \\
&\quad \left. + \sum_{r=0}^{s} b_{r,s-r} x_1^r x_2^{s-r} \xi^n \right] \varphi_n(z_1, z_2)\, dz_1 dz_2.
\end{aligned}
\tag{21}
$$

For further simplification of the equation and ease of calculation, the method of integration by parts is used [20]. This excludes the variables x_1 and x_2, since the surface S is closed. As a result of integration by parts, all derivatives of x_1 and x_2 are replaced by the corresponding values on the boundary of the surface S. This eliminates the dependence on internal variables and greatly simplifies further calculations, in integrals over z_1 and z_2. Incorporating these considerations, equation (17) takes the form (22):

$$
\begin{aligned}
L_x(u_0) &= \sum_{n=0}^{\infty} \int_S \left[Q_z^{n*}(\varphi_n) \xi^n + R_z^{n*}(\varphi_n) \xi^{n+s} \right] dz_1 dz_2 \\
&= \sum_{n=0}^{s-1} \int_S Q_z^{n*}(\varphi_n) \xi^n\, dz_1 dz_2 + \sum_{n=0}^{\infty} \int_S \left[Q_z^{(n+s)*}(\varphi_{n+s}) + R_z^{n*}(\varphi_n) \right] \xi^{n+s}\, dz_1 dz_2,
\end{aligned}
\tag{22}
$$

where (23, 24):

$$
Q_z^{n*}(\varphi_n) = \sum_{r=0}^{k} \sum_{q=0}^{k-r} \sum_{m=0}^{r+q} (-1)^{r+q} a_m^{r,q} \frac{\partial^{r+q}(z_1^r z_2^q \varphi_n)}{\partial z_1^m \partial z_2^{r+q-m}}, \tag{23}
$$

$$
R_z^{n*}(\varphi_n) = \frac{(-1)^s\, n!}{(n+s)!} \sum_{r=0}^{s} b_{r,s-r} \frac{\partial^s \varphi_n}{\partial z_1^r \partial z_2^{s-r}}, \tag{24}
$$

The functions $\varphi_n(z_1, z_2)$, which have important properties that can be used to write the equations for them in the form of a system of conditions, were considered. It is assumed that these functions satisfy equations (25, 26):

$$
Q_z^{n*}(\varphi_n) = 0, \quad n = \overline{0, s-1}, \tag{25}
$$

$$
Q_z^{n*}(\varphi_n) = -R_z^{(n-s)*}(\varphi_{n-s}), \quad n > s. \tag{26}
$$

If the series in relation (20) is uniformly convergent, this can be used to construct solutions for equation (17) in the form of (20). Uniform convergence of the series is a key condition, as it ensures that when calculating the sum of the series for any value of the index n, the accuracy of the results does not deteriorate. This means that the computational process is stable and provides reliable results, which is important for reaching correct mathematical conclusions. Under conditions of uniform convergence, each term in the series affects the result equally, which ensures no large fluctuations or errors that can occur with uneven convergence.

In addition, thanks to uniform convergence, it is possible to ensure that the functions that form this series are stable and provides accurate solutions for more complex mathematical models. This approach simplifies the problem, transforming it into a more manageable form, which greatly facilitates further calculations and analysis.

However, after constructing solutions of equation (17) in the form of (20), the next step was to construct solutions of the Helmholtz-type equations [21], which are more complex and require additional conditions and characteristics, such as special boundary conditions or conditions at infinity. The Helmholtz-type equation, which is often used to model wave processes in physics, has specific features related to the study of wave propagation in media with different properties. Therefore, the next step after constructing solutions to equation (17) is to consider in more detail the problem that involves the integration and use of the special properties of Helmholtz functions for more accurate modelling of physical processes.

The solutions of the Helmholtz equation, which can be written, according to (17), in the form (27), are found:

$$x_1 x_2 \frac{\partial^2 u}{\partial x_1^2} + x_1 x_2 \frac{\partial^2 u}{\partial x_2^2} + x_1 x_2 u = 0. \tag{27}$$

To demonstrate the inclusion of boundary conditions, consider the partial differential equation where $x_1 x_2$ multiplied by the second partial derivative of u with respect to x_1 squared plus $x_1 x_2$ multiplied by the second partial derivative of u with respect to x_2 squared plus $x_1 x_2$ times u equals zero. This equation can be extended to a boundary value problem in a specific domain, such as a unit disk or rectangle, by imposing Dirichlet or Neumann conditions on the boundary. In this framework, the boundary conditions determine the values or behavior of the auxiliary functions ϕ_n on the contour, which in turn guides the selection and deformation of the integration contours in the two-dimensional complex space. Consequently, the choice of contour and the properties of ϕ_n are directly influenced by the imposed boundary conditions, ensuring that the series of contour integrals satisfies both the differential equation and the conditions on the domain boundary. This approach allows the boundary value problem to be systematically reduced to constraints on the auxiliary functions and the contour geometry, making it possible to construct explicit solutions that fully incorporate the physical or geometric limits of the domain.

The specific case, when in relations (18) and (19) $k = 2$ and $s = 2$ was considered, employing the following values of the coefficients: $a_1^{2.0} = a_1^{0.2} = b_{1.1} = 1$, while all other coefficients are equal to zero. These values are chosen to simplify the calculations and emphasise the important aspects of the expressions for the corresponding differential forms. In this case, the following expressions for the corresponding forms $Q_z^{(n*)}$ and $R_z^{(n*)}$ (28, 29) was obtained:

$$Q_z^{(n*)}(\phi_n) = \frac{\partial^2[(z_1^2 + z_2^2)\phi_n]}{\partial z_1 \partial z_2}, \tag{28}$$

$$R_z^{(n*)}(\phi_n) = \frac{n!}{(n+2)!} \frac{\partial^2 \phi_n}{\partial z_1 \partial z_2}, \tag{29}$$

Furthermore, according to (25) and (26), the equations for determining the unknown functions $\phi_n(z_1, z_2)$ take the form (30, 31):

$$\frac{\partial^2[(z_1^2 + z_2^2)\phi_n]}{\partial z_1 \partial z_2} = 0, \quad n = 0, 1, \tag{30}$$

$$\frac{\partial^2[(z_1^2 + z_2^2)\phi_n]}{\partial z_1 \partial z_2} = -\frac{(n-2)!}{n!} \frac{\partial^2 \phi_{n-2}}{\partial z_1 \partial z_2}, \quad n \geq 2. \tag{31}$$

Note that in equation (31) the shift index is $n - 2$, since the corresponding equation (27) has the degree of polynomial coefficients $s = 2$. The authors consider a set G, which is a sphere with a centre at the origin and a boundary S, whose equation is $|z_1|^2 + |z_2|^2 = \rho^2$, where $0 < \rho < \infty$. The singular solutions of equations (30, 31), which have special points inside G, are as follows (32, 33):

$$\phi_0(z_1, z_2) = \frac{g_1(z_1) + g_2(z_2)}{z_1^2 + z_2^2}, \tag{32}$$

$$\phi_{2l}(z_1, z_2) = -\frac{(2l-2)!}{(2l)!} \frac{\phi_{2l-2}}{z_1^2 + z_2^2} = \frac{(-1)^l}{(2l)!} \frac{g_1(z_1) + g_2(z_2)}{(z_1^2 + z_2^2)^{l+1}}, \tag{33}$$

where $g_1(z_1), g_2(z_2)$ are arbitrary analytical functions.

