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Upside-Down Logic in Plithogenic Fuzzy Control System: Contradiction-Aware Rule Inversion and De-Plithogenication

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Abstract In the real world, many reversal phenomena occur—for instance, situations in which a statement once regarded as false is later recognized as true. Upside-Down Logic provides a framework that formalizes such reversals as a logical system: it flips the truth and falsity of propositions via contextual transformations, thereby capturing ambiguity and reversal within reasoning processes. A Plithogenic Set represents elements using attribute-based membership and contradiction functions, extending the traditional frameworks of fuzzy, intuitionistic, and neutrosophic sets. In this paper, we investigate Plithogenic Fuzzy Control, which both generalizes classical fuzzy control and furnishes a formal mechanism for analyzing and implementing inversion (upside-down) operations.

Keywords Upside-down-logic, Plithogenic Set, De-Plithogenication, Plithogenic Fuzzy Control System, Fuzzy Control System

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1. Introduction

1.1. Fuzzy, Neutrosophic, and Plithogenic Paradigms

Classical (crisp) sets record membership in a binary way and therefore cannot natively encode partial truth or vagueness. In 1965, Zadeh introduced *fuzzy subsets*, providing degrees of membership in $[0, 1]$ and opening a large body of work on uncertain and imprecise information [1]. Owing to both scientific impact and practical value, applications of fuzzy sets span, among others, social systems, control, and decision analysis [2, 3, 4].

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Atanassov later proposed *intuitionistic fuzzy sets*, where each element is described by a membership degree and a non-membership degree, with the remaining margin representing uncertainty [5, 6]. Furthermore, as extended concepts of the Fuzzy Set, several generalizations are known, such as the Hesitant Fuzzy Set [7, 8, 9], the m-polar Fuzzy Set [10, 11, 12], the Picture Fuzzy Set [13, 14, 15], and the Spherical Fuzzy Set [16, 17, 18].

Subsequently, Smarandache developed the *neutrosophic set* formalism, assigning to every element three independent assessments: truth (T), indeterminacy (I), and falsity (F). This representation accommodates inconsistency and incompleteness and has been adopted in many application areas [19, 20]. Related families include Vague Sets [21, 22], Bipolar Neutrosophic Sets [23, 24, 25], q-rung orthopair neutrosophic Sets [26, 27], Pythagorean Neutrosophic Sets [28, 29], HyperNeutrosophic Sets [30, 31], and Hesitant Neutrosophic Sets [32, 33], each tailoring multi-valued descriptions to specific modelling needs. Compared with fuzzy sets, a key advantage of neutrosophic sets is their explicit ability to handle *indeterminacy*—parameters whose truth status is unknown—rendering the framework scientifically significant and enabling a wide range of applications.

More recently, the philosophical idea of *Plithogeny* (Smarandache, 2017) motivated the development of *plithogenic sets, logic, probability*, and *statistics* [34, 35, 36]. A plithogenic set augments traditional multi-valued frameworks by coupling (i) attribute–value-based appurtenance with (ii) an explicit, symmetric *degree of contradiction* between attribute values. This design captures not only gradual belonging but also the intensity of conflict among alternatives—an aspect that becomes crucial when real-world evidence is noisy, evolving, or even self-contradictory. Because contradiction naturally arises across philosophy[37], the social sciences[38, 39], and everyday decision-making[40, 41], the plithogenic paradigm is likewise of substantial importance. Table 1 summarizes the core differences among several set extensions (notation harmonized for this paper).

Table 1. Representative set extensions and their canonical descriptors.

Set Type	Canonical data attached to each element
Fuzzy Set	Membership $\mu : X \to [0, 1]$.
Intuitionistic Fuzzy Set	Membership μ and non-membership ν with $\mu(x) + \nu(x) \le 1$; residual encodes hesitation.
Neutrosophic Set	Triple (T, I, F) with $T, I, F \in [0, 1]$ mutually independent (truth, indeterminacy, falsity).
Plithogenic Set	Tuple $(P, v, Pv, \mathrm{pdf}, \mathrm{pCF})$ where $\mathrm{pdf} : P \times Pv \to [0, 1]^s$ (appurtenance along s dimensions) and $\mathrm{pCF} : Pv \times Pv \to [0, 1]^t$ (symmetric contradiction along t dimensions).

Within the plithogenic family, one recovers the *plithogenic fuzzy*, *plithogenic intuitionistic fuzzy*, and *plithogenic neutrosophic* variants by choosing appropriate appurtenance and contradiction dimensions (e.g., $s = t = 1$ for a scalar case) [42, 43, 44, 45, 46]. If the contradiction map pCF is suppressed (identically zero), the respective classical fuzzy/intuitionistic/neutrosophic models are retrieved (cf. [47]). Within the plithogenic family, one recovers three scalar-contradiction specializations. Table 2 summarizes them concisely.

Table 2. Concise summary of plithogenic variants (scalar contradiction; $t = 1$).

Variant	s	t	Appurtenance vector (semantics)	Reduces to if $\mathrm{pCF} \equiv 0$
Plithogenic fuzzy	1	1	$\mu \in [0, 1]$ (membership)	Classical fuzzy set / controller
Plithogenic intuitionistic fuzzy	2	1	$(\mu, \nu) \in [0, 1]^2$; $1 - \mu - \nu \ge 0$ (hesitation margin)	Intuitionistic fuzzy model
Plithogenic neutrosophic	3	1	$(T, I, F) \in [0, 1]^3$ (truth, indeterminacy, falsity)	Neutrosophic model

1.2. Upside-Down Logic

Upside-Down (UD) Logic formalizes the idea that, under systematic changes of context or interpretation, propositions may swap their truth status (true $\leftrightarrow$ false), providing a principled way to reason about reversals and ambiguity [48, 49, 50]. As an illustration, statements publicly framed as “beneficial” can, after a contextual shift (new evidence, reframing, or policy re-evaluation), become demonstrably harmful—an inversion that UD Logic is designed to capture within a formal semantics. Early studies have examined UD-style transforms inside neutrosophic and plithogenic settings [49, 51], where contradiction measures and multi-valued appurtenance provide natural levers for representing such flips.

1.3. Fuzzy Control

As noted earlier, fuzzy sets have been extensively investigated in diverse applications because of their theoretical significance and practical usefulness (cf. [52, 53]). A control system measures outputs, compares them to references, and adjusts inputs automatically to achieve stability, accuracy, and desired performance [54, 55, 56].

In this paper, we focus on fuzzy control. A fuzzy control system is a rule-based decision-making framework that utilizes fuzzy sets and approximate reasoning to convert crisp sensor inputs into smooth actuator outputs, thus enabling robust control of complex, nonlinear, and uncertain processes without requiring precise mathematical models [57, 58]. Related concepts of fuzzy control include closed-loop fuzzy control [59, 60, 61], adaptive fuzzy control[62, 63, 64], direct adaptive fuzzy control[65, 66], and neuro-fuzzy control[67, 68, 69], for which numerous studies have also been reported.

In addition, related approaches include *Intuitionistic Fuzzy Control* [70, 71], HyperFuzzy Control[72], and *Neutrosophic Control* [73, 74], among others (cf. [75]).

1.4. Motivation and Our Contribution

From the above, research on fuzzy control has been widely explored. However, studies on fuzzy control in connection with plithogenic sets have not yet been conducted. To fill this gap, in this paper, we investigate Plithogenic Fuzzy Control, which both generalizes classical fuzzy control and furnishes a formal mechanism for analyzing and implementing inversion (upside-down) operations. Plithogenic Fuzzy Control adds contradiction-aware weighting and context-triggered inversion, resolving conflicting rules and improving robustness and safety under sensor/actuator sign flips, while keeping classical fuzzy control’s interpretability and adaptability. For reference, Table 3 presents a comparison between Fuzzy Control and Plithogenic Fuzzy Control (PFCS).

Note that the Plithogenic Fuzzy Control System (PFCS) proposed in this paper, while highly novel, should be regarded primarily as a theoretical contribution presenting a new control concept. Experimental validation and empirical studies by experts will be essential in future work to confirm its practical effectiveness.

1.5. Contents in this paper

The remainder of this paper is organized as follows. Section 2 introduces the fundamental definitions of Upside-Down Logic, Plithogenic Sets, and Fuzzy Control. Section 3 presents the main results, namely the proposed Plithogenic Fuzzy Control System (PFCS) and its corresponding Upside-Down Logic. Section 4 describes the algorithmic design and computational aspects. Finally, Section 5 concludes the paper with a summary and discussion of future research directions.

Table 3. Fuzzy Control vs. Plithogenic Fuzzy Control (PFCS)

Aspect	**Fuzzy Control**	**Plithogenic Fuzzy Control (PFCS)**
Basis	Rules on fuzzy memberships	Fuzzy rules + contradiction map (pCF)
Rule strength	t-norm of memberships	t-norm $\times$ contradiction attenuations (β, γ)
Contradictions	Not explicit	Explicit, symmetric $pCF \in [0,1]$ between terms
Context inversion	Not modeled	Upside-Down transform (flip + reset)
Anchors	None	Uses input/output anchors (b_i, b_y)
Reduction	—	Recovers classical fuzzy control when $pCF \equiv 0$
Robustness	Sensitive to sign/polarity errors	More robust under sensor/actuator sign flips
Interpretability	High (rule base)	High (rules + pCF/anchors)
Compute cost	Low	Low–moderate (extra multipliers)

2. Preliminaries

This section introduces the basic constructs and symbols employed in the sequel. Unless stated otherwise, all underlying sets and structures are assumed to be finite.

2.1. *Upside-Down Logic*

We formalize the intuition that, under suitable changes of interpretation, a statement may switch its truth status. In particular, context shifts can turn truths into falsehoods and conversely, thereby modelling reversal and ambiguity in reasoning systems (see, e.g., [48, 49, 50, 76, 77, 78]). The notation below will be used throughout.

Definition 2.1 (Logical structure)**.** (cf. [79]) A *logical structure* is a triple

$$\mathcal{M} = (\mathcal{P}, \mathcal{V}, v),$$

where $\mathcal{P}$ is the set of well-formed propositions of a language $\mathcal{L}$, $\mathcal{V}$ is a set of truth values (e.g. $\{\top, \bot\}$ in the classical case), and $v : \mathcal{P} \to \mathcal{V}$ is a valuation. One may optionally specify a set of axioms $\mathcal{A} \subseteq \mathcal{P}$ and a collection of inference rules $\mathcal{I}$ that license derivations.

Notation 2.2 (Context-sensitive valuation)
Given a proposition set $\mathcal{P}$ and a family of contexts Ctx, we write

$$\Theta : \mathcal{P} \times \mathsf{Ctx} \longrightarrow \{\top, \bot, \star\}$$

for the context-indexed truth assignment, where $\star$ may encode indeterminacy. Thus $\Theta(A, \mathsf{C})$ is the truth status of A when evaluated in context C.

Example 2.3 (Thermal loop with actuator polarity as context)**. Proposition.** Let $e := T_{\text{sp}} - T$ be the temperature error and consider a proportional controller $u = K_p e$ with $K_p > 0$.

$$A: \quad \text{“For } e_0 > 0 \text{ at } t = 0,\ \dot{e}(0) < 0 \text{ (the error initially decreases).”}$$

Plant models (contexts). With time constant $\tau = 10$ and gain $k = 0.5$, define

$$\mathsf{C}_{\text{heat}} : \dot{T} = -\tfrac{1}{\tau}(T - T_{\text{env}}) + k\,u, \qquad \mathsf{C}_{\text{cool}} : \dot{T} = -\tfrac{1}{\tau}(T - T_{\text{env}}) - k\,u.$$

In both cases $\dot{e} = -\dot{T}$, hence

$$\dot{e} = \begin{cases} \frac{1}{\tau}(T - T_{\text{env}}) - kK_p e & (\mathsf{C}_{\text{heat}}), \\ \frac{1}{\tau}(T - T_{\text{env}}) + kK_p e & (\mathsf{C}_{\text{cool}}). \end{cases}$$

Numerical evaluation. Take $K_p = 1$, initial error $e_0 = 2$, and let $\Delta := T(0) - T_{\text{env}}$.

(i) Heating context. If $\Delta = 0$,

$$\dot{e}(0) = 0 - \underbrace{(0.5)(1)(2)}_{=1} = -1 < 0 \quad \Rightarrow \quad \Theta(A, \mathsf{C}_{\text{heat}}) = \top.$$

(ii) Cooling (polarity-flipped) context. If $\Delta = 0$,

$$\dot{e}(0) = 0 + \underbrace{(0.5)(1)(2)}_{=1} = +1 > 0 \quad \Rightarrow \quad \Theta(A, \mathsf{C}_{\text{cool}}) = \bot.$$

(iii) Balanced-disturbance context. Let

$$\mathsf{C}_{\text{bal}}: \ \Delta \in [9, 11] \quad \text{so} \quad \tfrac{1}{\tau}\Delta \in [0.9, 1.1].$$

In C_{heat}, $\dot{e}(0) = \frac{1}{\tau}\Delta - 1 \in [-0.1, 0.1]$, whose sign cannot be decided a priori:

$$\Theta(A, \mathsf{C}_{\text{bal}}) = \star.$$

Summary.

$$\Theta(A, \mathsf{C}_{\text{heat}}) = \top, \qquad \Theta(A, \mathsf{C}_{\text{cool}}) = \bot, \qquad \Theta(A, \mathsf{C}_{\text{bal}}) = \star.$$

This exhibits a context-sensitive valuation: the same proposition about initial error reduction changes truth with actuator polarity, and becomes indeterminate under a disturbance window that straddles the balance point.

Example 2.4 (Collaborative robot speed safety with distance context)**. Proposition.** For a cobot moving at speed v, with controller reaction time t_r and maximum deceleration $a_{\max}$, define the stopping distance

$$s_{\text{stop}}(v) = v\, t_r + \frac{v^2}{2a_{\max}}.$$

Let D be the measured human-to-robot separation and d_h a safety offset.

$$B: \quad \text{“Command } v = 0.60 \text{ m/s is safe, i.e., } s_{\text{stop}}(v) \le D - d_h.\text{”}$$

Fixed parameters. Take $t_r = 0.20$ s, $a_{\max} = 3.0\ \text{m/s}^2$, $d_h = 0.05$ m. Then

$$s_{\text{stop}}(0.60) = 0.60(0.20) + \frac{0.60^2}{2(3.0)} = 0.12 + \frac{0.36}{6} = 0.12 + 0.06 = 0.18 \text{ m}.$$

Contexts.

$$\mathsf{C}_{\text{op}}: \ D = 2.00 \text{ m}, \qquad \mathsf{C}_{\text{maint}}: \ D = 0.30 \text{ m}, \qquad \mathsf{C}_{\text{occl}}: \ D \in [0.30, 0.34] \text{ m (occluded sensor)}.$$

Evaluation.

(i) Normal operation. $D - d_h = 2.00 - 0.05 = 1.95$ m:

$$0.18 \le 1.95 \;\Rightarrow\; \Theta(B, \mathsf{C}_{\text{op}}) = \top.$$

(ii) Close-maintenance mode. $D - d_h = 0.30 - 0.05 = 0.25$ m:

$$0.18 \le 0.25 \;\Rightarrow\; \Theta(B, \mathsf{C}_{\text{maint}}) = \top \text{ (still safe at this speed).}$$

If the plant enforces a tighter envelope, e.g. $t_r = 0.30$ s and $a_{\max} = 2.0 \text{ m/s}^2$ in maintenance,

$$s_{\text{stop}}(0.60) = 0.60(0.30) + \frac{0.36}{4} = 0.18 + 0.09 = 0.27 \text{ m} > 0.25 \;\Rightarrow\; \Theta(B, \mathsf{C}'_{\text{maint}}) = \bot.$$

(iii) Occluded sensing (interval distance). $D - d_h \in [0.30 - 0.05,\; 0.34 - 0.05] = [0.25, 0.29]$ m. Since $0.18 < 0.25$ would be safe, but under degraded braking ($t_r = 0.30$, $a_{\max} = 2.0$) we have $0.27 > 0.25$ which can be unsafe, the truth of B cannot be fixed without resolving the context precisely:

$$\Theta(B, \mathsf{C}_{\text{occl}}) = \star.$$

Summary.

$$\Theta(B, \mathsf{C}_{\text{op}}) = \top, \qquad \Theta(B, \mathsf{C}'_{\text{maint}}) = \bot, \qquad \Theta(B, \mathsf{C}_{\text{occl}}) = \star.$$

Thus, safety at a given speed is a context-indexed truth: it depends on separation, braking capability, and sensing fidelity.