Substituting expressions (32, 33) of the functions $\phi_0(z_1, z_2)$ and $\phi_{2l}(z_1, z_2)$ into solution (20), the following expression for the function $u_0(x_1, x_2)$ is obtained. This can be used to express the solution as an infinite sum of integrals including boundary conditions on the surface S. After appropriate substitutions and transformations, the following is obtained (34):

$$\begin{aligned} u_0(x_1,x_2) &= \sum_{l=0}^{\infty} \int_S (\omega + x_1 z_1 + x_2 z_2)^{2l} \phi_{2l}(z_1,z_2)\, dz_1 dz_2 \\ &= \sum_{l=0}^{\infty} \frac{(-1)^l}{(2l)!} \int_S (\omega + x_1 z_1 + x_2 z_2)^{2l} \frac{g_1(z_1) + g_2(z_2)}{(z_1^2 + z_2^2)^{l+1}}\, dz_1 dz_2. \end{aligned} \tag{34}$$

New coordinates (35) are entered:

$$\xi_1 = z_1 + iz_2, \quad \xi_2 = z_1 - iz_2. \tag{35}$$

This change of coordinates allows us to move to a more convenient form of representing functions and calculating integrals. By using complex variables ξ_1 and ξ_2, it is possible to simplify some calculations and make the formulas more compact. Then in the new coordinates, the following is present (36):

$$z_1 = \frac{1}{2}(\xi_1 + \xi_2), \quad z_2 = \frac{1}{2i}(\xi_1 - \xi_2), \quad z_1^2 + z_2^2 = \xi_1 \xi_2, \quad dz_1 dz_2 = \frac{i}{2} d\xi_1 d\xi_2. \tag{36}$$

The equation of the sphere S in the new variables is $|\xi_1|^2 + |\xi_2|^2 = 2\rho^2$. Each ξ_k plane intersects the sphere in the circles. In particular, ξ_1 plane, when $\xi_2 = 0$, intersects the sphere in a circle $D_1 = \{|\xi_1| = \sqrt{2}\rho; \xi_2 = 0\}$, while ξ_2 plane when $\xi_1 = 0$, according to $D_2 = \{|\xi_2| = \sqrt{2}\rho; \xi_1 = 0\}$.

The power series of the functions $g_1(z_1)$ and $g_2(z_2)$ can be expressed as follows (37, 38):

$$g_1(z_1) = \sum_{j=0}^{\infty} c_j z_1^j = \sum_{j=0}^{\infty} \frac{c_j}{2^j} (\xi_1 + \xi_2)^j = \sum_{j=0}^{\infty} \frac{c_j}{2^j} \sum_{m=0}^{j} C_j^m \xi_1^m \xi_2^{j-m}, \tag{37}$$

$$g_2(z_2) = \sum_{j=0}^{\infty} d_j z_2^j = \sum_{j=0}^{\infty} \frac{d_j}{(2i)^j} (\xi_1 - \xi_2)^j = \sum_{j=0}^{\infty} \frac{d_j}{(2i)^j} \sum_{m=0}^{j} (-1)^{j-m} C_j^m \xi_1^m \xi_2^{j-m}, \tag{38}$$

where c_j, d_j are certain constants.

According to (37, 38), the expression $g_1(z_1) + g_2(z_2)$ in new variables ξ_1 and ξ_2 can be expressed as follows (39):

$$\begin{aligned} g_1(z_1) + g_2(z_2) &= \sum_{j=0}^{\infty} \sum_{m=0}^{j} q_{j,m} C_j^m \xi_1^m \xi_2^{j-m} \\ &= \sum_{m=0}^{\infty} \sum_{j=m}^{\infty} q_{j,m} C_j^m \xi_1^m \xi_2^{j-m} \\ &= \sum_{m=0}^{\infty} \sum_{s=0}^{\infty} q_{s+m,m} C_{s+m}^m \xi_1^m \xi_2^s, \end{aligned} \tag{39}$$

where $q_{j,m} = \frac{1}{2^j} \left(c_j + (-1)^{j-m} \frac{d_j}{i^j} \right)$.

The following notation was introduced (40):

$$z = x_1 + ix_2, \quad \overline{z} = x_1 - ix_2. \tag{40}$$

It is possible to consider the transformation of the integral solution of equation (27) to simplify the expression, while retaining only those terms that form terms with negative powers of variables in the subintegral expression [22, 23]. After that, the expression for the function $u_0(x_1, x_2)$ is as follows (41):

$$\begin{aligned} u_0(x_1,x_2) &= \sum_{l=0}^{\infty} \frac{(-1)^l}{2^{2l}(2l)!} \sum_{m=0}^{l}\sum_{s=0}^{l} C_{m+s}^{m} \tilde{q}_{m,s} \int_{\Gamma_1}\int_{\Gamma_2} (\tilde{\omega} + \overline{z}\xi_1 + z\xi_2)^{2l} \frac{\xi_1^m \xi_2^s}{(\xi_1\xi_2)^{l+1}}\, d\xi_1 d\xi_2 \\ &= \sum_{l=0}^{\infty}\sum_{m=0}^{l}\sum_{s=0}^{l} \frac{C_{2l-m-s}^{l-m}\tilde{q}_{l-m,l-s}}{2^{2l}} \sum_{r=0}^{2l}\sum_{k=r}^{2l} \frac{(-1)^l \tilde{\omega}^{2l-k} C_k^r}{(2l-k)!\,k!} \overline{z}^r z^{k-r} \int_{\Gamma_1}\int_{\Gamma_2} \frac{d\xi_1 d\xi_2}{\xi_1^{m-r+1}\xi_2^{s-k+r+1}} \\ &= (2\pi i)^2 \sum_{l=0}^{\infty}\sum_{m=0}^{l}\sum_{s=0}^{l} \frac{C_{2l-m-s}^{l-m}\tilde{q}_{l-m,l-s}}{2^{2l}} \frac{(-1)^l \tilde{\omega}^{2l-m-s} C_{m+s}^{m}}{(2l-m-s)!\,(m+s)!} \overline{z}^m z^s \\ &= (2\pi i)^2 \sum_{m=0}^{\infty}\sum_{l=m}^{\infty}\sum_{s=0}^{l} \frac{(-1)^l \tilde{\omega}^{m+s}\tilde{q}_{m,s}}{2^{2l} s!\,m!\,(l-m)!\,(l-s)!} \overline{z}^{l-m} z^{l-s} \\ &= \sum_{m=0}^{\infty}\sum_{s=0}^{\infty} \tilde{\tilde{q}}_{m,s} \sum_{l=\max(m,s)}^{\infty} \frac{(-1)^l \overline{z}^{l-m} z^{l-s}}{2^{2l}(l-m)!\,(l-s)!}, \end{aligned} \tag{41}$$