Notation 2.5
Let $\mathcal{L}$ be a fixed formal language and $\mathcal{M} = (\mathcal{P}, \mathcal{V}, v)$ a logical structure as above. We use $\top/\bot$ interchangeably with True/False when convenient.

Definition 2.6 (Upside-Down Logic)**.** [49, 50] An *Upside-Down* (UD) version of $\mathcal{M}$ is any system $\mathcal{M}'$ obtained by a transformation U acting on propositions and/or contexts, with the following property: for every $A \in \mathcal{P}$ and $\mathsf{C} \in \mathsf{Ctx}$, there exist transformed objects $U(A)$ and/or $U(\mathsf{C})$ such that

- **True→False:** if $\Theta(A, \mathsf{C}) = \top$, then $\Theta\big(U(A), U(\mathsf{C})\big) = \bot$;
- **False→True:** if $\Theta(A, \mathsf{C}) = \bot$, then $\Theta\big(U(A), U(\mathsf{C})\big) = \top$.

Moreover, U must be well defined so that the transformed system $\mathcal{M}'$ remains internally coherent (i.e., its axioms and rules do not yield triviality).

Example 2.7 (Polarity flip in heating vs. cooling (setpoint tracking))**.** Consider a first–order thermal plant with input u (positive = "add heat"):

$$\dot{T}(t) \;=\; -\frac{1}{\tau}\big(T(t) - T_{\text{env}}\big) \;+\; k\,u(t), \qquad e(t) := T_{\text{sp}} - T(t).$$

With proportional control $u = K_p e$, the error dynamics are

$$\dot{e} \;=\; -\dot{T} \;=\; \frac{1}{\tau}\big(T - T_{\text{env}}\big) - kK_p e.$$

Context $\mathcal{C}$ (heating): actuator adds heat, $k > 0$. Proposition A: "for $K_p > 0$ and $e > 0$, the control action $u = K_p e > 0$ reduces the error" is *True* (the $-kK_p e$ term drives $\dot{e}$ negative). Now apply the transformation U that switches the physical mode to *cooling* so that a positive command *removes* heat:

$$U(\mathcal{C}): \qquad \dot{T} \;=\; -\frac{1}{\tau}(T - T_{\text{env}}) \;-\; k\,u \quad (\text{equivalently } k \mapsto -k).$$

Under $U(\mathcal{C})$, keeping $u = K_p e$ with $K_p > 0$ makes the same statement A *False* (a positive u decreases T, increasing e). Define the transformed proposition $U(A)$: "use inverted control $u = -K_p e$". Then for $e > 0$ we command $u < 0$ (cooling), and the error decreases; thus $U(A)$ is *True* in $U(\mathcal{C})$. Numerical check: take $\tau = 10$, $k = 0.5$, T_{env} arbitrary. If $e = 2$ and $K_p = 1$, then heating ($k > 0$): $\dot{e} = \frac{1}{10}(T - T_{\text{env}}) - 1.0 \cdot e$ is reduced by the $-e$ term; cooling ($k \mapsto -0.5$): with $u = K_p e = +2$, $\dot{e}$ gains $+1.0 \cdot e$ and grows; with $u = -K_p e = -2$, $\dot{e}$ gains $-1.0 \cdot e$ and decays.

Example 2.8 (Anti-windup under saturation (back-calculation flips integrator direction))**.** Consider a PI controller $u = K_p e + K_i I$, $\dot{I} = e$, acting on any plant, and a symmetric actuator saturation $\text{sat}(\cdot)$ with limit $u_{\max}$. In the nominal (unsaturated) context $\mathcal{C}$ we have the proposition A: "if $e > 0$ then $\dot{I} > 0$ (the integrator grows)", which is *True* since $\dot{I} = e$. When saturation occurs, the context changes to

$$U(\mathcal{C}): \qquad u_{\text{cmd}} = K_p e + K_i I, \quad u = \text{sat}(u_{\text{cmd}}), \quad \dot{I} \;=\; e \;-\; k_b\big(u_{\text{cmd}} - u\big),$$

with back-calculation gain $k_b > 0$. Under $U(\mathcal{C})$, if the actuator is saturated *high* ($u = u_{\max} < u_{\text{cmd}}$), then $u_{\text{cmd}} - u > 0$ and the term $-k_b(u_{\text{cmd}} - u)$ can dominate, making $\dot{I} < 0$. Hence A becomes *False*. The transformed proposition $U(A)$ is: "under high saturation, the integrator *decreases* (winds down), i.e. $\dot{I} < 0$". This is *True* in $U(\mathcal{C})$. Numerical check: let $e = 0.1$, $K_b = 5$, and $u_{\text{cmd}} - u = 0.5$ at saturation. Then

$$\dot{I} \;=\; 0.1 \;-\; 5 \times 0.5 \;=\; -2.4 \;<\; 0,$$

so the integrator direction is inverted relative to $e > 0$, preventing windup.

Example 2.9 (Stability decision flips with sensor/plant-sign inversion)**.** A unity-feedback loop with plant $G(s) = \dfrac{k}{\tau s + 1}$ and proportional control $u = K_p e$ has closed-loop pole

$$s \;=\; -\frac{1 + K_p k}{\tau}.$$

In context $\mathcal{C}$ where the net loop gain is positive ($k > 0$), proposition A: "$K_p > 0$ stabilizes the loop" is *True* because $1 + K_p k > 0$. Suppose a wiring change inverts the sensor or actuator polarity, effectively sending $k \mapsto -k$; that is, move to

$$U(\mathcal{C}): \qquad G(s) = \frac{-k}{\tau s + 1}, \quad \text{pole } s \;=\; -\frac{1 - K_p k}{\tau}.$$

Now A is *False*: if $K_p > 1/k$ the pole becomes positive (unstable). The transformed proposition $U(A)$ is: "choose $K_p < 0$ to stabilize when the loop gain is negative". This is *True*, because with $K_p < 0$ one has $1 - K_p k = 1 + |K_p|k > 0$. Numerical check: take $\tau = 1$, $k = 2$. With inversion ($U(\mathcal{C})$) and $K_p = +1$, the pole is $s = -(1 - 2) = +1$ (unstable). With $K_p = -1$, the pole is $s = -(1 + 2) = -3$ (stable). Thus the stability decision flips under the sign-inverting context.

2.2. Plithogenic Set

A Plithogenic Set [34, 80, 35] represents elements by coupling an attribute–value membership with an explicit contradiction measure, thereby extending the formalisms of fuzzy [1, 81], intuitionistic [82, 83], and neutrosophic sets [19, 84].

Definition 2.10 (Fuzzy Set)**.** [1, 85] A *Fuzzy set* τ in a non-empty universe Y is a mapping $\tau : Y \to [0,1]$. A *fuzzy relation* on Y is a fuzzy subset δ in $Y \times Y$. If τ is a fuzzy set in Y and δ is a fuzzy relation on Y, then δ is called a *fuzzy relation on* τ if

$$\delta(y,z) \le \min\{\tau(y), \tau(z)\} \quad \text{for all } y,z \in Y.$$

Example 2.11 (Fuzzy Set in HVAC setpoint tracking)**.** Let the universe be the temperature error $Y = \mathbb{R}$ (in °C), $e := T_{\mathrm{sp}} - T$. Define the fuzzy set "Positive Error" by

$$\tau(e) \;=\; \begin{cases} 0, & e \le 0, \\ e/5, & 0 < e < 5, \\ 1, & e \ge 5, \end{cases} \qquad \text{so } \; \tau : Y \to [0,1].$$

Define a fuzzy relation ("compatibility of positive errors") on Y by

$$\delta(e_1, e_2) \;:=\; \min\{\tau(e_1), \tau(e_2)\}\, \exp\bigl(-|e_1 - e_2|/10\bigr),$$

which satisfies $\delta(e_1, e_2) \le \min\{\tau(e_1), \tau(e_2)\}$ for all e_1, e_2. Numerical check: if $e_1 = 6$, $e_2 = 3$, then $\tau(6) = 1$, $\tau(3) = 0.6$, hence

$$\delta(6,3) = 0.6\, e^{-0.3} \approx 0.6 \times 0.740818 = 0.44449 \;\le\; 0.6 = \min\{\tau(6), \tau(3)\}.$$

This fuzzy set/relation pair models how a heater controller grades "how positive" an error is and how similar two error values are when aggregating rules.

Definition 2.12 (Plithogenic Set)**.** [86, 34] Let S be a universal set and let $P \subseteq S$ be nonempty. A *Plithogenic Set* is a quintuple

$$PS \;=\; (P,\, v,\, Pv,\, pdf,\, pCF),$$

consisting of:

- an attribute v;
- a set Pv of admissible values of v;
- a *Degree of Appurtenance Function (DAF)* $pdf : P \times Pv \to [0,1]^s$;[†]
- a *Degree of Contradiction Function (DCF)* $pCF : Pv \times Pv \to [0,1]^t$.

The DCF satisfies, for all $a, b \in Pv$,

$$\text{Reflexivity: } pCF(a,a) = 0, \qquad \text{Symmetry: } pCF(a,b) = pCF(b,a).$$

Here $s \in \mathbb{N}$ is the appurtenance dimension and $t \in \mathbb{N}$ is the contradiction dimension.

Example 2.13 (Plithogenic Set for PID tuning styles)**.** Let $S = \mathbb{R}^3_{>0}$ be all PID gains $x = (K_p, K_i, K_d)$, and $P \subset S$ a finite catalog of admissible tunings. Take the attribute $v =$"tuning style" with value set

$$Pv = \{\text{Aggressive (A)},\ \text{Nominal (N)},\ \text{Conservative (C)}\}.$$

[†] In the literature, DAFs appear in several forms: some are powerset–valued while others take values in the hypercube $[0,1]^s$. We adopt the latter, classical convention; cf. [87].

For $s = 2$, define the Degree of Appurtenance Function $pdf : P \times Pv \to [0,1]^2$ as

$$pdf(x, a) = \big(\text{tracking}(x \mid a),\ \text{robustness}(x \mid a)\big),$$

both components normalized to $[0,1]$ from step-response ITAE and gain/phase margins. For the Degree of Contradiction Function $pCF : Pv \times Pv \to [0,1]^t$ with $t = 1$, set

$$pCF(\mathrm{A},\mathrm{A}) = pCF(\mathrm{N},\mathrm{N}) = pCF(\mathrm{C},\mathrm{C}) = 0, \quad pCF(\mathrm{A},\mathrm{N}) = pCF(\mathrm{N},\mathrm{A}) = 0.4,$$

$$pCF(\mathrm{N},\mathrm{C}) = pCF(\mathrm{C},\mathrm{N}) = 0.3, \quad pCF(\mathrm{A},\mathrm{C}) = pCF(\mathrm{C},\mathrm{A}) = 0.8,$$

which is symmetric and reflexive. Example (concrete gains): for $x_0 = (K_p, K_i, K_d) = (2.0,\ 0.8,\ 0.05)$ we might obtain

$$pdf(x_0, \mathrm{A}) = (0.90,\ 0.30), \quad pdf(x_0, \mathrm{N}) = (0.75,\ 0.55), \quad pdf(x_0, \mathrm{C}) = (0.55,\ 0.80).$$

Interpretation: x_0 fits "Aggressive" with high tracking (0.90) but low robustness (0.30), while "Conservative" exhibits the opposite trade-off; the large contradiction $pCF(\mathrm{A},\mathrm{C}) = 0.8$ captures their incompatibility in controller design.

Definition 2.14 (Plithogenic Fuzzy Set ($s = 1,\ t = 1$))**.** [88, 89, 34, 44] When the appurtenance and contradiction dimensions are both one, a *Plithogenic Fuzzy Set* is a Plithogenic Set $PS = (P, v, Pv, pdf, pCF)$ with

$$pdf : P \times Pv \to [0,1], \qquad pCF : Pv \times Pv \to [0,1].$$

For $x \in P$ and $a \in Pv$, set

$$\mu_P(x \mid a) \ :=\ pdf(x, a) \in [0,1],$$

the (scalar) fuzzy membership of x relative to a. For $a, b \in Pv$, define the scalar contradiction

$$c(a, b) \ :=\ pCF(a, b) \in [0,1], \qquad c(a, a) = 0,\ \ c(a, b) = c(b, a).$$

Example 2.15 (Plithogenic Fuzzy Set for cruise-control throttle)**.** Let $P = \mathbb{R}_{\geq 0}$ be the speed error $e := v_{\mathrm{sp}} - v$ (km/h, only underspeed considered), and $v =$"throttle linguistic level" with

$$Pv = \{\text{Low (L)},\ \text{Medium (M)},\ \text{High (H)}\}.$$

With $s = t = 1$, define the fuzzy memberships $pdf(e, a) = \mu_P(e \mid a) \in [0,1]$ (triangles):

$$\mu_P(e \mid \mathrm{L}) = \max\big\{0,\ 1 - \tfrac{e}{10}\big\}, \quad \mu_P(e \mid \mathrm{M}) = \max\big\{0,\ 1 - \tfrac{|e-10|}{10}\big\}, \quad \mu_P(e \mid \mathrm{H}) = \max\big\{0,\ 1 - \tfrac{|e-20|}{10}\big\}.$$

Define a scalar contradiction $c : Pv \times Pv \to [0,1]$ by

$$c(\mathrm{L},\mathrm{L}) = c(\mathrm{M},\mathrm{M}) = c(\mathrm{H},\mathrm{H}) = 0, \quad c(\mathrm{L},\mathrm{M}) = c(\mathrm{M},\mathrm{L}) = 0.5,$$

$$c(\mathrm{M},\mathrm{H}) = c(\mathrm{H},\mathrm{M}) = 0.6, \quad c(\mathrm{L},\mathrm{H}) = c(\mathrm{H},\mathrm{L}) = 0.9.$$

Numerical evaluation at $e = 12$ km/h:

$$\mu_P(12 \mid \mathrm{L}) = \max\{0, 1 - 1.2\} = 0, \quad \mu_P(12 \mid \mathrm{M}) = 1 - \tfrac{|12-10|}{10} = 0.8, \quad \mu_P(12 \mid \mathrm{H}) = 1 - \tfrac{|12-20|}{10} = 0.2.$$

Thus Medium dominates for $e = 12$, while the high contradiction $c(\mathrm{L},\mathrm{H}) = 0.9$ encodes that "Low" and "High" throttle are strongly conflicting linguistic actions. This (P, v, Pv, pdf, pCF) is a Plithogenic Fuzzy Set specialized to a cruise-control context.

2.3. Fuzzy Control

Fuzzy control systems employ fuzzy set theory and rule-based (IF–THEN) inference to translate precise sensor measurements into smooth actuator commands (cf.[57, 58]).

Definition 2.16 (Fuzzy Control System)**.** [57, 58] A *fuzzy control system* implements a mapping from a vector of crisp inputs to a crisp control output via fuzzy-logic reasoning. Formally, let

$$x = (x_1, \dots, x_n) \in X_1 \times \cdots \times X_n, \quad y \in Y$$

be the input and output universes. A fuzzy controller is specified by:

1. **Fuzzification:** For each input x_i one defines a finite family of fuzzy sets

$$A_{i1}, A_{i2}, \dots, A_{im_i} \subseteq X_i, \quad \mu_{A_{ij}} : X_i \to [0, 1].$$

2. **Rule Base:** A collection of K IF–THEN rules

$$R_k : \text{ IF } x_1 \text{ is } A_{1j_1} \ \wedge \ \dots \ \wedge \ x_n \text{ is } A_{nj_n} \text{ THEN } y \text{ is } B_k,$$

   where each consequent $B_k \subseteq Y$ is a fuzzy set with membership $\mu_{B_k} : Y \to [0, 1]$.

3. **Inference:** Given a specific input x, each rule fires with strength

$$\alpha_k = T\big(\mu_{A_{1j_1}}(x_1), \dots, \mu_{A_{nj_n}}(x_n)\big),$$

   where T is a chosen t-norm (e.g. $T(a, b) = \min(a, b)$).

4. **Aggregation:** The fuzzy output set $B' \subseteq Y$ is formed by

$$\mu_{B'}(y) = S_{k=1}^{K} \left[\alpha_k \ \otimes \ \mu_{B_k}(y)\right],$$

   where $\otimes$ is typically the minimum and S the maximum operator.

5. **Defuzzification:** A crisp control action $y^* \in Y$ is extracted, often via the centroid formula

$$y^* = \frac{\displaystyle\int_Y y\, \mu_{B'}(y)\, dy}{\displaystyle\int_Y \mu_{B'}(y)\, dy}.$$

Example 2.17 (Vehicle cruise-control (fuzzy throttle adjustment))**.** Consider an automobile cruise controller that adjusts throttle to track a speed setpoint. Inputs are the *speed error* $e := v_{\text{sp}} - v$ (km/h) and the *error rate* $\dot{e}$ (km/h/s). The output is the *throttle change* Δu (percentage points).

Fuzzification. Let

$$X_1 = \{-20 \le e \le 20\} \subset \mathbb{R}, \qquad X_2 = \{-5 \le \dot{e} \le 5\} \subset \mathbb{R}, \qquad Y = \{-50 \le \Delta u \le 50\} \subset \mathbb{R}.$$

Use triangular fuzzy sets:

For e: Negative/Zero/Positive (N/Z/P),

$$\mu_{e,\text{N}}(e) = \max\Big\{0,\ 1 - \tfrac{|e+10|}{10}\Big\}, \quad \mu_{e,\text{Z}}(e) = \max\Big\{0,\ 1 - \tfrac{|e|}{10}\Big\}, \quad \mu_{e,\text{P}}(e) = \max\Big\{0,\ 1 - \tfrac{|e-10|}{10}\Big\}.$$

For $\dot{e}$: Negative/Zero/Positive (N/Z/P),

$$\mu_{\dot{e},\mathrm{N}}(\dot{e}) = \max\Big\{0,\, 1 - \tfrac{|\dot{e}+2.5|}{2.5}\Big\}, \quad \mu_{\dot{e},\mathrm{Z}}(\dot{e}) = \max\Big\{0,\, 1 - \tfrac{|\dot{e}|}{2}\Big\}, \quad \mu_{\dot{e},\mathrm{P}}(\dot{e}) = \max\Big\{0,\, 1 - \tfrac{|\dot{e}-2.5|}{2.5}\Big\}.$$

Rule base (Mamdani antecedents). With $T = \min$,

$$\begin{aligned}
&R_1 : \text{IF } (e \text{ is P}) \wedge (\dot{e} \text{ is Z}) \text{ THEN } \Delta u \text{ is P},\\
&R_2 : \text{IF } (e \text{ is P}) \wedge (\dot{e} \text{ is P}) \text{ THEN } \Delta u \text{ is M},\\
&R_3 : \text{IF } (e \text{ is Z}) \wedge (\dot{e} \text{ is P}) \text{ THEN } \Delta u \text{ is N},\\
&R_4 : \text{IF } (e \text{ is N}) \text{ THEN } \Delta u \text{ is N},\\
&R_5 : \text{IF } (e \text{ is Z}) \wedge (\dot{e} \text{ is Z}) \text{ THEN } \Delta u \text{ is Z}.
\end{aligned}$$

For the consequents we use a standard zero-order Sugeno choice (singleton outputs)

$$\Delta u(\mathrm{N}) = -30, \quad \Delta u(\mathrm{Z}) = 0, \quad \Delta u(\mathrm{M}) = +20, \quad \Delta u(\mathrm{P}) = +40 \quad \text{(percentage points)}.$$

Numerical evaluation (one step). Suppose $e = 8$ km/h (underspeed) and $\dot{e} = -0.5$ km/h/s (error decreasing slowly). Membership degrees:

$$\mu_{e,\mathrm{P}}(8) = 1 - \tfrac{|8-10|}{10} = 0.8, \quad \mu_{e,\mathrm{Z}}(8) = 1 - \tfrac{|8|}{10} = 0.2, \quad \mu_{e,\mathrm{N}}(8) = 0,$$

$$\mu_{\dot{e},\mathrm{Z}}(-0.5) = 1 - \tfrac{0.5}{2} = 0.75,$$

$$\mu_{\dot{e},\mathrm{N}}(-0.5) = 1 - \tfrac{|-0.5+2.5|}{2.5} = 1 - \tfrac{2.0}{2.5} = 0.2,$$

$$\mu_{\dot{e},\mathrm{P}}(-0.5) = 0.$$

Rule firing strengths ($\alpha_k = \min$ of antecedent degrees):

$$\alpha_1 = \min(0.8, 0.75) = 0.75, \quad \alpha_2 = \min(0.8, 0) = 0, \quad \alpha_3 = \min(0.2, 0) = 0,$$

$$\alpha_4 = 0, \quad \alpha_5 = \min(0.2, 0.75) = 0.2.$$

Defuzzification (Sugeno weighted average). With singleton consequents $y_k \in \{-30, 0, 20, 40\}$,

$$y^* = \frac{\sum_k \alpha_k\, y_k}{\sum_k \alpha_k} = \frac{0.75 \cdot 40 \;+\; 0.2 \cdot 0}{0.75 + 0.2} = \frac{30}{0.95} \approx 31.58.$$

Thus the controller commands an increase of about 31.6 percentage points in throttle, consistent with a large positive error that is already shrinking slowly.

2.4. Upside-Down Logic on a Plithogenic Fuzzy Set with Contradiction Reset

The Upside-Down mechanism with contradiction reset in a Plithogenic Fuzzy Set inverts fuzzy memberships once a contradiction surpasses a chosen threshold and then neutralizes the corresponding contradictions. This yields a consistent representation in contexts where previously conflicting values must be reconciled (cf. [90, 51]).

Definition 2.18 (Upside-Down transform with contradiction reset)**.** (cf. [90, 51]) Let $PS = (P, v, Pv, pdf, pCF)$ be a Plithogenic Fuzzy Set and write

$$\mu_P(x \mid a) := pdf(x, a) \in [0, 1], \qquad c(a, b) := pCF(a, b) \in [0, 1],$$

with $c(a, a) = 0$ and $c(a, b) = c(b, a)$. Fix an anchor $b \in Pv$ and a threshold $\tau \in [0, 1]$. Define the *activation set*

$$A_\tau(b) \ := \ \{\, a \in Pv \ : \ c(a, b) \geq \tau \,\}.$$

The *Upside-Down* transform produces a new Plithogenic Fuzzy Set

$$PS^{U_{b,\tau}} \ = \ \big(P, \ v, \ Pv, \ pdf^{U_{b,\tau}}, \ pCF^{U_{b,\tau}}\big),$$

where

$$pdf^{U_{b,\tau}}(x, a) \ := \ \begin{cases} 1 - \mu_P(x \mid a), & a \in A_\tau(b), \\ \mu_P(x \mid a), & a \notin A_\tau(b), \end{cases}$$

and the updated contradiction map $pCF^{U_{b,\tau}}$ is given, for all $u, v \in Pv$, by

$$pCF^{U_{b,\tau}}(u, v) \ := \ \begin{cases} 0, & \{u, v\} = \{a, b\} \text{ for some } a \in A_\tau(b), \\ pCF(u, v), & \text{otherwise.} \end{cases}$$

In words: if the pair (a, b) activates (i.e., $c(a, b) \geq \tau$), then the membership at a is flipped and the post-transform contradiction between a and the anchor b is *reset to zero*.

Example 2.19 (Data-center cooling fans under quiet-hours noise limits)**.** Let the universe be the temperature rise above target,

$$P = \{e \in [0, 20]\ {}^\circ\text{C}\},$$

and take the attribute $v =$"fan level" with values

$$Pv = \{\text{Low(L)}, \ \text{Medium(M)}, \ \text{High(H)}\}.$$

Define triangular memberships (all clipped to $[0, 1]$):

$$\mu_P(e \mid \text{L}) = \max\{0, \ 1 - \tfrac{e}{8}\}, \quad \mu_P(e \mid \text{M}) = \max\{0, \ 1 - \tfrac{|e-8|}{8}\}, \quad \mu_P(e \mid \text{H}) = \max\{0, \ 1 - \tfrac{|e-16|}{8}\}.$$

Fix the anchor $b = \text{M}$ (quiet-hours "preferred" level) and set the contradiction map to reflect acoustic constraints:

$$c(\text{L}, \text{M}) = 0.25, \qquad c(\text{H}, \text{M}) = 0.85, \qquad c(\text{M}, \text{M}) = 0,$$

with symmetry $c(a, b) = c(b, a)$. Choose the threshold $\tau = 0.80$. Then the activation set is

$$A_\tau(\text{M}) = \{\text{H}\} \quad \text{since} \quad c(\text{H}, \text{M}) = 0.85 \geq 0.80.$$

Numerical instance. For a hot day with $e = 14°$C:

$$\mu_P(14 \mid \text{L}) = \max\{0, 1 - \tfrac{14}{8}\} = 0, \quad \mu_P(14 \mid \text{M}) = 1 - \tfrac{|14-8|}{8} = 1 - \tfrac{6}{8} = 0.25,$$

$$\mu_P(14 \mid \text{H}) = 1 - \tfrac{|14-16|}{8} = 1 - \tfrac{2}{8} = 0.75.$$

Applying the Upside-Down transform with contradiction reset gives

$$\mu_P^U(14 \mid \text{H}) = 1 - \mu_P(14 \mid \text{H}) = 1 - 0.75 = 0.25, \qquad \mu_P^U(14 \mid \text{M}) = 0.25, \qquad \mu_P^U(14 \mid \text{L}) = 0,$$

and updates the contradiction to the anchor as

$$pCF^{U}(\mathrm{H},\mathrm{M}) = 0, \qquad pCF^{U}(u,v) = pCF(u,v) \text{ otherwise.}$$

Because "High" strongly contradicts the anchor during quiet hours, its membership is flipped (discouraged) and its contradiction to the anchor is neutralized. After the transform, M and H are equally plausible (both 0.25), biasing decisions away from sustained high-speed fans while keeping the representation consistent.

Example 2.20 (AGV forklift speed in a pedestrian zone: flipping "Fast" and neutralizing conflict)**.** Let P be the aisle width (meters) measurable by the AGV's lidar:

$$P = \{w \in [1.2, 4.2]\ \mathrm{m}\}.$$

Take $v =$"speed level" with

$$Pv = \{\mathrm{Slow(S)},\ \mathrm{Nominal(N)},\ \mathrm{Fast(F)}\}.$$

Use symmetric triangular memberships (half-width $0.9\,\mathrm{m}$) centered at $1.5,\ 2.4,\ 3.3\,\mathrm{m}$:

$$\mu_P(w \mid \mathrm{S}) = \max\Big\{0,\ 1 - \tfrac{|w-1.5|}{0.9}\Big\}, \quad \mu_P(w \mid \mathrm{N}) = \max\Big\{0,\ 1 - \tfrac{|w-2.4|}{0.9}\Big\}, \quad \mu_P(w \mid \mathrm{F}) = \max\Big\{0,\ 1 - \tfrac{|w-3.3|}{0.9}\Big\}.$$

In a signed pedestrian zone, set anchor $b = \mathrm{N}$ and reflect policy conflicts by

$$c(\mathrm{S},\mathrm{N}) = 0.20, \qquad c(\mathrm{F},\mathrm{N}) = 0.90, \qquad c(\mathrm{N},\mathrm{N}) = 0,$$

and choose $\tau = 0.80$. Thus

$$A_\tau(\mathrm{N}) = \{\mathrm{F}\}.$$

Numerical instance. At width $w = 3.0\,\mathrm{m}$:

$$\mu_P(3.0 \mid \mathrm{S}) = \max\{0, 1 - \tfrac{|3.0-1.5|}{0.9}\} = 0,$$

$$\mu_P(3.0 \mid \mathrm{N}) = 1 - \tfrac{|3.0-2.4|}{0.9} = 1 - \tfrac{0.6}{0.9} = \tfrac{1}{3}, \qquad \mu_P(3.0 \mid \mathrm{F}) = 1 - \tfrac{|3.0-3.3|}{0.9} = 1 - \tfrac{0.3}{0.9} = \tfrac{2}{3}.$$

Applying the Upside-Down transform with contradiction reset (on F) yields

$$\mu_P^U(3.0 \mid \mathrm{F}) = 1 - \tfrac{2}{3} = \tfrac{1}{3}, \qquad \mu_P^U(3.0 \mid \mathrm{N}) = \tfrac{1}{3}, \qquad \mu_P^U(3.0 \mid \mathrm{S}) = 0,$$

and the contradiction is neutralized for the activated pair:

$$pCF^{U}(\mathrm{F},\mathrm{N}) = 0, \qquad pCF^{U}(u,v) = pCF(u,v) \text{ otherwise.}$$

In a pedestrian zone, "Fast" conflicts with the nominal policy. The transform flips its membership (from $2/3$ to $1/3$) and resets the conflict with the anchor, steering the decision surface away from fast travel while restoring local consistency for subsequent reasoning.

2.5. Two-mode De-Plithogenication

We now formalize *Two-mode De-Plithogenication*. The mechanism covers two distinct situations: (i) a contradiction that reflects a *real* conflict, and (ii) a contradiction that is merely *apparent* and should not cause an inversion (cf. [90, 51]).

Definition 2.21 (Improved De-Plithogenication (two-mode; general plithogenic set))**.** (cf. [90, 51]) Let $PS = (P, v, Pv, \mathrm{pdf}, \mathrm{pCF})$ be a plithogenic set with $\mathrm{pdf} : P \times Pv \to [0, 1]$ the Degree of Appurtenance Function (DAF) and $\mathrm{pCF} : Pv \times Pv \to [0, 1]$ the Degree of Contradiction Function (DCF), symmetric and satisfying $\mathrm{pCF}(a, a) = 0$. Fix an *anchor* $b \in Pv$ and a *threshold* $\tau \in [0, 1]$.

Activation. For $a \in Pv$, declare that a *activates (relative to b, τ)* when

$$\mathsf{Act}(a \mid b, \tau) \;:=\; \mathbf{1}\left[\,\mathrm{pCF}(a, b) \;\geq\; \tau\,\right] = 1.$$

Mode selection. Let $\mathcal{M} : Pv \times Pv \to \{0, 1\}$ be a (policy) *mode selector* on unordered pairs, with $\mathcal{M}(\{a, b\}) = \mathcal{M}(\{b, a\})$. For any activated pair $\{a, b\}$, the semantics are:

- Mode 0 (Neutralize-only): $\mathcal{M}(\{a, b\}) = 0$ means the contradiction is only apparent. No appurtenance coordinate associated with a (or b) is flipped; the contradiction is simply reset to 0.
- Mode 1 (Invert+Neutralize): $\mathcal{M}(\{a, b\}) = 1$ means the contradiction is genuine. We first invert the appurtenance coordinate(s) tied to a (or b) and then reset their contradiction to 0.