where $\tilde{\omega} = 2\omega$, $\tilde{q}_{m,s} = \frac{i}{2} q_{m,s}$, $\tilde{\tilde{q}}_{m,s} = \frac{(2\pi i)^2 \tilde{\omega}^{m+s}\tilde{q}_{m,s}}{m!\,s!}$ are arbitrary constants.

Since there are linearly dependent subsystems of functions among the obtained solutions of equation (27), it is necessary to write down systems of linearly independent solutions for further analysis. This avoids duplication and simplifies mathematical expressions. For this purpose, the functions $U_{m,s}$ that are linearly independent solutions of equation (27) were considered. They have the form (42):

$$U_{m,s} = \sum_{l=\max(m,s)}^{\infty} \frac{(-1)^l \overline{z}^{\,l-m} z^{l-s}}{2^{2l}(l-m)!\,(l-s)!}. \tag{42}$$

If $m = s + k = \max(m, s)$ or $s = m + k = \max(m, s)$, then two possible expressions for the solutions are present, depending on the correlation between indices m and s. In each case, functions $U_{s+k,s}$ and $U_{m,m+k}$ can also be represented as power series. Then, accordingly, the following was obtained (43, 44):

$$U_{s+k,s} = \frac{(-1)^{s+k}}{2^{2(s+k)}} z^k \sum_{l=0}^{\infty} \frac{(-1)^l (z\overline{z})^l}{l!\,(l+k)!}, \tag{43}$$

$$U_{m,m+k} = \frac{(-1)^{m+k}}{2^{2(m+k)}} \overline{z}^k \sum_{l=0}^{\infty} \frac{(-1)^l (z\overline{z})^l}{2^{2l}\, l!\,(l+k)!}. \tag{44}$$

Thus, as a result of solving equation (27), a linearly independent system of solutions that can effectively describe behaviour under the relevant conditions was obtained. Hence, the following linearly independent system of solutions to equation (45) can be chosen:

$$U_k(x_1,x_2) + iV_k(x_1,x_2) = \sum_{l=0}^{\infty} \frac{(-1)^l z^{l+k} \overline{z}^l}{2^{2l+k}\, l!\,(l+k)!}. \tag{45}$$

To provide a practical assessment of convergence, the truncated solution $U_k^L + iV_k^L$ was numerically evaluated for $\rho = 5$ and $k = 0$. The approximation obtained with $L = 25$ was used as a reference solution. These values are significantly smaller than the theoretical upper bound 2.1×10^{-9} given by estimate (45a), which confirms rapid practical convergence and high numerical accuracy of the proposed method.

For the solution (45) truncated to L terms of the series, the absolute error obeys the estimate (45a):

$$\left|(U_k + iV_k) - (U_k^L + iV_k^L)\right| \leq \frac{(\rho/2)^{2L+k+2}}{((L+1)!\,(L+k+1)!)}, \tag{45a}$$

where $\rho = \sqrt{x_1^2 + x_2^2}$. For example, in a typical case with $\rho = 5$, $k = 0$, $L = 10$ the right-hand side of inequality is approximately 2.1×10^{-9}, which confirms the high accuracy of the method with a relatively small number of series terms.

To prove that the system of functions defined by relation (45) satisfies the Helmholtz equation (27), the equation is noted as follows (46):

$$4\frac{\partial^2 u}{\partial z \partial \overline{z}} + u = 0, \tag{46}$$

where variables $z, \overline{z}$ are provided by formula (40).

Substituting the functions from (45) into the Helmholtz equation (46), the following is obtained (47):

$$\begin{aligned} 4\frac{\partial^2}{\partial z \partial \overline{z}}\left[U_k(x_1,x_2) + iV_k(x_1,x_2)\right] &= 4\left[\sum_{l=1}^{\infty} \frac{(-1)^l z^{l-1+k} \overline{z}^{\,l-1}}{2^{2l+k}(l-1+k)!\,(l-1)!}\right] \\ &= -\left[U_k(x_1,x_2) + iV_k(x_1,x_2)\right]. \end{aligned} \tag{47}$$

Notably, the functions (45) for negative values of the parameter k are also solutions of equation (27). For instance, taking the functions for negative values of k, the following equality was obtained (48):

$$\begin{aligned} U_{-k}(x_1,x_2) + iV_{-k}(x_1,x_2) &= \sum_{l=k}^{\infty} \frac{(-1)^l}{2^{2l-k}(l-k)!\,l!} \overline{z}^{\,l} z^{l-k} \\ &= \sum_{l=0}^{\infty} \frac{(-1)^{l+k}}{2^{2l+k} l!\,(l+k)!} \overline{z}^{\,l+k} z^{l} \\ &= (-1)^k \left(U_k(x_1,x_2) - iV_k(x_1,x_2)\right). \end{aligned} \tag{48}$$

Using appropriate special functions, system (45) can also be written in the form (49):

$$U_k(x_1,x_2) + iV_k(x_1,x_2) = \left(\frac{z}{2}\right)^k \sum_{l=0}^{\infty} \frac{(-1)^l (z\overline{z})^l}{2^{2l}(l+k)!\,l!} = z^k J_k^*(\rho), \tag{49}$$

where $\rho = \sqrt{x_1^2 + x_2^2}$, $J_k^*(\rho) = \left(\frac{2}{\rho}\right)^k J_k(\rho)$, $J_k(\rho) = \sum_{l=0}^{\infty} \frac{(-1)^l \rho^{2l+k}}{2^{2l+k}(l+k)!\,l!}$ is power series expansion of the Bessel function [24, 25].

Furthermore, introducing a polar coordinate system $z = \rho e^{i\varphi}$, where $0 \leq \rho < \infty$, $-\pi < \varphi \leq \pi$, then the expressions for functions in polar coordinates can be used to construct the following system of functions: $\{e^{ik\varphi} \cdot J_k(\rho),\, e^{ik\varphi} \cdot J_k(\rho)\}_{-\infty}^{\infty}$. Each of these functions is a solution to equation (27). The introduction of polar coordinates decomposed the functions into orthogonal components depending on the angle, which provides a more convenient way to express solutions to the Helmholtz equation (27) in this system.