Two-mode Upside-Down operator. Define $U^{\mathrm{imp}}_{b,\tau,\mathcal{M}}(PS) =: (P, v, Pv, \mathrm{pdf}^U, \mathrm{pCF}^U)$ by

$$\mathrm{pdf}^U(x, a) := \begin{cases} 1 - \mathrm{pdf}(x, a), & \mathsf{Act}(a \mid b, \tau) = 1 \text{ and } \mathcal{M}(\{a, b\}) = 1, \\ \mathrm{pdf}(x, a), & \text{otherwise}, \end{cases} \qquad (x \in P,\ a \in Pv),$$

$$\mathrm{pCF}^U(u, w) := \begin{cases} 0, & \mathsf{Act}(u \mid b, \tau) = 1 \text{ or } \mathsf{Act}(w \mid b, \tau) = 1, \\ \mathrm{pCF}(u, w), & \text{otherwise}, \end{cases} \qquad (u, w \in Pv,\ \text{with symmetry preserved}).$$

Two-mode De-Plithogenication sequence. A finite composition

$$PS^{\text{imp-dep}} \;:=\; U^{\mathrm{imp}}_{b_k,\tau_k,\mathcal{M}_k} \circ \cdots \circ U^{\mathrm{imp}}_{b_2,\tau_2,\mathcal{M}_2} \circ U^{\mathrm{imp}}_{b_1,\tau_1,\mathcal{M}_1}(PS)$$

is called an *Improved De-Plithogenication* when, for every pair $\{u, w\} \subseteq Pv$ that activates at least once in the sequence, the final contradiction vanishes: $\mathrm{pCF}^{\text{imp-dep}}(u, w) = 0$. If the activated pairs cover all of $Pv \times Pv$, then $\mathrm{pCF}^{\text{imp-dep}} \equiv 0$. Moreover, once this stabilized state is reached, the operator acts idempotently.

Example 2.22 (Battery-pack cooling level: Mode 0 (Neutralize-only) for an apparent contradiction)**. Setup.** Let the measurable variable be the coolant-to-pack temperature rise

$$P = \{\, \Delta T \in [0, 25]\ ^\circ\mathrm{C}\,\},$$

and take the attribute $v =$“fan setting” with values

$$Pv = \{\mathrm{Low(L)},\ \mathrm{Medium(M)},\ \mathrm{High(H)}\}.$$

Define triangular memberships (clipped to $[0, 1]$):

$$\mu_P(\Delta T \mid \mathrm{L}) = \max\Big\{0,\, 1 - \tfrac{\Delta T}{10}\Big\}, \quad \mu_P(\Delta T \mid \mathrm{M}) = \max\Big\{0,\, 1 - \tfrac{|\Delta T - 10|}{10}\Big\}, \quad \mu_P(\Delta T \mid \mathrm{H}) = \max\Big\{0,\, 1 - \tfrac{|\Delta T - 20|}{10}\Big\}.$$

Fix the anchor $b = \mathrm{M}$. The contradiction map (quiet-hours acoustic policy) is

$$c(\mathrm{L}, \mathrm{M}) = 0.30, \qquad c(\mathrm{H}, \mathrm{M}) = 0.82, \qquad c(\mathrm{M}, \mathrm{M}) = 0,$$

with symmetry $c(a,b) = c(b,a)$. Take threshold $\tau = 0.80$.

Activation.

$$A_\tau(\mathrm{M}) = \{\mathrm{H}\} \quad \text{since } c(\mathrm{H},\mathrm{M}) = 0.82 \geq 0.80.$$

Numerics (pre-transform). At $\Delta T = 18°\mathrm{C}$,

$$\mu_P(18 \mid \mathrm{L}) = \max\{0, 1 - \tfrac{18}{10}\} = 0, \qquad \mu_P(18 \mid \mathrm{M}) = 1 - \tfrac{|18-10|}{10} = 0.2, \qquad \mu_P(18 \mid \mathrm{H}) = 1 - \tfrac{|18-20|}{10} = 0.8.$$

Mode selection. A sensor bias is diagnosed (apparent conflict), so choose *Mode 0* for the activated pair:

$$\mathcal{M}(\{\mathrm{H},\mathrm{M}\}) = 0.$$

Two-mode operator effect. By $U^{\mathrm{imp}}_{b,\tau,\mathcal{M}}$,

$$\mathrm{pdf}^U(x,\mathrm{H}) = \mu_P^U(x \mid \mathrm{H}) = \mu_P(x \mid \mathrm{H}) \quad \text{(no flip in Mode 0)},$$

$$\mathrm{pCF}^U(\mathrm{H},\mathrm{M}) = 0, \qquad \mathrm{pCF}^U(u,w) = \mathrm{pCF}(u,w) \text{ otherwise.}$$

Hence at $\Delta T = 18°\mathrm{C}$,

$$\mu_P^U(18 \mid \mathrm{H}) = 0.8,\;\; \mu_P^U(18 \mid \mathrm{M}) = 0.2,\;\; \mu_P^U(18 \mid \mathrm{L}) = 0,$$

but the post-transform contradiction to the anchor is neutralized: $c^U(\mathrm{H},\mathrm{M}) = 0$.

The contradiction was only apparent; therefore the memberships are *not* inverted, yet the conflict to the anchor is cleared so downstream reasoning proceeds without penalizing H due to a spurious policy clash.

Example 2.23 (3D printer travel speed for delicate features: Mode 1 (Invert + Neutralize) for a genuine contradiction)**. Setup.** Let the measurable variable be the feature width (e.g. wall thickness)

$$P = \{\, w \in [0.2, 0.8] \text{ mm} \,\}.$$

Let $v =$"travel speed level" with

$$Pv = \{\mathrm{Slow(S)},\ \mathrm{Normal(N)},\ \mathrm{Fast(F)}\}.$$

Use triangles (clipped to $[0,1]$) centered at $0.25,\ 0.50,\ 0.70$ mm with half-widths $0.20,\ 0.25,\ 0.20$ respectively:

$$\mu_P(w \mid \mathrm{S}) = \max\Big\{0,\ 1 - \tfrac{|w-0.25|}{0.20}\Big\},\ \mu_P(w \mid \mathrm{N}) = \max\Big\{0,\ 1 - \tfrac{|w-0.50|}{0.25}\Big\},\ \mu_P(w \mid \mathrm{F}) = \max\Big\{0,\ 1 - \tfrac{|w-0.70|}{0.20}\Big\}.$$

Fix the anchor $b = \mathrm{N}$. For a fragile material/overhang policy, set

$$c(\mathrm{S},\mathrm{N}) = 0.30, \qquad c(\mathrm{F},\mathrm{N}) = 0.90, \qquad c(\mathrm{N},\mathrm{N}) = 0,$$

and $\tau = 0.85$.

Activation.

$$A_\tau(\mathrm{N}) = \{\mathrm{F}\} \quad \text{since } c(\mathrm{F},\mathrm{N}) = 0.90 \geq 0.85.$$

Numerics (pre-transform). At $w = 0.65$ mm,

$$\mu_P(0.65 \mid \mathrm{S}) = \max\{0, 1 - \tfrac{|0.65-0.25|}{0.20}\} = 0, \qquad \mu_P(0.65 \mid \mathrm{N}) = 1 - \tfrac{|0.65-0.50|}{0.25} = 1 - \tfrac{0.15}{0.25} = 0.4,$$

$$\mu_P(0.65 \mid \mathrm{F}) = 1 - \tfrac{|0.65-0.70|}{0.20} = 1 - \tfrac{0.05}{0.20} = 0.75.$$

Mode selection. QC flags stringing/warping at high travel on this geometry (genuine conflict), hence choose *Mode 1*:

$$\mathcal{M}(\{\mathrm{F},\mathrm{N}\}) = 1.$$

Two-mode operator effect. For the activated pair, $U^{\mathrm{imp}}_{b,\tau,\mathcal{M}}$ flips the appurtenance and resets the contradiction:

$$\mu_P^U(0.65 \mid \mathrm{F}) = 1 - \mu_P(0.65 \mid \mathrm{F}) = 1 - 0.75 = 0.25, \qquad \mu_P^U(0.65 \mid \mathrm{N}) = 0.4, \qquad \mu_P^U(0.65 \mid \mathrm{S}) = 0,$$
$$\mathrm{pCF}^U(\mathrm{F},\mathrm{N}) = 0, \qquad \mathrm{pCF}^U(u,w) = \mathrm{pCF}(u,w) \text{ otherwise}.$$

Because the contradiction is *real*, Mode 1 suppresses the previously dominant "Fast" by inversion (from 0.75 to 0.25) while neutralizing its conflict with the anchor. Subsequent decision-making is steered toward safer speeds for delicate features without residual penalty artifacts.

3. Main Results

In this section, we present and explain the results of this paper.

3.1. Plithogenic Fuzzy Control System

We formalize a *Plithogenic Fuzzy Control System* (PFCS) by enriching each linguistic variable with (i) an attribute–value set and (ii) a contradiction function that modulates rule firing.

Definition 3.1 (Plithogenic Fuzzy Control System (PFCS))**.** Fix input universes $X_1, \dots, X_n$ and an output universe Y. For each input $i \in \{1, \dots, n\}$, let

$$\big(X_i,\ v_i,\ Pv_i,\ \mu_{i,\cdot},\ pCF_i\big)$$

be a plithogenic fuzzy variable, where:

- v_i is an attribute of X_i with admissible values Pv_i (linguistic terms);
- for each $a \in Pv_i$, $\mu_{i,a} : X_i \to [0,1]$ is the (scalar) membership function;
- $pCF_i : Pv_i \times Pv_i \to [0,1]$ is the (symmetric, reflexive) contradiction function with $pCF_i(a,a) = 0$.

Similarly, the output variable is $\big(Y,\ v_y,\ Pv_y,\ \mu_{y,\cdot},\ pCF_y\big)$ with $a \in Pv_y \mapsto \mu_{y,a} : Y \to [0,1]$ and $pCF_y : Pv_y \times Pv_y \to [0,1]$.

A *rule base* is a finite set

$$\mathcal{R} \ \subseteq\ Pv_1 \times \cdots \times Pv_n \times Pv_y, \quad r = (a_1, \dots, a_n \Rightarrow b).$$

Let $T : [0,1]^n \to [0,1]$ be a t-norm for antecedent aggregation, $S : [0,1]^K \to [0,1]$ an s-norm for rule aggregation, and $\otimes$ a shaping operator (typically $\min$) acting on $(\text{firing}, \text{consequent } \mu)$ pairs. Let D be a defuzzifier (e.g. centroid).

Anchors $b_1, \ldots, b_n \in Pv_1 \times \cdots \times Pv_n$ and $b_y \in Pv_y$ are fixed design parameters (contextual reference values). For input $x = (x_1, \ldots, x_n)$ and a rule $r = (a_1, \ldots, a_n \Rightarrow b) \in \mathcal{R}$ define:

$$\alpha_r^{\text{base}}(x) \;:=\; T\Big(\mu_{1,a_1}(x_1), \ldots, \mu_{n,a_n}(x_n)\Big),$$

$$\beta_r \;:=\; \prod_{i=1}^{n}\Big(1 - pCF_i(a_i, b_i)\Big) \in [0,1], \qquad \gamma_r \;:=\; 1 - pCF_y(b, b_y) \in [0,1],$$

and the *plithogenic firing strength*

$$\alpha_r(x) \;:=\; \alpha_r^{\text{base}}(x) \cdot \beta_r \cdot \gamma_r \;\in [0,1].$$

The aggregated output fuzzy set $B'_x \subseteq Y$ has membership

$$\mu_{B'_x}(y) \;:=\; S_{r\in\mathcal{R}} \Big[\alpha_r(x) \;\otimes\; \mu_{y,\,b_r}(y)\Big],$$

where b_r denotes the consequent of r. Finally, the crisp control action is

$$y^*(x) \;:=\; D\big(\mu_{B'_x}\big).$$

We call

$$\mathsf{PFCS} \;:=\; \Big((X_i, v_i, Pv_i, \mu_{i,\cdot}, pCF_i)_{i=1}^{n},\; (Y, v_y, Pv_y, \mu_{y,\cdot}, pCF_y),\; \mathcal{R},\; T, S, \otimes,\; D,\; b_1, \ldots, b_n,\; b_y\Big)$$

a *Plithogenic Fuzzy Control System*.

Remark 3.2. The scalars β_r and γ_r attenuate rule firing according to pairwise contradictions between the rule's linguistic values and the anchors. Choosing b_i (resp. b_y) is a design choice capturing the prevailing context/reference for input i (resp. output).

Example 3.3 (Adaptive cruise control with plithogenic attenuation (safety–aware throttle))**.** Consider a PFCS with two inputs and one output.

Inputs:

$$x_1 = e := v_{\text{sp}} - v \in [-20, 20] \text{ (km/h)}, \qquad x_2 = \dot{e} \in [-5, 5] \text{ (km/h/s)}.$$

For x_1 use $Pv_1 = \{\text{N}, \text{Z}, \text{P}\}$ (Negative/Zero/Positive) with triangular memberships

$$\mu_{1,\text{N}}(e) = \max\Big\{0, 1 - \tfrac{|e+10|}{10}\Big\},\; \mu_{1,\text{Z}}(e) = \max\Big\{0, 1 - \tfrac{|e|}{10}\Big\},\; \mu_{1,\text{P}}(e) = \max\Big\{0, 1 - \tfrac{|e-10|}{10}\Big\}.$$

For x_2 use $Pv_2 = \{\text{N}, \text{Z}, \text{P}\}$ with

$$\mu_{2,\text{N}}(\dot{e}) = \max\Big\{0, 1 - \tfrac{|\dot{e}+2.5|}{2.5}\Big\},\; \mu_{2,\text{Z}}(\dot{e}) = \max\Big\{0, 1 - \tfrac{|\dot{e}|}{2}\Big\},\; \mu_{2,\text{P}}(\dot{e}) = \max\Big\{0, 1 - \tfrac{|\dot{e}-2.5|}{2.5}\Big\}.$$

Output (throttle increment Δu in percentage points):

$$Pv_y = \{\text{N}, \text{Z}, \text{P}\}, \quad \Delta u(\text{N}) = -30,\; \Delta u(\text{Z}) = 0,\; \Delta u(\text{P}) = +40 \text{ (Sugeno singletons)}.$$

Anchors and contradictions (safety–aware design):

$$b_1 = \text{Z}, \quad b_2 = \text{Z}, \quad b_y = \text{Z},$$

$$pCF_1(\mathrm{N},\mathrm{Z}) = pCF_1(\mathrm{P},\mathrm{Z}) = 0.2, \quad pCF_1(\mathrm{Z},\mathrm{Z}) = 0,$$
$$pCF_2(\mathrm{N},\mathrm{Z}) = pCF_2(\mathrm{P},\mathrm{Z}) = 0.1, \quad pCF_2(\mathrm{Z},\mathrm{Z}) = 0,$$
$$pCF_y(\mathrm{N},\mathrm{Z}) = pCF_y(\mathrm{P},\mathrm{Z}) = 0.3, \quad pCF_y(\mathrm{Z},\mathrm{Z}) = 0.$$

Rule base (Mamdani antecedents, $T = \min$; Sugeno aggregation/defuzzification):

$$\begin{aligned} &R_1 : (\mathrm{P},\mathrm{Z}) \Rightarrow \mathrm{P}, \\ &R_2 : (\mathrm{P},\mathrm{P}) \Rightarrow \mathrm{M}\ \text{(not used; omit)}, \\ &R_3 : (\mathrm{Z},\mathrm{P}) \Rightarrow \mathrm{N}, \\ &R_4 : (\mathrm{N},*) \Rightarrow \mathrm{N}, \\ &R_5 : (\mathrm{Z},\mathrm{Z}) \Rightarrow \mathrm{Z}. \end{aligned}$$

Numerical evaluation at

$$e = 8\ \text{km/h}, \qquad \dot{e} = -0.5\ \text{km/h/s}.$$

Memberships:

$$\mu_{1,\mathrm{P}}(8) = 1 - \tfrac{|8-10|}{10} = 0.8, \quad \mu_{1,\mathrm{Z}}(8) = 1 - \tfrac{|8|}{10} = 0.2, \quad \mu_{1,\mathrm{N}}(8) = 0,$$
$$\mu_{2,\mathrm{Z}}(-0.5) = 1 - \tfrac{0.5}{2} = 0.75, \quad \mu_{2,\mathrm{N}}(-0.5) = 1 - \tfrac{|-0.5+2.5|}{2.5} = 0.2, \quad \mu_{2,\mathrm{P}}(-0.5) = 0.$$

Base firings ($\alpha_r^{\text{base}} = \min$ of antecedents):

$$\alpha_{R_1}^{\text{base}} = \min(0.8, 0.75) = 0.75, \quad \alpha_{R_5}^{\text{base}} = \min(0.2, 0.75) = 0.2, \quad \text{others } = 0.$$

Plithogenic attenuations:

$$\beta_{R_1} = (1-0.2)(1-0) = 0.8, \quad \gamma_{R_1} = 1 - 0.3 = 0.7,$$
$$\beta_{R_5} = (1-0)(1-0) = 1, \quad \gamma_{R_5} = 1 - 0 = 1.$$

Plithogenic firings:

$$\alpha_{R_1} = 0.75 \times 0.8 \times 0.7 = 0.42, \qquad \alpha_{R_5} = 0.2 \times 1 \times 1 = 0.2.$$

Sugeno output:

$$\Delta u^* = \frac{0.42 \cdot 40 + 0.2 \cdot 0}{0.42 + 0.2} = \frac{16.8}{0.62} = \frac{840}{31} \approx 27.10\ \text{pp}.$$

Thus PFCS attenuates rule R_1 (“go high throttle”) via pCF against the neutral anchor, yielding a safer command than classical fuzzy control.

Example 3.4 (Robot arm pick-and-place: speed command with contradiction to precision anchor)**.** Inputs:

$$x_1 = e_{\text{pos}} \in [0, 6]\ \text{(mm)}, \qquad x_2 = v_{\text{RMS}} \in [0, 1]\ \text{(g; vibration severity)}.$$

For x_1 use $Pv_1 = \{\mathrm{S},\mathrm{M},\mathrm{L}\}$ with triangles centered at $0, 3, 6$ mm (half–width 3 mm):

$$\mu_{1,\mathrm{S}}(e) = \max\Big\{0, 1 - \tfrac{|e-0|}{3}\Big\},\ \mu_{1,\mathrm{M}}(e) = \max\Big\{0, 1 - \tfrac{|e-3|}{3}\Big\},\ \mu_{1,\mathrm{L}}(e) = \max\Big\{0, 1 - \tfrac{|e-6|}{3}\Big\}.$$

For x_2 use $Pv_2 = \{\mathrm{L},\mathrm{M},\mathrm{H}\}$ with triangles centered at $0, 0.5, 1$ g (half–width 0.5 g):

$$\mu_{2,\mathrm{L}}(v) = \max\Big\{0, 1 - \tfrac{|v-0|}{0.5}\Big\},\ \mu_{2,\mathrm{M}}(v) = \max\Big\{0, 1 - \tfrac{|v-0.5|}{0.5}\Big\},\ \mu_{2,\mathrm{H}}(v) = \max\Big\{0, 1 - \tfrac{|v-1|}{0.5}\Big\}.$$

Output (end–effector speed s in m/s):

$$Pv_y = \{\text{Slow}, \text{Medium}, \text{Fast}\}, \quad s(\text{Slow}) = 0.1,\ s(\text{Medium}) = 0.3,\ s(\text{Fast}) = 0.6.$$

Anchors (favor precision):

$$b_1 = \text{M}, \quad b_2 = \text{M}, \quad b_y = \text{Medium}.$$

Contradictions (higher conflict for extremal choices):

$$pCF_1(\text{S}, \text{M}) = 0.4, \quad pCF_1(\text{L}, \text{M}) = 0.6, \quad pCF_1(\text{M}, \text{M}) = 0,$$

$$pCF_2(\text{L}, \text{M}) = 0.3, \quad pCF_2(\text{H}, \text{M}) = 0.5, \quad pCF_2(\text{M}, \text{M}) = 0,$$

$$pCF_y(\text{Slow}, \text{Medium}) = 0.2, \quad pCF_y(\text{Fast}, \text{Medium}) = 0.7, \quad pCF_y(\text{Medium}, \text{Medium}) = 0.$$

Rule base ($T = \min$):

$$\begin{aligned} &R_1 : (\text{L}, \text{L}) \Rightarrow \text{Fast}, \\ &R_2 : (\text{L}, \text{M}) \Rightarrow \text{Medium}, \\ &R_3 : (\text{M}, \text{M}) \Rightarrow \text{Medium}, \\ &R_4 : (\text{M}, \text{H}) \Rightarrow \text{Slow}, \\ &R_5 : (\text{S}, *) \Rightarrow \text{Slow}. \end{aligned}$$

Numerical evaluation at $e_{\text{pos}} = 3.2$ mm and $v_{\text{RMS}} = 0.7$ g:

$$\mu_{1,\text{L}} = 1 - \tfrac{|3.2-6|}{3} = 1 - \tfrac{2.8}{3} = \tfrac{1}{15} \approx 0.0666667, \quad \mu_{1,\text{M}} = 1 - \tfrac{|3.2-3|}{3} = 1 - \tfrac{0.2}{3} = \tfrac{14}{15} \approx 0.9333333, \quad \mu_{1,\text{S}} = 0,$$

$$\mu_{2,\text{L}} = 0, \quad \mu_{2,\text{M}} = 1 - \tfrac{|0.7-0.5|}{0.5} = 0.6, \quad \mu_{2,\text{H}} = 1 - \tfrac{|0.7-1|}{0.5} = 0.4.$$

Base firings:

$$\alpha^{\text{base}}_{R_2} = \min\Big(\tfrac{1}{15}, 0.6\Big) = \tfrac{1}{15}, \quad \alpha^{\text{base}}_{R_3} = \min\Big(\tfrac{14}{15}, 0.6\Big) = 0.6, \quad \alpha^{\text{base}}_{R_4} = \min\Big(\tfrac{14}{15}, 0.4\Big) = 0.4, \quad \text{others} = 0.$$

Plithogenic attenuations:

$$\beta_{R_2} = (1 - 0.6)(1 - 0) = 0.4, \quad \gamma_{R_2} = 1,$$

$$\beta_{R_3} = (1 - 0)(1 - 0) = 1, \quad \gamma_{R_3} = 1,$$

$$\beta_{R_4} = (1 - 0)(1 - 0.5) = 0.5, \quad \gamma_{R_4} = 1 - 0.2 = 0.8.$$

Plithogenic firings:

$$\alpha_{R_2} = \tfrac{1}{15} \cdot 0.4 \cdot 1 = \tfrac{0.4}{15} \approx 0.0266667, \quad \alpha_{R_3} = 0.6 \cdot 1 \cdot 1 = 0.6, \quad \alpha_{R_4} = 0.4 \cdot 0.5 \cdot 0.8 = 0.16.$$

Sugeno output:

$$s^* = \frac{\alpha_{R_2} \cdot 0.3 + \alpha_{R_3} \cdot 0.3 + \alpha_{R_4} \cdot 0.1}{\alpha_{R_2} + \alpha_{R_3} + \alpha_{R_4}} = \frac{0.008 + 0.18 + 0.016}{\frac{0.4}{15} + 0.6 + 0.16} = \frac{0.204}{0.786\overline{6}} = \frac{153}{590} \approx 0.2593 \text{ m/s}.$$

Thus PFCS down–weights risky choices (“Fast” under significant vibration and non–medium error) via pCF, biasing the decision toward safer speeds.

Theorem 3.5 (PFCS generalizes fuzzy control)
Fix a PFCS with $(T, S, \otimes, D)$ as above and the same rule base $\mathcal{R}$. If

$$pCF_i \equiv 0 \quad \text{for all } i = 1, \dots, n, \qquad \text{and} \qquad pCF_y \equiv 0,$$

then, for every input x, the PFCS output $y^*(x)$ coincides with the output of the classical fuzzy control system that uses the same $\mu_{i,\cdot}$, $\mu_{y,\cdot}$, $T, S, \otimes, D$, and $\mathcal{R}$.

Proof
Under the stated hypothesis, for every rule $r = (a_1, \dots, a_n \Rightarrow b)$,

$$\beta_r = \prod_{i=1}^{n} \big(1 - pCF_i(a_i, b_i)\big) = \prod_{i=1}^{n} (1 - 0) = 1, \qquad \gamma_r = 1 - pCF_y(b, b_y) = 1 - 0 = 1.$$

Hence $\alpha_r(x) = \alpha_r^{\text{base}}(x) \cdot 1 \cdot 1 = \alpha_r^{\text{base}}(x)$ for all r and all x. Therefore the aggregated fuzzy output satisfies

$$\mu_{B'_x}(y) = S_{r\in\mathcal{R}}\Big[\alpha_r(x) \otimes \mu_{y,b_r}(y)\Big] = S_{r\in\mathcal{R}}\Big[\alpha_r^{\text{base}}(x) \otimes \mu_{y,b_r}(y)\Big],$$

which is precisely the classical fuzzy inference surface produced by $(T, S, \otimes)$ and $\mathcal{R}$. Applying the same defuzzifier D yields identical crisp output:

$$y^*(x) = D(\mu_{B'_x}) \quad \text{(PFCS)} \;=\; D(\mu_{B'_x}) \quad \text{(classical)}.$$

Thus PFCS reduces to classical fuzzy control when contradiction is identically zero, proving that PFCS is a strict generalization. □

Lemma 3.6 (Singleton reduction)
If, for each i, Pv_i is a singleton and Pv_y is a singleton, then regardless of the values of pCF_i, pCF_y we have $\beta_r = \gamma_r = 1$ for every rule r, and PFCS reduces to classical fuzzy control with the given $(T, S, \otimes, D)$.

Proof
If $Pv_i = \{b_i\}$, then in any rule r we necessarily have $a_i = b_i$, hence $pCF_i(a_i, b_i) = pCF_i(b_i, b_i) = 0$ by reflexivity; thus each factor in β_r equals 1. Similarly, $Pv_y = \{b_y\}$ forces $b = b_y$ in any rule, so $\gamma_r = 1$. The rest follows as in Theorem 3.5. □

Notation 3.7
Unless otherwise stated, we assume a zero–order Sugeno defuzzifier with rule–wise singletons $\{y_r\}_{r\in\mathcal{R}} \subset \mathbb{R}$. For a rule $r = (a_1, \dots, a_n \Rightarrow b)$ and input $x = (x_1, \dots, x_n)$ we recall

$$\alpha_r(x) = \alpha_r^{\text{base}}(x) \cdot \beta_r \cdot \gamma_r, \qquad \beta_r = \prod_{i=1}^{n} \big(1 - pCF_i(a_i, b_i)\big), \qquad \gamma_r = 1 - pCF_y(b, b_y).$$

The crisp output is

$$y^*(x) = \frac{\sum_{r\in\mathcal{R}} \alpha_r(x)\, y_r}{\sum_{r\in\mathcal{R}} \alpha_r(x)} \;=:\; \frac{N(x)}{W(x)}.$$

Theorem 3.8 (Uniform attenuation cancels in zero–order Sugeno)
Suppose there exists a constant $\lambda \in (0, 1]$ such that for all $r \in \mathcal{R}$, $\beta_r \gamma_r = \lambda$ (i.e., all rules are attenuated by the same factor, independent of the antecedent/consequent values). Then $y^*(x)$ coincides with the output of the classical Sugeno controller (no contradictions):

$$y^*(x) = \frac{\sum_r \alpha_r^{\text{base}}(x)\, y_r}{\sum_r \alpha_r^{\text{base}}(x)}.$$

Proof
Under the hypothesis, $\alpha_r = \lambda\,\alpha_r^{\text{base}}$. Hence $N = \sum_r \lambda\,\alpha_r^{\text{base}}\,y_r = \lambda\,N_{\text{base}}$ and $W = \sum_r \lambda\,\alpha_r^{\text{base}} = \lambda\,W_{\text{base}}$. Therefore $y^* = N/W = N_{\text{base}}/W_{\text{base}}$. □

Corollary 3.9 (A simple sufficient condition for Theorem 3.8)
If for each input i there is a constant $c_i \in [0,1)$ with $pCF_i(a_i,b_i) = c_i$ for every $a_i \in Pv_i$, and on the output side there is a constant $c_y \in [0,1)$ with $pCF_y(b,b_y) = c_y$ for every $b \in Pv_y$, then $\beta_r\gamma_r = \lambda := \bigl(\prod_{i=1}^{n}(1-c_i)\bigr)(1-c_y)$ for all r, and Theorem 3.8 applies.

Theorem 3.10 (Monotonicity and exact sensitivity w.r.t. a contradiction parameter)
Fix a rule r_0 and one contradiction entry $c := pCF_i(a_i,b_i)$ that appears in β_{r_0}. Treating all other quantities as constants, we have

$$\frac{\partial \alpha_{r_0}}{\partial c} \;=\; -\frac{\alpha_{r_0}}{1-c}\,.$$

For the zero–order Sugeno output,

$$\frac{\partial y^*}{\partial c} \;=\; -\frac{\alpha_{r_0}}{(1-c)\,W}\,\bigl(y_{r_0} - y^*\bigr).$$

Consequently:

i) If $y_{r_0} > y^*$, then $\partial y^*/\partial c < 0$ (increasing the contradiction on an "above-average" rule pushes y^* down).

ii) If $y_{r_0} < y^*$, then $\partial y^*/\partial c > 0$ (increasing the contradiction on a "below-average" rule pushes y^* up).

Proof
Write $\alpha_{r_0} = \alpha_{r_0}^{\text{base}}\Bigl(\prod_{j\neq i}(1-pCF_j)\Bigr)\cdot(1-c)\cdot\gamma_{r_0} =: K\,(1-c)$ with $K>0$ constant in c. Then $\partial\alpha_{r_0}/\partial c = -K = -\alpha_{r_0}/(1-c)$. For $y^* = N/W$ and only α_{r_0} depending on c,

$$\frac{\partial y^*}{\partial c} = \frac{\bigl(\partial\alpha_{r_0}/\partial c\bigr)\,y_{r_0}\,W - \bigl(\partial\alpha_{r_0}/\partial c\bigr)\,N}{W^2} = \frac{\partial\alpha_{r_0}}{\partial c}\cdot\frac{y_{r_0}-y^*}{W} = -\frac{\alpha_{r_0}}{(1-c)\,W}\,(y_{r_0}-y^*).$$

The sign conclusions follow immediately. □

Remark 3.11 (Concrete sensitivity check)**.** In the *adaptive cruise control* example (numbers already given), take rule R_1 with $y_{R_1} = 40$, $\alpha_{R_1} = 0.42$, the active contradiction $c = pCF_1(\mathrm{P},\mathrm{Z}) = 0.2$, and $W = \alpha_{R_1} + \alpha_{R_5} = 0.62$. Then

$$\frac{\partial y^*}{\partial c} = -\frac{0.42}{(1-0.2)\cdot 0.62}\,(40-27.10) \approx -\frac{0.42}{0.496}\cdot 12.90 \approx -10.94\;\frac{\text{pp}}{\text{unit of } c}.$$

Thus a $+0.05$ increase in c would (locally) decrease y^* by about 0.55 percentage points.

Theorem 3.12 (Continuity of the PFCS mapping)
Assume each membership $x_i \mapsto \mu_{i,a_i}(x_i)$ is continuous and that the chosen t–norm T, s–norm S, shaping operator $\otimes$, and defuzzifier D are continuous in their arguments (e.g. $T=\min$, $S=\max$, $\otimes=\min$, D = centroid, or zero–order Sugeno). Then $x \mapsto y^*(x)$ is continuous in x and continuous in each contradiction entry $pCF_i(a_i,b_i)$ and $pCF_y(b,b_y)$.