Using the formula (20), it is possible to construct a system of solutions for equation (50):

$$\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + (x_1^2 + x_2^2)u = 0, \tag{50}$$

or, according to formula (17), the following equation (51):

$$x_1 x_2 \frac{\partial^2 u}{\partial x_1^2} + x_1 x_2 \frac{\partial^2 u}{\partial x_2^2} + x_1 x_2 (x_1^2 + x_2^2) u = 0. \tag{51}$$

Substituting in (18, 19) $k = 2$, $s = 4$, $a_1^{2.0} = a_1^{0.2} = b_{3.1} = b_{1.3} = 1$ and setting all other coefficients to zero, the expressions of the corresponding forms (52, 53) was obtained:

$$Q_z^{(n*)}(\phi_n) = \frac{\partial^2[(z_1^2 + z_2^2)\phi_n]}{\partial z_1 \partial z_2}, \tag{52}$$

$$R_z^{(n*)}(\phi_n) = \frac{n!}{(n+4)!} \frac{\partial^2}{\partial z_1 \partial z_2} \left(\frac{\partial^2 \phi_n}{\partial z_1^2} + \frac{\partial^2 \phi_n}{\partial z_2^2} \right). \tag{53}$$

Therefore, equations (25, 26) for the unknown functions $\phi_n(z_1, z_2)$ will take the form (54, 55):

$$\frac{\partial^2[(z_1^2 + z_2^2)\phi_n]}{\partial z_1 \partial z_2} = 0, \quad 0 \leq n \leq 3, \tag{54}$$

$$\frac{\partial^2[(z_1^2 + z_2^2)\phi_n]}{\partial z_1 \partial z_2} = -\frac{(n-4)!}{n!} \frac{\partial^2}{\partial z_1 \partial z_2} \left(\frac{\partial^2 \phi_{n-4}}{\partial z_1^2} + \frac{\partial^2 \phi_{n-4}}{\partial z_2^2} \right), \quad n \geq 4. \tag{55}$$

In equation (55) the shift $n-4$ arises because in equation (51) the degree is $s = 4$. Thus, the magnitude of the shift is determined by the order of the polynomial coefficients in the original PDE. Using the variables defined in (35), the relation (36) and the equality $\frac{\partial^2 \phi}{\partial z_1^2} + \frac{\partial^2 \phi}{\partial z_2^2} = 4 \frac{\partial^2 \phi}{\partial \xi_1 \partial \xi_2}$, it is possible to obtain solutions to equations (54, 55) in the form (56):

$$\begin{aligned}
\phi_0 &= \frac{g_1(z_1) + g_2(z_2)}{z_1^2 + z_2^2} = \frac{g_1\left(\frac{\xi_1+\xi_2}{2}\right) + g_2\left(\frac{\xi_1-\xi_2}{2}\right)}{\xi_1 \xi_2} = \frac{g(\xi_1, \xi_2)}{\xi_1 \xi_2}, \\
\phi_4 &= -\frac{1}{4!} \frac{1}{z_1^2 + z_2^2} \left(\frac{\partial^2 \phi_0}{\partial z_1^2} + \frac{\partial^2 \phi_0}{\partial z_2^2} \right) = -\frac{4}{4!} \frac{1}{\xi_1 \xi_2} \frac{\partial^2 \phi_0}{\partial \xi_1 \partial \xi_2}, \\
&\dots, \\
\phi_{4l} &= -\frac{(4l-4)!}{(4l)!} \left(\frac{\partial^2 \phi_{4l-4}}{\partial z_1^2} + \frac{\partial^2 \phi_{4l-4}}{\partial z_2^2} \right) = -\frac{(4l-4)!}{(4l)!} \frac{4}{\xi_1 \xi_2} \frac{\partial^2 \phi_{4l-4}}{\partial \xi_1 \partial \xi_2},
\end{aligned} \tag{56}$$

where $g_1(z_1), g_2(z_2), g(\xi_1, \xi_2)$ are arbitrary analytical functions. Assuming that $g(\xi_1, \xi_2) = \xi_2^{2m}$ for $m = 0, 1, \ldots$, it is possible to find singular solutions of equations (54, 55) that have special points inside the ball G. These solutions will take the form (57):

$$\phi_{4(m+l)} = \frac{(-1)^l 4^{m+l}}{(4m+4l)!} \frac{(2m+2l)!\,(2m)!\,(2l)!}{(2m+2l)!!\,(2m)!!\,(2l)!!} \frac{1}{\xi_1^{2m+2l+1} \xi_2^{2l+1}} \quad (l = 0, 1, \ldots). \tag{57}$$

Substituting the expressions for the functions $\phi_{4(m+l)}$ in (20), a general form of the solution u_0 (58) was obtained:

$$\begin{aligned}
u_0 &= \sum_{l=0}^{\infty} \int_S \xi^{4(m+l)} \phi_{4(m+l)}(\xi_1, \xi_2)\, d\xi_1 d\xi_2 \\
&= \sum_{l=0}^{\infty} \int_S \xi^{4(m+l)} \frac{(-1)^l 4^{m+l}}{(4m+4l)!} \frac{(2m+2l)!\,(2m)!\,(2l)!}{(2m+2l)!!\,(2m)!!\,(2l)!!} \frac{d\xi_1 d\xi_2}{\xi_1^{2m+2l+1} \xi_2^{2l+1}}.
\end{aligned} \tag{58}$$

Employing the definitions ξ and formulae (40), it is possible to find the expression for u_0. The initial expressions expanded the general form of the function, considering all possible variants of the variables and their impact on the final result. Thus, by substituting the necessary values and using the definition ξ, the following was obtained (59):