Proof
Each $\alpha_r^{\text{base}}(x)$ is continuous by continuity of the μ's and T. The factors β_r and γ_r are continuous polynomials in the pCF's. Hence α_r depends continuously on $(x,\{pCF\})$. The fuzzy aggregation and defuzzification are continuous by hypothesis, thus y^* is continuous. □

Theorem 3.13 (Representation as a weighted Sugeno controller for $T = \prod$)
If T is the product t–norm, then a PFCS is equivalent to a *weighted* zero–order Sugeno controller with rule weights

$$w_r \;:=\; \Big(\prod_{i=1}^{n}\big(1 - pCF_i(a_i, b_i)\big)\Big) \cdot \big(1 - pCF_y(b, b_y)\big),$$

in the sense that

$$\alpha_r \;=\; w_r \cdot \underbrace{\prod_{i=1}^{n} \mu_{i,a_i}(x_i)}_{=\,\alpha_r^{\text{base}}} \quad \text{and} \quad y^*(x) = \frac{\sum_r w_r\, \alpha_r^{\text{base}}(x)\, y_r}{\sum_r w_r\, \alpha_r^{\text{base}}(x)}.$$

Proof
For $T = \prod$, $\alpha_r^{\text{base}} = \prod_i \mu_{i,a_i}(x_i)$. By definition, $\alpha_r = \alpha_r^{\text{base}} \cdot \beta_r \cdot \gamma_r = \big(\prod_i \mu_{i,a_i}\big) \cdot w_r$. Substitute into the Sugeno formula. □

Theorem 3.14 (Preservation of the winner under a sufficient margin)
Let $r^\star$ maximize the base firing: $\alpha_{r^\star}^{\text{base}} = \max_r \alpha_r^{\text{base}}$, and suppose it is unique. Define

$$\Lambda \;:=\; \min_{r \neq r^\star} \frac{\alpha_{r^\star}^{\text{base}}}{\alpha_r^{\text{base}}} \;>\; 1.$$

If the contradiction factors satisfy

$$\frac{\beta_r \gamma_r}{\beta_{r^\star} \gamma_{r^\star}} \;\le\; \Lambda \qquad \text{for every } r \neq r^\star,$$

then $r^\star$ also maximizes the *plithogenic* firing α_r.

Proof
For any $r \neq r^\star$,

$$\frac{\alpha_{r^\star}}{\alpha_r} = \frac{\alpha_{r^\star}^{\text{base}}}{\alpha_r^{\text{base}}} \cdot \frac{\beta_{r^\star}\gamma_{r^\star}}{\beta_r \gamma_r} \;\ge\; \Lambda \cdot \frac{\beta_{r^\star}\gamma_{r^\star}}{\beta_r \gamma_r} \;\ge\; 1,$$

so $\alpha_{r^\star} \ge \alpha_r$ for all r. □

Theorem 3.15 (Output perturbation bound under contradiction changes)
Let $y^* = N/W$ be the PFCS output (Sugeno). If the rule weights change from $\{\alpha_r\}$ to $\{\alpha_r + \Delta\alpha_r\}$ (e.g. by changing some pCF's), then

$$\Delta y^* \;:=\; y^*_{\text{new}} - y^*_{\text{old}} \;=\; \frac{1}{W} \sum_{r \in \mathcal{R}} \Delta\alpha_r\, (y_r - y^*) \quad \Rightarrow \quad \big|\Delta y^*\big| \;\le\; \frac{R}{W} \sum_{r \in \mathcal{R}} \big|\Delta\alpha_r\big|,$$

where $R := \max_r y_r - \min_r y_r$ and $W = \sum_r \alpha_r$ is the original total weight.

Proof
Using $y^* = N/W$, a direct quotient rule computation gives $\Delta y^* = (\Delta N\, W - N\, \Delta W)/W^2 = \sum_r \Delta\alpha_r (y_r - y^*)/W$. Since $|y_r - y^*| \le R$, the bound follows. □

3.2. Upside-Down Logic on a Plithogenic Control System

We lift the Upside-Down transform with contradiction reset to the entire controller.

Definition 3.16 (Upside-Down transform of a PFCS)**.** Fix anchors $b_1, \dots, b_n, b_y$ and thresholds $\tau_1, \dots, \tau_n, \tau_y \in [0,1]$. For each input i and value $a \in Pv_i$ define activation

$$\mathsf{Act}_i(a \mid b_i, \tau_i) \ := \ \mathbf{1}\,[\,pCF_i(a, b_i) \ge \tau_i\,].$$

Construct a transformed controller $U_{\boldsymbol{b},\boldsymbol{\tau}}(\mathsf{PFCS})$ by replacing, for each activated a,

$$\mu_{i,a} \ \mapsto \ \mu^U_{i,a} := \begin{cases} 1-\mu_{i,a}, & \mathsf{Act}_i(a \mid b_i, \tau_i) = 1, \\ \mu_{i,a}, & \text{otherwise}, \end{cases}$$

$$pCF_i(u,w) \ \mapsto \ pCF^U_i(u,w) := \begin{cases} 0, & \{u,w\} = \{a, b_i\} \text{ for some activated } a, \\ pCF_i(u,w), & \text{otherwise.} \end{cases}$$

Apply the analogous operation to $(v_y, Pv_y, \mu_{y,\cdot}, pCF_y)$ using (b_y, τ_y).

Proposition 3.17 (Local consistency after reset)
For any rule $r = (a_1, \dots, a_n \Rightarrow b)$, if a_i activates for some i (or b activates on the output side), then in $U_{\boldsymbol{b},\boldsymbol{\tau}}(\mathsf{PFCS})$ the corresponding contradiction factor(s) that enter β_r (and/or γ_r) vanish to 0 *after* the flip, hence the post-transform rule firing is computed in a contradiction-neutral local neighborhood relative to the anchor(s).

Proof
By construction, whenever $\mathsf{Act}_i(a_i \mid b_i, \tau_i) = 1$ we set $pCF^U_i(a_i, b_i) = 0$ while possibly flipping μ_{i,a_i} to $1 - \mu_{i,a_i}$. Therefore the ith factor in β_r equals $1 - pCF^U_i(a_i, b_i) = 1$. The same reasoning holds on the output side for γ_r. □

Example 3.18 (Reversible conveyor position loop: UD transform fixes a sign inversion)**.** Consider a one–input PFCS for setpoint tracking of a conveyor's carriage position. The input is the position error $e := x_{\mathrm{sp}} - x \in [-1,1]$ (m) with linguistic values $Pv_e = \{\mathrm{N}, \mathrm{Z}, \mathrm{P}\}$ (Negative/Zero/Positive). The output is the motor voltage $u \in [-10, 10]$ (V) with consequents $Pv_y = \{\mathrm{N}, \mathrm{Z}, \mathrm{P}\}$ and zero–order Sugeno singletons $u(\mathrm{N}) = -10$, $u(\mathrm{Z}) = 0$, $u(\mathrm{P}) = +10$. Let the anchor values be $b_e = \mathrm{Z}$ and $b_y = \mathrm{Z}$.

Fuzzy memberships for e (triangular, peaks at $-0.5, 0, +0.5$ with base width 1):

$$\mu_{e,\mathrm{N}}(e) = \max\bigl\{0,\ 1 - \tfrac{|e+0.5|}{0.5}\bigr\}, \quad \mu_{e,\mathrm{Z}}(e) = \max\bigl\{0,\ 1 - \tfrac{|e|}{0.5}\bigr\}, \quad \mu_{e,\mathrm{P}}(e) = \max\bigl\{0,\ 1 - \tfrac{|e-0.5|}{0.5}\bigr\}.$$

Plithogenic contradiction (to the anchor) is fixed as

$$pCF_e(\mathrm{N},\mathrm{Z}) = pCF_e(\mathrm{P},\mathrm{Z}) = 0.9, \qquad pCF_e(\mathrm{Z},\mathrm{Z}) = 0, \qquad pCF_y \equiv 0 \text{ (for simplicity).}$$

Thus the plithogenic firing for a rule with antecedent $a \in \{\mathrm{N}, \mathrm{Z}, \mathrm{P}\}$ is

$$\alpha_a(e) \ = \ \underbrace{\mu_{e,a}(e)}_{\alpha_a^{\mathrm{base}}} \cdot \underbrace{\bigl(1 - pCF_e(a, b_e)\bigr)}_{\beta_a},$$

and the Sugeno output is

$$u^*(e) \ = \ \frac{\alpha_{\mathrm{N}}(-10) + \alpha_{\mathrm{Z}} \cdot 0 + \alpha_{\mathrm{P}}(+10)}{\alpha_{\mathrm{N}} + \alpha_{\mathrm{Z}} + \alpha_{\mathrm{P}}}.$$

Suppose a wiring mistake reverses the encoder sign, so the controller *measures* $e = -0.6$ while the true error is $+0.6$. Evaluate before applying the Upside-Down transform. Memberships at $e = -0.6$:

$$\mu_{e,\mathrm{N}}(-0.6) = 1 - \tfrac{|-0.6+0.5|}{0.5} = 1 - \tfrac{0.1}{0.5} = 0.8, \quad \mu_{e,\mathrm{Z}}(-0.6) = \max\{0, 1 - \tfrac{0.6}{0.5}\} = 0, \quad \mu_{e,\mathrm{P}}(-0.6) = 0.$$

Contradiction attenuations: $\beta_{\mathrm{N}} = \beta_{\mathrm{P}} = 1 - 0.9 = 0.1$, $\beta_{\mathrm{Z}} = 1$. Hence

$$\alpha_{\mathrm{N}} = 0.8 \times 0.1 = 0.08, \qquad \alpha_{\mathrm{Z}} = 0 \times 1 = 0, \qquad \alpha_{\mathrm{P}} = 0 \times 0.1 = 0.$$

Crisp output (before transform):

$$u^* = \frac{0.08 \cdot (-10)}{0.08} = -10 \text{ V} \quad \text{(wrong sign)}.$$

Now apply the Upside-Down transform $U_{b_e,\tau}$ with threshold $\tau = 0.8$. Since $pCF_e(\mathrm{N},\mathrm{Z}) = pCF_e(\mathrm{P},\mathrm{Z}) = 0.9 \geq \tau$, both N and P *activate*. The transform flips memberships and resets the activated contradictions to zero:

$$\mu^U_{e,a}(e) = 1 - \mu_{e,a}(e) \ \ (a \in \{\mathrm{N},\mathrm{P}\}), \qquad pCF^U_e(a,\mathrm{Z}) = 0 \text{ for activated } a.$$

At $e = -0.6$ this yields

$$\mu^U_{e,\mathrm{N}}(-0.6) = 1 - 0.8 = 0.2, \qquad \mu^U_{e,\mathrm{P}}(-0.6) = 1 - 0 = 1, \qquad \mu^U_{e,\mathrm{Z}}(-0.6) = 0 \text{ (unchanged)},$$

and the new attenuations are $\beta^U_{\mathrm{N}} = \beta^U_{\mathrm{P}} = \beta^U_{\mathrm{Z}} = 1$. Thus

$$\alpha^U_{\mathrm{N}} = 0.2, \qquad \alpha^U_{\mathrm{Z}} = 0, \qquad \alpha^U_{\mathrm{P}} = 1.$$

Crisp output (after transform):

$$u^{*U} = \frac{0.2 \cdot (-10) + 1 \cdot (+10)}{0.2 + 1} = \frac{-2 + 10}{1.2} = \frac{8}{1.2} \approx 6.67 \text{ V} \quad \text{(correct sign)}.$$

Therefore the Upside-Down transform of the PFCS recovers the correct actuation under a sensor sign inversion and simultaneously neutralizes the contradictions to the anchor.

Example 3.19 (Inverted camera gimbal (yaw control): UD transform under frame flip)**.** A single–input PFCS stabilizes a camera gimbal's yaw to a heading setpoint. Let the yaw error $e_\psi := \psi_{\mathrm{sp}} - \psi \in [-30^\circ, 30^\circ]$ with $Pv_e = \{\mathrm{N}, \mathrm{Z}, \mathrm{P}\}$ defined by triangles centered at $-15^\circ, 0^\circ, 15^\circ$ with half–width 15°:

$$\mu_{e,\mathrm{N}}(\theta) = \max\Big\{0,\, 1 - \tfrac{|\theta+15|}{15}\Big\}, \quad \mu_{e,\mathrm{Z}}(\theta) = \max\Big\{0,\, 1 - \tfrac{|\theta|}{15}\Big\}, \quad \mu_{e,\mathrm{P}}(\theta) = \max\Big\{0,\, 1 - \tfrac{|\theta-15|}{15}\Big\}.$$

Output r is the commanded yaw rate (deg/s) with Sugeno singletons $r(\mathrm{N}) = -20$, $r(\mathrm{Z}) = 0$, $r(\mathrm{P}) = +20$. Anchors are $b_e = \mathrm{Z}$, $b_y = \mathrm{Z}$, and we take

$$pCF_e(\mathrm{N},\mathrm{Z}) = pCF_e(\mathrm{P},\mathrm{Z}) = 0.9, \quad pCF_e(\mathrm{Z},\mathrm{Z}) = 0, \quad pCF_y \equiv 0.$$

Thus $\alpha_a(\theta) = \mu_{e,a}(\theta) \cdot (1 - pCF_e(a, b_e))$ and

$$r^*(\theta) = \frac{\alpha_{\mathrm{N}}(-20) + \alpha_{\mathrm{Z}} \cdot 0 + \alpha_{\mathrm{P}}(+20)}{\alpha_{\mathrm{N}} + \alpha_{\mathrm{Z}} + \alpha_{\mathrm{P}}}.$$

When the gimbal flips upside–down (e.g. switching camera frames from NED to ENU), the sign of the measured yaw error is reversed. Suppose the *measured* error is $\theta = -15^\circ$ while the true physical error is $+15^\circ$. Before applying the transform:

$$\mu_{e,\mathrm{N}}(-15) = 1, \quad \mu_{e,\mathrm{Z}}(-15) = 0, \quad \mu_{e,\mathrm{P}}(-15) = 0,$$

$$\beta_{\mathrm{N}} = \beta_{\mathrm{P}} = 1 - 0.9 = 0.1,\ \beta_{\mathrm{Z}} = 1 \ \Rightarrow\ \alpha_{\mathrm{N}} = 1 \times 0.1 = 0.1, \quad \alpha_{\mathrm{Z}} = 0, \quad \alpha_{\mathrm{P}} = 0.$$

Hence $r^* = (-20 \cdot 0.1)/(0.1) = -20$ deg/s (wrong direction).

Apply the Upside-Down transform with threshold $\tau = 0.8$. Both N and P activate ($0.9 \geq \tau$), so

$$\mu^U_{e,\mathrm{N}}(\theta) = 1 - \mu_{e,\mathrm{N}}(\theta), \quad \mu^U_{e,\mathrm{P}}(\theta) = 1 - \mu_{e,\mathrm{P}}(\theta), \quad pCF^U_e(\mathrm{N},\mathrm{Z}) = pCF^U_e(\mathrm{P},\mathrm{Z}) = 0.$$

At $\theta = -15°$:

$$\mu^U_{e,\mathrm{N}}(-15) = 1 - 1 = 0, \qquad \mu^U_{e,\mathrm{P}}(-15) = 1 - 0 = 1, \qquad \mu^U_{e,\mathrm{Z}}(-15) = 0 \text{ (unchanged)},$$

so the attenuations become $\beta^U_{\mathrm{N}} = \beta^U_{\mathrm{P}} = \beta^U_{\mathrm{Z}} = 1$ and

$$\alpha^U_{\mathrm{N}} = 0, \qquad \alpha^U_{\mathrm{Z}} = 0, \qquad \alpha^U_{\mathrm{P}} = 1.$$

Therefore

$$r^{*U} \ = \ \frac{1 \cdot (+20)}{1} \ = \ +20 \text{ deg/s},$$

which correctly commands a positive yaw rate to reduce the *true* positive yaw error. The transform both flips the effective semantics of the (mis-sensed) input and neutralizes the contradictions to the anchor, restoring consistent PFCS behavior under frame inversion.

Notation 3.20
We work with the Upside–Down (UD) transform $U_{\boldsymbol{b},\boldsymbol{\tau}}(\mathsf{PFCS})$ already defined. For a rule $r = (a_1, \ldots, a_n \Rightarrow b)$ and input $x = (x_1, \ldots, x_n)$ we abbreviate

$$\alpha_r = \alpha_r(x), \qquad \alpha_r^{\text{base}} = \alpha_r^{\text{base}}(x), \qquad c_i(a_i, b_i) = pCF_i(a_i, b_i), \qquad c_y(b, b_y) = pCF_y(b, b_y).$$

The post-transform quantities are decorated by ^{U}. Activation sets are $A_{\tau_i}(b_i) = \{a \in Pv_i :\ c_i(a, b_i) \geq \tau_i\}$.

Theorem 3.21 (Finite termination and idempotence of UD on PFCS)
Consider any sequence of UD steps that, at each step, chooses one activated pair $\{a_i, b_i\}$ (or $\{b, b_y\}$), flips the corresponding membership(s) and resets the chosen contradiction to 0 as per the UD definition. Let

$$N_{\max} = \sum_{i=1}^{n} \#\bigl\{a \in Pv_i \setminus \{b_i\} : c_i(a, b_i) \geq \tau_i\bigr\} + \#\bigl\{b' \in Pv_y \setminus \{b_y\} : c_y(b', b_y) \geq \tau_y\bigr\}.$$

Then the process terminates after at most $N_{\max}$ steps. Moreover, one full pass to termination is idempotent: applying it again effects no change.

Proof
Define the integer potential $\Phi = \sum_{i=1}^n \#A_{\tau_i}(b_i) + \#A_{\tau_y}(b_y) \in \mathbb{N}$. Each UD step resets exactly one activated pair to $0 < \tau$, strictly decreasing Φ by 1 and never creating new activations. Since initially $\Phi \leq N_{\max}$ and $\Phi \geq 0$, termination occurs in at most $N_{\max}$ steps. At termination, no activation predicate holds, hence a subsequent pass performs no update (idempotence). □

Theorem 3.22 (Closed-form post-UD firing ratio for $T = \prod$)
Assume the product t-norm $T = \prod$. For a fixed rule $r = (a_1, \ldots, a_n \Rightarrow b)$ let $I = \{i :\ a_i \in A_{\tau_i}(b_i)\}$ and $\sigma_y =$

$\mathbf{1}[b \in A_{\tau_y}(b_y)]$ denote the activated indices (inputs and possibly the output side). Then

$$\frac{\alpha_r^U}{\alpha_r} = \left(\prod_{i \in I} \frac{1 - \mu_{i,a_i}(x_i)}{\mu_{i,a_i}(x_i)} \right) \cdot \left(\prod_{i \in I} \frac{1}{1 - c_i(a_i, b_i)} \right) \cdot \begin{cases} \frac{1}{1 - c_y(b, b_y)}, & \sigma_y = 1, \\ 1, & \sigma_y = 0. \end{cases}$$

In particular, for each activated input coordinate $i \in I$ the UD step multiplies the rule firing by the factor $\frac{1 - \mu_{i,a_i}}{\mu_{i,a_i}} \cdot \frac{1}{1 - c_i}$, and for an activated output consequent by $\frac{1}{1 - c_y}$.

Proof
For $T = \prod$, $\alpha_r = \left(\prod_{i=1}^n \mu_{i,a_i}(x_i)\right) \cdot \left(\prod_{i=1}^n (1 - c_i(a_i, b_i))\right) \cdot (1 - c_y(b, b_y))$. Under UD, for each $i \in I$ we replace $\mu_{i,a_i} \mapsto 1 - \mu_{i,a_i}$ and set $c_i(a_i, b_i) \mapsto 0$; for $i \notin I$ both terms are unchanged. If $\sigma_y = 1$ we also set $c_y(b, b_y) \mapsto 0$; otherwise unchanged. Taking the quotient α_r^U / α_r gives exactly the stated factors. □

Remark 3.23 (Numerical check of Theorem 3.22)**.** In the conveyor example (measured $e = -0.6$), for the rule with antecedent N we had $\mu_{e,\mathrm{N}} = 0.8$ and $c_e(\mathrm{N}, \mathrm{Z}) = 0.9$; pre-UD $\alpha_{\mathrm{N}} = 0.08$. The ratio factor is $\frac{1 - 0.8}{0.8} \cdot \frac{1}{1 - 0.9} = \frac{0.2}{0.8} \cdot 10 = 2.5$, hence $\alpha_{\mathrm{N}}^U = 0.08 \times 2.5 = 0.2$, matching the explicit computation.

Theorem 3.24 (A sufficient condition for post-UD firing improvement when $T = \min$)
Assume $T = \min$. Fix a rule $r = (a_1, \ldots, a_n \Rightarrow b)$ and let exactly one antecedent, say a_i, activate. Write

$$\mu := \mu_{i,a_i}(x_i), \qquad m := \min_{j \neq i} \mu_{j,a_j}(x_j), \qquad c := c_i(a_i, b_i).$$

Then

$$\alpha_r^U \geq \alpha_r \quad \Longleftarrow \quad \min(1 - \mu,\, m) \;\geq\; (1 - c)\, \min(\mu,\, m).$$

In particular, the easy-to-check sufficient condition

$$\mu \;\leq\; \frac{1}{2 - c}$$

guarantees $\alpha_r^U \geq \alpha_r$ regardless of m.

Proof
Before UD, the antecedent aggregator is $\min(\mu, m)$ and the contradiction factor is $1 - c$; after UD they are $\min(1 - \mu, m)$ and 1, respectively. Thus $\alpha_r = (1 - c)\min(\mu, m) \cdot (\text{constants})$, $\alpha_r^U = \min(1 - \mu, m) \cdot (\text{same constants})$. The first inequality is immediate by cancelling the common positive factor. For the sufficient condition, require $1 - \mu \geq (1 - c)\mu$, i.e. $1 \geq \mu(2 - c)$. □

Theorem 3.25 (Winner swap and sign correction under polarity inversion)
Consider a one-input, three-term PFCS with symmetric memberships around 0:

$$\mu_{\mathrm{N}}(-x) = \mu_{\mathrm{P}}(x), \qquad \mu_{\mathrm{Z}}(-x) = \mu_{\mathrm{Z}}(x) \qquad (\forall x),$$

and zero-order Sugeno consequents $y(\mathrm{N}) = -u_0$, $y(\mathrm{Z}) = 0$, $y(\mathrm{P}) = +u_0$ with $u_0 > 0$. Let anchors be $b_e = \mathrm{Z}$, $b_y = \mathrm{Z}$, and suppose both $c_e(\mathrm{N}, \mathrm{Z})$ and $c_e(\mathrm{P}, \mathrm{Z})$ exceed τ so that N, P activate and are flipped; contradictions to the anchor are reset to 0. If the *measured* error equals the negative of the *true* error, $e_{\mathrm{m}} = -e_{\mathrm{t}}$, then the post-UD crisp command satisfies

$$\mathrm{sign}\big(y^{*U}(e_{\mathrm{m}})\big) = \mathrm{sign}(e_{\mathrm{t}})$$

whenever $\mu_{\mathrm{Z}}(e_{\mathrm{m}}) < \max\{\mu_{\mathrm{N}}(e_{\mathrm{m}}), \mu_{\mathrm{P}}(e_{\mathrm{m}})\}$.

Proof
Write $\mu_N := \mu_{\mathrm{N}}(e_{\mathrm{m}})$ and $\mu_P := \mu_{\mathrm{P}}(e_{\mathrm{m}})$. By symmetry with $e_{\mathrm{m}} = -e_{\mathrm{t}}$,

$$\mu_N = \mu_{\mathrm{P}}(e_{\mathrm{t}}), \qquad \mu_P = \mu_{\mathrm{N}}(e_{\mathrm{t}}).$$

After UD, N and P are flipped and contradiction factors to b_e are 1, so (up to the same positive normalizing denominator W^U)

$$y^{*U} \;\propto\; (-u_0)\,(1-\mu_N) \;+\; (+u_0)\,(1-\mu_P) \;=\; u_0\big[(1-\mu_P)-(1-\mu_N)\big] \;=\; u_0(\mu_N-\mu_P).$$

Thus $\mathrm{sign}(y^{*U}) = \mathrm{sign}(\mu_N - \mu_P)$. If $e_{\mathrm{t}} > 0$, then by symmetry $\mu_N = \mu_{\mathrm{P}}(e_{\mathrm{t}}) \ge \mu_{\mathrm{N}}(e_{\mathrm{t}}) = \mu_P$, so $\mu_N - \mu_P \ge 0$ and $\mathrm{sign}(y^{*U}) = +1 = \mathrm{sign}(e_{\mathrm{t}})$. If $e_{\mathrm{t}} < 0$ the inequalities reverse and the sign is -1. The stated condition on μ_{Z} ensures Z does not dominate the average. □

Theorem 3.26 (Convex-hull preservation under UD)
In zero-order Sugeno inference with consequents $\{y_r\}_{r\in\mathcal{R}} \subset \mathbb{R}$, the UD transform preserves the output range:

$$y^{*U}(x) \;\in\; [\min_r y_r,\; \max_r y_r] \qquad (\forall x).$$

Proof
Both pre- and post-UD outputs are convex combinations: $y^* = \sum_r \omega_r y_r$, $y^{*U} = \sum_r \omega_r^U y_r$ with $\omega_r, \omega_r^U \ge 0$ and $\sum_r \omega_r = \sum_r \omega_r^U = 1$. The claim follows. □

Theorem 3.27 (No-change outside the activation set)
If $A_{\tau_i}(b_i) = \varnothing$ for all i and $A_{\tau_y}(b_y) = \varnothing$, then $U_{\boldsymbol{b},\boldsymbol{\tau}}(\mathsf{PFCS})$ acts as the identity on the controller: all μ's, pCF's, rule firings α_r, and the crisp map $y^*(\cdot)$ remain unchanged.

Proof
By hypothesis no predicate $\mathsf{Act}_i(\cdot \mid b_i, \tau_i)$ or $\mathsf{Act}_y(\cdot \mid b_y, \tau_y)$ returns 1, so the definition of the UD transform performs no membership flip and no contradiction reset. All subsequent computations are identical. □

Theorem 3.28 (Output perturbation bound for a UD pass)
Let $y^* = N/W$ and $y^{*U} = N^U/W^U$ denote, respectively, the pre- and post-UD Sugeno outputs. Set $R = \max_r y_r - \min_r y_r$. Then

$$\big|y^{*U} - y^*\big| \;\le\; \frac{R}{\min\{W, W^U\}} \sum_{r\in\mathcal{R}} \big|\alpha_r^U - \alpha_r\big|.$$

If $T = \prod$, the differences $\alpha_r^U - \alpha_r$ factorize by Theorem 3.22.

Proof
Write $y^{*U} - y^* = (N^U/W^U) - (N/W)$. Adding and subtracting N/W^U yields $y^{*U} - y^* = (N^U - N)/W^U + N(1/W^U - 1/W)$. Using $N = \sum_r \alpha_r y_r$, $N^U = \sum_r \alpha_r^U y_r$ and rearranging gives

$$y^{*U} - y^* = \frac{1}{W^U} \sum_r (\alpha_r^U - \alpha_r)(y_r - y^{*U}).$$

Taking absolute values and using $|y_r - y^{*U}| \le R$ yields $|y^{*U} - y^*| \le \frac{R}{W^U} \sum_r |\alpha_r^U - \alpha_r|$. Symmetrizing W^U with W gives the stated bound. □

3.3. Two-mode De-Plithogenication on a Plithogenic Control System

We incorporate the mode selector $\mathcal{M}$ from the preliminaries to distinguish genuine from apparent contradictions at the controller level.

Definition 3.29 (Two-mode De-Plithogenication of PFCS)**.** Let $\mathcal{M}_i : \{\{a, b_i\} : a \in Pv_i\} \to \{0, 1\}$ and $\mathcal{M}_y : \{\{b, b_y\} : b \in Pv_y\} \to \{0, 1\}$ be mode selectors. Define the operator $U^{\text{imp}}_{\boldsymbol{b},\boldsymbol{\tau},\mathcal{M}}$ by:

$$\mu_{i,a} \mapsto \begin{cases} 1 - \mu_{i,a}, & \mathsf{Act}_i(a \mid b_i, \tau_i) = 1 \text{ and } \mathcal{M}_i(\{a, b_i\}) = 1, \\ \mu_{i,a}, & \text{otherwise,} \end{cases} \qquad pCF_i(a, b_i) \mapsto 0 \text{ whenever } \mathsf{Act}_i(a \mid b_i, \tau_i) = 1,$$

and analogously for the output side. A finite composition of such operators is an *improved de-plithogenication sequence* of the controller.

Proposition 3.30 (Idempotent stabilization)
If, along a finite improved de-plithogenication sequence, every activated pair has its contradiction reset (as above), then a subsequent application of $U^{\text{imp}}_{\boldsymbol{b},\boldsymbol{\tau},\mathcal{M}}$ leaves the controller unchanged. Consequently, once stabilized, the inference map $x \mapsto y^*(x)$ is fixed under further de-plithogenication steps.

Proof
After the first pass, all activated pairs $\{a, b_i\}$ (and $\{b, b_y\}$) satisfy $pCF(a, b_\cdot) = 0$. Hence no further activations occur at the same thresholds, and neither flips nor resets are triggered. Therefore the mapping is unchanged; applying the operator again is the identity. □

Example 3.31 (HVAC VAV damper: Mode 0 (Neutralize–only) for an apparent contradiction)**.** A single–input PFCS regulates a zone's supply airflow via a damper. Input is temperature error $e := T_{\text{sp}} - T \in [-4, 4]$ (°C) with linguistic values $Pv_e = \{\text{N}, \text{Z}, \text{P}\}$ (Negative/Zero/Positive); anchor $b_e = \text{Z}$. Output is damper increment $\Delta u \in [-30, 30]$ (percentage points) with Sugeno singletons $\Delta u(\text{N}) = -30$, $\Delta u(\text{Z}) = 0$, $\Delta u(\text{P}) = +30$; $pCF_y \equiv 0$.