$$
\begin{aligned}
u_0 &= \sum_{l=0}^{\infty} \sum_{k=0}^{4m+4l} \frac{(-1)^l 4^{m+l} \omega^{4m+4l-k}}{2^{4(m+l)}(4m+4l-k)!\, k!} \frac{(2m+2l)!\,(2m)!\,(2l)!}{(2m+2l)!!\,(2m)!!\,(2l)!!} \\
&\quad \times \int_{\Gamma_1}\int_{\Gamma_2} \frac{(\overline{z}\xi_1 + z\xi_2)^k}{\xi_1^{2m+2l+1}\xi_2^{2l+1}}\, d\xi_1 d\xi_2 \\
&= \sum_{l=0}^{\infty} \sum_{r=0}^{4m+4l} \sum_{k=r}^{4m+4l} \frac{(-1)^l \omega^{4m+4l-k}}{4^{m+l}(4m+4l-k)!\, k!} \frac{(2m+2l)!\,(2m)!\,(2l)!\, C_k^r}{(2m+2l)!!\,(2m)!!\,(2l)!!} \\
&\quad \times \overline{z}^r z^{k-r} \int_{\Gamma_1}\int_{\Gamma_2} \frac{d\xi_1 d\xi_2}{\xi_1^{2m+2l-r+1}\xi_2^{2l-k+r+1}} \\
&= \sum_{l=0}^{\infty} \sum_{k=2m+2l}^{4m+4l} \frac{(-1)^l \omega^{4m+4l-k}}{4^{m+l}(4m+4l-k)!\, k!} \frac{(2m+2l)!\,(2m)!\,(2l)!\, C_k^{2m+2l}}{(2m+2l)!!\,(2m)!!\,(2l)!!} \\
&\quad \times \overline{z}^{2m+2l} z^{k-2m-2l} \int_{\Gamma_1}\int_{\Gamma_2} \frac{d\xi_1 d\xi_2}{\xi_1^{l}\xi_2^{4l+2m-k+1}} \\
&= \frac{(2\pi i)^2 \omega^{2m}}{(2m)!!} \sum_{l=0}^{\infty} \frac{(-1)^l \overline{z}^{2m+2l} z^{2l}}{4^{m+l}(2m+2l)!!\,(2l)!!} \\
&= \frac{(2\pi i)^2 \omega^{2m}}{2^m m!} \sum_{l=0}^{\infty} \frac{(-1)^l}{2^{2m+2l}(2l+2m)!!\,(2l)!!} \overline{z}^{2m+2l} z^{2l}.
\end{aligned}
\tag{59}
$$

Hence, using the known relations for the factorial, solutions for arbitrary integers m (60) were considered:

$$
u_m = \sum_{l=0}^{\infty} \frac{(-1)^l}{2^{m+3l}(2l+m)!!\, l!} \overline{z}^{2l+m} z^{2l}, \tag{60}
$$

and show that they satisfy equation (50). To do so, this equation will be transformed to a simpler form that facilitates analysis and solution.

In complex variables (40), equation (50) takes the form (61):

$$
4\frac{\partial^2 u}{\partial z \partial \overline{z}} + z\overline{z}u = 0. \tag{61}
$$

To show that functions (60) are solutions to equation (61), they were substituted into this equation. To do this, the mixed partial derivatives (62) was found:

$$
\begin{aligned}
4\frac{\partial^2 u_m}{\partial z \partial \overline{z}} &= 8z\overline{z} \sum_{l=1}^{\infty} \frac{(-1)^l}{2^{m+3l}(2l+m-2)!!\,(l-1)!} \overline{z}^{2l+m-2} z^{2l-2} \\
&= -z\overline{z} \sum_{l=0}^{\infty} \frac{(-1)^l \overline{z}^{2l+m} z^{2l}}{2^{m+3l}(2l+m)!!\, l!} \\
&= -z\overline{z}\, u_m.
\end{aligned}
\tag{62}
$$

Equality (62) proves that the functions u_m are solutions to equation (61). It is easy to check that the real and imaginary parts of functions (60) are also solutions of equation (61).

A system of solutions to the differential equation (63) was found:

$$
\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + (x_1^2 + x_2^2)^{p-1} u = 0, \quad p = 1, 2, \ldots \tag{63}
$$

Multiplying the equation (63) by $x_1 x_2$, we reduce it to the canonical form (17), which can be solved by the method under consideration. Thus, a more convenient form is obtained for further analysis and solution of this differential equation. A system of independent solutions of equation (63) is considered, which can be written in the form of a series (64):

$$u_m = U_m^{(p-1)} + iV_m^{(p-1)} = \sum_{l=0}^{\infty} \frac{(-1)^l}{2^{2l+m}(pr+m)_l\,(pr)_l} \overline{z}^{pl} z^{pl+m} = \left(\frac{z}{2}\right)^m \sum_{l=0}^{\infty} \frac{(-1)^l}{2^{2l}(pr+m)_l\,(pr)_l} (z\overline{z})^{pl}, \qquad (64)$$

where $m = 0, 1, \ldots$, $(pr+m)_l = \prod_{r=0}^{l}(pr+m)$ for $m \neq 0$ and $(pr+0)_l = p^l l!$ for $m = 0$.

Using the complex variables (40), it is possible to rewrite equation (63) in a more convenient form for analysis (65):

$$4\frac{\partial^2 u}{\partial z \partial \overline{z}} + (z\overline{z})^{p-1} u = 0. \qquad (65)$$

To prove that the functions (64) satisfy the above equation, let us find the second-order mixed derivative in the variables z and $\overline{z}$ (66):

$$\begin{aligned} 4\frac{\partial^2 u_m}{\partial z \partial \overline{z}} &= 4\sum_{l=1}^{\infty} \frac{(-1)^l}{2^{2l+m}(pr+m)_{l-1}\,(pr)_{l-1}} \overline{z}^{p(l-1)+p-1} z^{p(l-1)+m+p-1} \\ &= -(z\overline{z})^{p-1} \sum_{l=0}^{\infty} \frac{(-1)^l}{2^{2l+m}(pr+m)_l\,(pr)_l} \overline{z}^{pl} z^{pl+m} \\ &= -(z\overline{z})^{p-1} u_m. \end{aligned} \qquad (66)$$

If $m = pn$, where n is an integer, the expression for $(pr+m)_l$ can be found. Replacing $m = pn$, the following is obtained (67):

$$(pr+m)_l = p^l \prod_{r=0}^{l}(r+n) = \frac{p^l (l+n)!}{(n-1)!}. \qquad (67)$$

Then the solutions of (64) can be noted as (68):

$$u_{pn} = U_{pn}^{(p-1)} + iV_{pn}^{(p-1)} = (n-1)!\, z^{pn} \sum_{l=0}^{\infty} \frac{(-1)^l}{2^{2l+m} p^{2l} (l+n)!\, l!} (z\overline{z})^{pl}. \qquad (68)$$

It is worth noting that the solution obtained by formula (60) for equation (61) has significant differences from the solution expressed in the form of (64) for $p = 2$. For $p = 2$, the difference between these solutions is manifested only in the presence of a constant factor that does not change the general form of the solution but affects its value. This factor can be due to specific properties of the equation or dependencies inherent in a particular value of the parameter p, which provided useful information for further calculations and analysis.

We consider the following process associated with the use of conformal mapping. Let the $z = w(\xi)$ be the conformal mapping of domain M on a complex plane $(\xi = t_1 + it_2)$ to a domain D of the complex plane be $(z = x_1 + ix_2)$. New variables $\xi = w^{-1}(z)$, $\overline{\xi} = w^{-1}(\overline{z})$ were added. Since $w'(\xi) \neq 0$, $\xi \in M$, then equation (46) can be rewritten as (69):

$$4\frac{\partial^2 u}{\partial \xi \partial \overline{\xi}} + w'\overline{w'} u = 0, \qquad (69)$$

where $w = w'(\xi)$, $\overline{w'} = \overline{w'(\xi)}$.

To obtain a system of solutions to equation (69), formula (45) was used, where it is assumed that the variables z and $\overline{z}$ are replaced by the corresponding expressions through conformal mapping: $z = w(\xi)$ and $\overline{z} = \overline{w(\xi)}$. This transformed the original solution system in a new coordinate system connected through the mapping w. Substituting these variables into equation (45), a system of solutions for equation (69) in the form of (70) was obtained:

$$u_k(t_1, t_2) = \sum_{l=0}^{\infty} \frac{(-1)^l w^{l+k} \overline{w}^l}{2^{2l+k} (l+k)!\, l!}. \tag{70}$$

To show that functions (70) satisfy equation (69), the functions $u_k(t_1, t_2)$ were substituted into equation (69) and verified to determine whether the equality holds. The expression for second-order mixed derivatives is as follows:

$$\begin{aligned} 4\frac{\partial^2 u}{\partial \xi \partial \overline{\xi}} &= 4\frac{\partial^2}{\partial \xi \partial \overline{\xi}} \sum_{l=0}^{\infty} \frac{(-1)^l w^{l+k} \overline{w}^l}{2^{2l+k} (l+k)!\, l!} \\ &= 4w'\overline{w'} \sum_{l=1}^{\infty} \frac{(-1)^l w^{l-1+k} \overline{w}^{\,l-1}}{2^{2l+k} (l+k-1)!\, (l-1)!} \\ &= -w'\overline{w'}u. \end{aligned} \tag{71}$$

We consider a specific case when the variable z takes the form $z = \xi^p$. Such a replacement of variables is typical for problems where power mappings are used to conveniently transform equations into simpler forms, which simplifies calculations and solutions [26]. Substituting $z = \xi^p$ into equation (69), the following was acquired (72):

$$4\frac{\partial^2 u}{\partial \xi \partial \overline{\xi}} + p^2 (\xi\overline{\xi})^{p-1} u = 0. \tag{72}$$

The system of solutions to equation (72) is as follows:

$$u_k(t_1, t_2) = \sum_{l=0}^{\infty} \frac{(-1)^l \xi^{p(l+k)} \overline{\xi}^{pl}}{2^{2l+k} (l+k)!\, l!}, \quad k = 0, 1, \ldots \tag{73}$$

This expression gives a system of solutions for the equation in the new variables ξ and $\overline{\xi}$, which arise after applying a conformal mapping that transforms the domain M on the complex plane into the domain D. The solution in the form of a series, where each term depends on the degree of the variables ξ and its complex conjugation, obtaining a more convenient form for further calculations or analysis.

Such systems of solutions are often used to model physical processes described by differential equations with complex geometry or highly nonlinear characteristics. Sun et al. [27] fractional differential equations (FDEs) of variable-order (VO), the order of which depends on time (t), space (x), or other variables were used to study dynamics that vary in time and/or space. However, analytical solutions for VO-FDE models are extremely difficult to obtain, so the use of efficient numerical or approximate solution methods is of great practical importance. Comparing this approach with the present study, several key differences and common aspects can be observed. Both approaches are focused on solving complex partial differential equations, but in the study of Sun et al. [27], the emphasis is on fractional differential equations, which enables modelling processes with a variable order of differentiation. Despite the differences in approaches, a common feature of both studies is the need to use numerical or approximate methods to obtain solutions in cases where analytical methods are too complex or unsuitable for solving equations. Importantly, in both the case of fractional differential equations and the integration approach, the need to use numerical methods emphasises the importance of computer computing and numerical modelling for solving complex scientific problems in various fields. Thanks to conformal mappings and appropriate substitutions of variables, solutions that can be convenient for numerical calculations or further mathematical analysis were obtained.

For differential equations that use a conformal mapping, there is a universal solution that can be obtained by substituting $z = \xi^p$. This solution is represented as a series with certain coefficients and is used for various problems in theoretical physics and mathematics, regardless of the value of p.

To ensure the correctness and convergence of the series solution obtained via conformal mapping, the coefficients of the differential equation must be analytic within the domain under consideration. In practice, this can be verified by examining the functional form of the coefficients, ensuring the absence of poles, discontinuities, or other singularities. Failure to meet this requirement may result in a divergent series or physically incorrect solutions, preventing accurate representation of the solution. The integration contour in the complex plane must be chosen so as to avoid proximity to singularities of auxiliary functions, including poles and branch points. In practice, this involves analyzing the locations of these singularities and adjusting the contour accordingly. Neglecting this criterion can lead to significant numerical errors or invalid evaluation of the integrals, compromising the analytical accuracy of the method.

The geometry of the domain must be regular and compatible with the conformal mapping, allowing the domain to be mapped onto a standard region, such as a half-plane or disk, without discontinuities. Practically, this is verified by analyzing the boundaries of the domain and confirming its mappability onto a simple geometric region. Irregular or highly complex geometries hinder the construction of a convergent series and may lead to loss of solution accuracy. The type of boundary conditions must be compatible with the series representation, meaning that the conditions can be expressed in a form suitable for expansion in the chosen basis functions. In practice, this is checked by examining the form of the boundary conditions and confirming that all series coefficients can be determined. If this requirement is not met, the solution becomes indeterminate, and the series representation may be incomplete or invalid. The method works best for polynomial or otherwise analytic coefficients, when the integration contour can be selected with sufficient distance from singularities of auxiliary functions, the domain geometry is regular, and the boundary conditions are compatible with the series expansion. Adherence to these criteria ensures the correctness, convergence, and high accuracy of the series solution.

4. Discussion

The work Sukhorolsky et al. [28] extended the contour integral approach to more general boundary value problems, obtaining solutions for specific PDEs in one variable. Their work demonstrated the effectiveness of contour integration for linear equations with polynomial or analytic coefficients and clarified convergence properties for one-dimensional problems. Nonetheless, the approach does not address PDEs with two independent complex variables, limiting its practical application for multidimensional problems. In contrast, the current study's methodology allows representation of solutions in two-dimensional complex space, providing a broader framework for Helmholtz-type equations and PDEs with polynomial coefficients in two variables. In the present study, the methods of the theory of functions of a complex variable are applied to boundary value problems, demonstrating their effectiveness in expanding the analytical tools for solving equations in mathematics and physics. These results align with the work of Batal et al. [29], who investigated the application of the unified transform (Fokas) method to linear boundary value problems involving mixed spatial derivatives. The authors derived explicit integral representations of solutions based on complex analysis and contour deformation techniques, demonstrating that such problems can be solved without reducing them by coordinate transformations or separation of variables.The use of contour integrals makes it possible to present to present solutions to problems in a form that is convenient for further analysis and numerical calculation. This is especially important for problems where classical methods may not be effective enough or not applicable at all. The research formulated a general approach to representing solutions of boundary value problems by contour integrals in one variable and obtained solutions to several problems for ordinary differential equations and PDEs. The formulas for some linear initial-boundary value problems in half-space containing mixed spatial derivatives were obtained using the unified transformation method (UTM). This method, also known as the Fokas method, is substantial for solving differential equations arising in mathematical physics and other fields of science and technology, and containing complex conditions at the boundaries and in the middle of the domain. For biharmonic problems, where cross terms cannot be eliminated by linear replacement of variables, the main emphasis is on the precise determination of the analyticity conditions critical for the Fokas method. The Fokas method has proven to be effective in solving partial differential equations where traditional methods may not be powerful enough or complex to implement. This approach can be used to address problems of different nature and complexity, in particular, problems with boundary conditions defined at infinity or on complex contours.

Similarly, the main study considers the use of contour integrals in a two-dimensional complex space to solve partial differential equations. Since a two-dimensional complex space is considered, the emphasis is on the use of complex analysis to solve problems. Despite the common features, there are significant differences between these studies. Gunning and Rossi [30] analysed biharmonic problems and solving problem where standard variable replacement methods cannot be applied to eliminate cross terms. At the same time, a study that considers the representation of solutions using integrals in complex space provides a more general approach that can be used to solve various types of partial differential equations, including those with more complex boundary conditions and geometries. Of considerable interest are the solutions of the PDEs, which are represented in the form of series whose terms are contour integrals of two complex variables. This approach can be used to decompose complex functions into simpler components, which is key to understanding their behaviour and properties. Furthermore, it enables the use the powerful tools of the theory of functions of a complex variable to analyse and construct solutions.

In the present study, explicit solutions of Helmholtz-type equations are obtained using special functions of two variables, which enables their direct application to the analysis of physical phenomena such as wave propagation and membrane oscillations. These findings are consistent with the results presented by Boyce et al. [31], where classical analytical techniques for solving ordinary differential equations and boundary value problems were systematically developed, including the use of special functions, eigenfunction expansions, and Sturm–Liouville theory. The present results extend this foundational framework by demonstrating that the incorporation of complex variable methods and contour integral representations allows for a more general and flexible treatment of applied boundary value problems.

Anderson et al. [32] presented general approaches for constructing closed solutions of linear partial differential equations (PDEs) with constant coefficients and polynomial right-hand sides in two and three dimensions. To solve some complex systems where only a partial solution is possible, the method of potential representation is applied, in particular, the Helmholtz or Poisson equations are used. Some of the cases considered, such as the Stokes flow, Maxwell's equations, and linearised Navier-Stokes equations, naturally include divergence constraints on the solutions. Comparing this with the present study, it can be noted that both approaches are focused on solving complex differential equations, but with different techniques and approaches. Instead, the researchers' work focuses on constructing closed-form solutions using classical potential methods, which are important for problems of a physical nature, such as the Stokes flow or the Navier-Stokes equation. Whereas the study in complex space uses a more general mathematical apparatus, applying series and integrals to analyse solutions under specific conditions, adapting to various physical situations.

Each of these approaches has advantages depending on the context of the problem, and their application creates new possibilities for solving complex differential equations that appear in various fields of science and technology. Potential techniques provide particularly useful tools for describing physical phenomena where the nature of the system enables the application of these classical methods, while series and integral methods open wider possibilities for modelling more general and complex cases. Murari et al. [19] obtained new results on the structure of transcendental integral functions, which are solutions of nonlinear partial differential equations of high order, similar to the Fermat equation, in the domain of two complex variables. This study focuses on solutions to complex systems of equations, which adds additional complexity to solving these problems. In the context of this study, which uses series methods with integrals as terms to solve PDEs, a certain interaction and interconnection between these approaches is present. In particular, the use of series or integrals helps to simplify calculations and provide more accurate solutions, especially when equations have complex, nonlinear or variable coefficients. This approach can be used to adapt integration and approximation methods to various types of problems, which is also relevant in the context of researchers' work, where the solutions are transcendental and require complex construction for different orders of derivatives.

The obtained solutions were critically analyzed in terms of their strengths and limitations. For instance, solution forty-five provides a complete basis of solutions in terms of Bessel functions, which is highly advantageous for spectral methods and allows accurate representation of wave-type phenomena. Similarly, other derived solutions offer explicit series representations that facilitate the study of PDEs with polynomial coefficients and Helmholtz-type equations, providing a clear framework for constructing solutions in a two-dimensional complex space. These results demonstrate the practical applicability of the proposed methodology and its effectiveness in producing convergent and stable solutions under the considered conditions.

However, the methodology also exhibits certain limitations. The convergence of the series can be slow for large arguments, particularly in problems involving high-frequency components or extended domains. The approach requires the regularity of the spatial domain and the analyticity of the functions involved, which may restrict its application to more general or irregular geometries. Additionally, the selection of integration contours and the solution of auxiliary systems of PDEs can be computationally intensive, and the method may become inefficient for highly nonlinear or multidimensional extensions. These constraints highlight the importance of carefully evaluating the applicability of the proposed method in practical scenarios and suggest directions for future improvements and optimizations. Recent research in the field of mathematical equations and operator theory has expanded the range of techniques available for solving complex differential and functional equations. Riveros and Corro [20] investigated the generalized Helmholtz equation, providing analytical constructions that extend classical solutions to more general boundary conditions and higher-dimensional contexts, which facilitates the modeling of wave and potential phenomena in physics. Similarly, Veselovska et al. [21] focused on the Helmholtz equation in cylindrical coordinate systems, constructing solutions as homogeneous polynomials within two biorthogonal systems of functions, demonstrating the applicability of algebraic and orthogonal function methods for precise solution generation in structured geometries. Lu et al. [22] proposed a novel approach through DeepONet, which leverages the universal approximation theorem for nonlinear operators to identify differential equations, enabling the computational learning of solution operators from data and providing a bridge between classical analytical methods and modern machine learning techniques.

Functional calculus approaches have also been advanced, with Dupire [23] developing functional Itô calculus, which allows differential operators to act on functionals of stochastic processes, enhancing the analytical toolkit for financial mathematics and stochastic modeling. In the context of high-energy physics, Fontana and Peraro [24] introduced methods for reducing complex integrals to master integrals through intersection numbers and polynomial expansions, facilitating precise computations in perturbative quantum field theories and highlighting the utility of algebraic techniques in simplifying intricate integral structures. Meanwhile, Das [25] presented a path integral approach to field theory, offering a foundational method for quantifying physical systems through functional integration, which complements both analytical and numerical solution techniques. Korn and Korn [33] provided an extensive reference framework of mathematical definitions, theorems, and formulas, which continues to serve as a fundamental resource for applied problem-solving across engineering and scientific domains. Finally, Lassas et al. [34] examined inverse problems for elliptic equations with power-type nonlinearities, extending classical methods to the reconstruction of unknown coefficients and sources from boundary measurements, which has direct implications for imaging, control, and diagnostic applications.

Each of these approaches presents advantages depending on the context of the problem, and their integration facilitates the solution of increasingly complex differential and functional equations. The combination of classical analytical techniques, algebraic constructions, functional calculus, and computational learning methods opens new possibilities for both theoretical exploration and practical applications, particularly in physical, engineering, and computational sciences. In the framework of this study, these methodologies collectively inform the selection of series, integral, and operator-based techniques, providing adaptable tools for tackling nonlinear, variable-coefficient, and high-dimensional systems.

5. Conclusions

As a result of this study, a general approach to representing the solutions of PDEs by contour integrals in two complex variables was formulated. The most important stage in constructing solutions is to find singular solutions of the system (15) of ordinary differential equations. Then the PDE solutions are written in the form of contour integrals.

In the case of Helmholtz equations, the solutions are obtained explicitly through special functions of two complex variables. The construction of solutions to these equations can be used to apply methods of the theory of analytic functions to construct solutions to boundary value problems. Similar boundary value problems for the considered partial differential equations in the half-plane, strip and circular domains are considered in the works, the construction of the corresponding solution systems which is based on the use of contour integrals in one complex variable. At the same time, systems of biorthogonal functions on curves in a complex domain are used to formulate boundary conditions on coordinate lines.

Explicit solutions of the differential equations (27) and (50) can also be obtained if instead of the contour integrals (20) in variables z_1, z_2, contour integrals in variables ξ_1, ξ_2 were introduced as per formula:

$$u = \sum_{n=0}^{\infty} \int_{\Gamma_1} \int_{\Gamma_2} (\omega + \overline{z}\xi_1 + z\xi_2)^n \phi_n(\xi_1, \xi_2)\, d\xi_1 d\xi_2. \tag{74}$$

This representation is also consistent with the general scheme of using contour integrals to construct PDE solutions.

From the point of view of practical results, the study demonstrates the effectiveness of using complex integration methods to construct solutions to problems that could be solved by other approaches. However, a much wider range of issues related to the application of contour integrals to the construction of solutions to boundary value problems for partial differential equations requires further research. These include the formulation and solution of boundary value problems for the equations under consideration in arbitrary domains. It is also possible to investigate the properties of systems of functions that are biorthogonal to functions (45), (60), and (70).

The results demonstrate that the proposed method surpasses traditional contour integration techniques by combining multi-dimensional complex analysis with series expansions, yielding a highly adaptable representation of solutions with controllable convergence and error bounds. The approach provides new tools for addressing problems with complex geometries and variable coefficients, enabling the explicit construction of singular solutions and the systematic analysis of their influence on the domain. Practically, this allows for more accurate and efficient numerical computations of partial differential equations, extending the range of solvable problems and offering a framework that integrates theoretical rigor with computational applicability, thereby opening avenues for both analytical and numerical advancements beyond the scope of classical methods.

In future research, the proposed method could be extended to solve nonlinear partial differential equations, opening opportunities for analyzing more complex physical and engineering systems where linear approaches are insufficient. Additionally, applying the method in multi-dimensional complex spaces allows consideration of interactions among multiple variables and the study of complex geometric and boundary effects. Developing approaches for non-polynomial coefficients creates new avenues for modeling media with variable properties or nonlinear materials. Furthermore, potential software implementation of the algorithm would enable automated solution construction, ensure computational scalability, and facilitate applications in practical problems in engineering, physics, theoretical mechanics, and other fields requiring high accuracy and error control.

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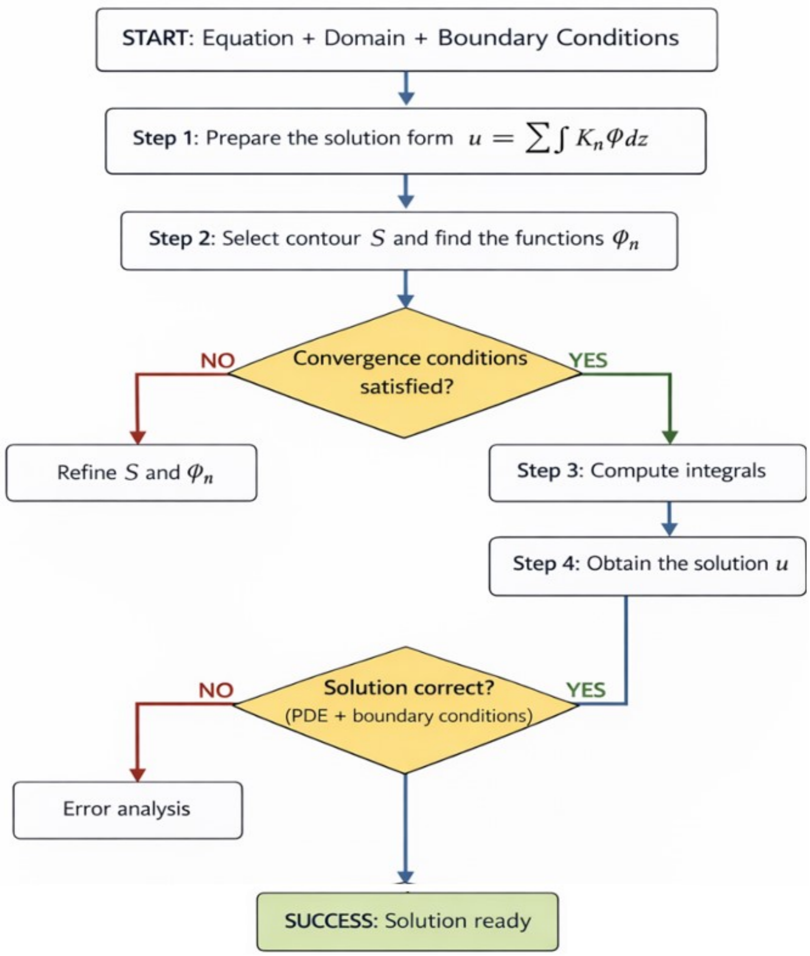
APPENDIX A



Source: compiled by the author based on original research using Matplotlib [26].