Fuzzy memberships (triangular, centers at $-2, 0, +2$ with half–width 2):

$$\mu_{e,\text{N}}(e) = \max\Big\{0,\ 1 - \tfrac{|e+2|}{2}\Big\}, \quad \mu_{e,\text{Z}}(e) = \max\Big\{0,\ 1 - \tfrac{|e|}{2}\Big\}, \quad \mu_{e,\text{P}}(e) = \max\Big\{0,\ 1 - \tfrac{|e-2|}{2}\Big\}.$$

Plithogenic contradiction to the anchor:

$$pCF_e(\text{N}, \text{Z}) = pCF_e(\text{P}, \text{Z}) = 0.8, \qquad pCF_e(\text{Z}, \text{Z}) = 0.$$

Hence, with $T = \min$ (here identity since $n = 1$), the plithogenic firing weights are

$$\alpha_a(e) = \mu_{e,a}(e) \cdot \beta_a, \qquad \beta_a := 1 - pCF_e(a, \text{Z}) \in \{1, 0.2\}.$$

Numerics at $e = +1.2$ °C:

$$\mu_{e,\text{P}} = 1 - \tfrac{|1.2-2|}{2} = 1 - \tfrac{0.8}{2} = 0.6, \quad \mu_{e,\text{Z}} = 1 - \tfrac{|1.2|}{2} = 0.4, \quad \mu_{e,\text{N}} = 0.$$

Before de–plithogenication:

$$\beta_\text{P} = 0.2,\ \beta_\text{Z} = 1\ \Rightarrow\ \alpha_\text{P} = 0.6 \cdot 0.2 = 0.12, \quad \alpha_\text{Z} = 0.4.$$

Sugeno output:

$$\Delta u^* = \frac{0.12 \cdot 30 + 0.4 \cdot 0}{0.12 + 0.4} = \frac{3.6}{0.52} \approx 6.923.$$

Suppose an occupancy sensor glitch is diagnosed (apparent conflict). Take threshold $\tau = 0.75$, so $\mathrm{Act}_e(\mathrm{P} \mid \mathrm{Z}, \tau) = 1$ and we choose Mode 0: $\mathcal{M}_e(\{\mathrm{P}, \mathrm{Z}\}) = 0$. By the two–mode operator $U^{\mathrm{imp}}_{\boldsymbol{b},\boldsymbol{\tau},\mathcal{M}}$,

$$\mu_{e,\mathrm{P}} \text{ unchanged}, \qquad pCF_e(\mathrm{P}, \mathrm{Z}) \mapsto 0 \ (\Rightarrow \beta_{\mathrm{P}} \mapsto 1).$$

After de–plithogenication:

$$\alpha^U_{\mathrm{P}} = 0.6 \cdot 1 = 0.6, \quad \alpha^U_{\mathrm{Z}} = 0.4, \quad \Delta u^{*U} = \frac{0.6 \cdot 30 + 0.4 \cdot 0}{0.6 + 0.4} = \frac{18}{1} = 18.$$

Effect: without flipping memberships (since the contradiction was only *apparent*), neutralizing pCF removes undue attenuation and restores a stronger, correct opening command ($6.923 \to 18$ percentage points).

Example 3.32 (Quadcopter altitude hold: Mode 1 (Invert + Neutralize) for a genuine contradiction)**.** A PFCS commands vertical speed v_c to hold altitude. Input is altitude error $e_z := z_{\mathrm{sp}} - z \in [-2, 2]$ m with $Pv_e = \{\mathrm{N}, \mathrm{Z}, \mathrm{P}\}$ and anchor $b_e = \mathrm{Z}$. Output uses Sugeno singletons $v_c(\mathrm{N}) = -2$, $v_c(\mathrm{Z}) = 0$, $v_c(\mathrm{P}) = +2$ (m/s); $pCF_y \equiv 0$.

Fuzzy memberships (triangular, centers at $-1, 0, +1$ with half–width 1):

$$\mu_{e,\mathrm{N}}(e) = \max\Big\{0,\ 1 - \tfrac{|e+1|}{1}\Big\}, \quad \mu_{e,\mathrm{Z}}(e) = \max\Big\{0,\ 1 - \tfrac{|e|}{1}\Big\}, \quad \mu_{e,\mathrm{P}}(e) = \max\Big\{0,\ 1 - \tfrac{|e-1|}{1}\Big\}.$$

Due to a maintenance error, motor mapping is reversed: a *positive* v_c makes the vehicle go *down*. This is a *genuine* contradiction. Set

$$pCF_e(\mathrm{N}, \mathrm{Z}) = pCF_e(\mathrm{P}, \mathrm{Z}) = 0.85, \qquad pCF_e(\mathrm{Z}, \mathrm{Z}) = 0,$$

with threshold $\tau = 0.8$ so both N, P activate.

At $e_z = +0.8$ m (we are below the setpoint), before de–plithogenication:

$$\mu_{e,\mathrm{P}} = 1 - |0.8 - 1| = 0.8, \quad \mu_{e,\mathrm{Z}} = 1 - |0.8| = 0.2, \quad \mu_{e,\mathrm{N}} = 0,$$

$$\beta_{\mathrm{P}} = \beta_{\mathrm{N}} = 1 - 0.85 = 0.15,\ \beta_{\mathrm{Z}} = 1 \ \Rightarrow\ \alpha_{\mathrm{P}} = 0.8 \cdot 0.15 = 0.12,\ \alpha_{\mathrm{Z}} = 0.2,\ \alpha_{\mathrm{N}} = 0.$$

Sugeno output:

$$v_c^* = \frac{0.12 \cdot (+2) + 0.2 \cdot 0}{0.12 + 0.2} = \frac{0.24}{0.32} = 0.75 \text{ m/s} \quad \text{(WRONG sign given the reversed mapping)}.$$

Apply $U^{\mathrm{imp}}_{\boldsymbol{b},\boldsymbol{\tau},\mathcal{M}}$ with *Mode 1* on both activated pairs: $\mathcal{M}_e(\{\mathrm{P}, \mathrm{Z}\}) = \mathcal{M}_e(\{\mathrm{N}, \mathrm{Z}\}) = 1$. Then

$$\mu^U_{e,\mathrm{P}} = 1 - \mu_{e,\mathrm{P}} = 1 - 0.8 = 0.2, \qquad \mu^U_{e,\mathrm{N}} = 1 - \mu_{e,\mathrm{N}} = 1 - 0 = 1, \qquad \mu^U_{e,\mathrm{Z}} = \mu_{e,\mathrm{Z}} = 0.2,$$

and the activated contradictions are reset:

$$pCF^U_e(\mathrm{P}, \mathrm{Z}) = pCF^U_e(\mathrm{N}, \mathrm{Z}) = 0 \ \Rightarrow\ \beta^U_{\mathrm{P}} = \beta^U_{\mathrm{N}} = \beta^U_{\mathrm{Z}} = 1.$$

Post–transform firings and output:

$$\alpha^U_{\mathrm{N}} = 1, \quad \alpha^U_{\mathrm{P}} = 0.2, \quad \alpha^U_{\mathrm{Z}} = 0.2,$$

$$v_c^{*U} = \frac{1 \cdot (-2) + 0.2 \cdot 0 + 0.2 \cdot (+2)}{1 + 0.2 + 0.2} = \frac{-2 + 0.4}{1.4} = -\frac{1.6}{1.4} \approx -1.1429 \text{ m/s}.$$

Mode 1 flips the semantics of the activated input terms (and neutralizes their contradiction to the anchor). The controller now issues a *negative* command, which— under the reversed motor mapping—produces an *upward* response, correctly reducing the positive altitude error.

Notation 3.33
Throughout, anchors $b_1, \dots, b_n$ and b_y, thresholds $\tau_1, \dots, \tau_n, \tau_y \in [0,1]$, and the mode selector $\mathcal{M}$ are fixed as in the definitions given earlier. We write $c_i(\cdot,\cdot)$ and $c_y(\cdot,\cdot)$ for the scalar contradiction maps pCF_i and pCF_y, respectively. For a rule $r = (a_1, \dots, a_n \Rightarrow b)$ we recall

$$\alpha_r^{\text{base}}(x) \;=\; T\big(\mu_{1,a_1}(x_1), \dots, \mu_{n,a_n}(x_n)\big), \qquad \beta_r \;=\; \prod_{i=1}^{n}\big(1 - c_i(a_i, b_i)\big), \qquad \gamma_r \;=\; 1 - c_y(b, b_y),$$

and $\alpha_r = \alpha_r^{\text{base}} \cdot \beta_r \cdot \gamma_r$.

Theorem 3.34 (Finite termination with an explicit step bound)
Fix a PFCS with finite value sets $Pv_1, \dots, Pv_n, Pv_y$ and anchors $(b_1, \dots, b_n, b_y)$. Consider any sequence of two-mode De-Plithogenication steps (arbitrary mixture of Mode 0 and Mode 1 choices) that, at each step, selects an *activated* pair $\{a_i, b_i\}$ (or $\{b, b_y\}$ on the output side) and applies the corresponding update (flip if Mode 1; no flip if Mode 0; then reset the chosen pair's contradiction to 0). Then the process terminates after at most

$$N_{\max} \;=\; \sum_{i=1}^{n} \#\big\{\, a \in Pv_i \setminus \{b_i\} \,:\, c_i(a, b_i) \;\ge\; \tau_i \,\big\} \;+\; \#\big\{\, b \in Pv_y \setminus \{b_y\} \,:\, c_y(b, b_y) \;\ge\; \tau_y \,\big\}$$

steps, and upon termination no pair satisfies the activation condition.

Proof
Define the potential

$$\Phi \;:=\; \sum_{i=1}^{n} \#\big\{\, a \in Pv_i \,:\, c_i(a, b_i) \ge \tau_i \,\big\} \;+\; \#\big\{\, b \in Pv_y \,:\, c_y(b, b_y) \ge \tau_y \,\big\}.$$

At each step one activated pair $\{a_i, b_i\}$ (or $\{b, b_y\}$) is chosen; the algorithm *resets* exactly that pair's contradiction to 0, hence $c_i^U(a_i, b_i) = 0 < \tau_i$ (resp. $c_y^U(b, b_y) = 0 < \tau_y$). Therefore Φ decreases by at least 1 per step and can never increase because the update never *creates* a new contradiction entry $\ge \tau$. Since Φ is a nonnegative integer and initially $\Phi \le N_{\max}$, after at most $N_{\max}$ steps we reach $\Phi = 0$. No pair remains with contradiction $\ge \tau$, i.e. no activation is possible, so the process terminates. □

Theorem 3.35 (Idempotence after termination)
Let U denote one full two-mode De-Plithogenication pass (any order, any modes) applied until termination as in Theorem 3.34. Then U is idempotent:

$$U \circ U \;=\; U.$$

Proof
By Theorem 3.34, after one full pass every contradiction entry that could activate has been driven below threshold. Reapplying U checks the same activation predicates; all fail, so no further change occurs. Hence $U(U(PS)) = U(PS)$. □

Proposition 3.36 (Order independence on disjoint pairs and across variables)
Fix a variable i and two distinct activated values $a, a' \in Pv_i \setminus \{b_i\}$ with $a \ne a'$. Let $U_{i,a}$ (resp. $U_{i,a'}$) denote a single-step two-mode update that handles only the pair $\{a, b_i\}$ (resp. $\{a', b_i\}$), with the same mode choices as prescribed by $\mathcal{M}$. Then

$$U_{i,a} \circ U_{i,a'} \;=\; U_{i,a'} \circ U_{i,a}.$$

Moreover, updates taken on different variables commute pairwise.

Proof
Both $U_{i,a}$ and $U_{i,a'}$ only (i) possibly flip the single membership coordinate $\mu_{i,a}$ or $\mu_{i,a'}$, and (ii) reset the *respective* contradiction entry $c_i(a, b_i)$ or $c_i(a', b_i)$ to 0. These operations act on disjoint coordinates and disjoint contradiction entries, hence commute. For different variables $i \neq j$, the state components $(\mu_{i,\cdot}, c_i)$ and $(\mu_{j,\cdot}, c_j)$ are also disjoint, so the same argument applies. □

Theorem 3.37 (Exact multiplicative gain under Mode 0)
Suppose, for a rule $r = (a_1, \ldots, a_n \Rightarrow b)$ and a fixed input x, that *only* the pair $\{a_i, b_i\}$ activates and is processed in Mode 0. Then

$$\alpha_r^{\text{post}} \;=\; \frac{1}{1 - c_i(a_i, b_i)}\, \alpha_r^{\text{pre}} \;\geq\; \alpha_r^{\text{pre}},$$

i.e. the rule firing increases by the exact factor $1/(1 - c_i(a_i, b_i))$. If several Mode 0 activations occur on distinct indices $i \in I$, the total gain is $\prod_{i \in I} \frac{1}{1 - c_i(a_i, b_i)}$.

Proof
Mode 0 does not alter *any* membership, hence α_r^{base} and all the β-factors except the ith remain unchanged. The ith factor changes from $(1 - c_i(a_i, b_i))$ to 1 due to the reset. Therefore

$$\alpha_r^{\text{post}} = \alpha_r^{\text{base}} \cdot \Big(\prod_{j \neq i} (1 - c_j(a_j, b_j))\Big) \cdot 1 \cdot \gamma_r = \frac{1}{1 - c_i(a_i, b_i)}\, \alpha_r^{\text{pre}}.$$

Iterating over $i \in I$ multiplies the gains. □

Theorem 3.38 (Mode 1: algebraic condition for firing improvement under $T = \min$)
Assume $T = \min$. Consider a rule $r = (a_1, \ldots, a_n \Rightarrow b)$ where *exactly one* antecedent value a_i activates and is processed in Mode 1. Let

$$\mu := \mu_{i,a_i}(x_i), \qquad m := \min_{j \neq i} \mu_{j,a_j}(x_j), \qquad \kappa := \Big(\prod_{j \neq i} (1 - c_j(a_j, b_j))\Big) \cdot \gamma_r,$$

so that $\alpha_r^{\text{pre}} = (1 - c_i(a_i, b_i)) \cdot \min(\mu, m) \cdot \kappa$ and $\alpha_r^{\text{post}} = 1 \cdot \min(1 - \mu, m) \cdot \kappa$. Then

$$\alpha_r^{\text{post}} \;\geq\; \alpha_r^{\text{pre}} \quad\Longleftrightarrow\quad \min(1 - \mu, m) \;\geq\; (1 - c_i(a_i, b_i))\, \min(\mu, m).$$

In particular, a simple sufficient condition is

$$1 - \mu \;\geq\; (1 - c_i(a_i, b_i))\, \mu \quad \text{i.e.} \quad \mu \;\leq\; \frac{1}{2 - c_i(a_i, b_i)}.$$

Proof
With $T = \min$ and only the ith coordinate flipped, the antecedent aggregator changes from $\min(\mu, m)$ to $\min(1 - \mu, m)$; all other multiplicative factors are as stated. Cancelling the common nonnegative factor κ yields the displayed inequality. The sufficient condition is obtained by requiring $1 - \mu \geq (1 - c)\mu$, where $c = c_i(a_i, b_i)$, which gives $1 \geq \mu(2 - c)$, i.e. $\mu \leq 1/(2 - c)$. □

Remark 3.39 (Numerical check of Theorem 3.38)**.** Let $c_i(a_i, b_i) = 0.9$ so $1 - c = 0.1$; take $\mu = 0.80$ and $m = 0.75$. Then

$$\alpha_r^{\text{pre}}/\kappa = (1 - c)\min(\mu, m) = 0.1 \times 0.75 = 0.075, \qquad \alpha_r^{\text{post}}/\kappa = \min(1 - \mu, m) = \min(0.20, 0.75) = 0.20,$$

so $\alpha_r^{\text{post}} > \alpha_r^{\text{pre}}$. The sufficient condition gives $\mu \leq 1/(2 - c) = 1/1.1 \approx 0.909\ldots$, which is satisfied by $\mu = 0.80$.

Theorem 3.40 (Convex-hull (range) preservation of crisp output)
Suppose the PFCS uses either (i) Mamdani inference with any nonnegative s-norm aggregation followed by centroid defuzzification, or (ii) zero-order Sugeno with singleton consequents $\{y_k\} \subset \mathbb{R}$. Under any sequence of two-mode De-Plithogenication steps, the crisp output remains in the convex hull of the consequents: in case (ii),

$$y^*(x) \in \big[\min_k y_k,\ \max_k y_k\big] \qquad \text{for every input } x.$$

Proof
All rule firing strengths are nonnegative by construction, and De-Plithogenication only *rescales* them via nonnegative multipliers and (under Mode 1) flips some memberships into $1-\mu \in [0,1]$ but never introduces negative weights. For (ii) zero-order Sugeno, the crisp output has the form

$$y^*(x) = \frac{\sum_k \alpha_k(x)\, y_k}{\sum_k \alpha_k(x)}, \qquad \alpha_k(x) \geq 0,\ \sum_k \alpha_k(x) > 0,$$

which is a convex combination of $\{y_k\}$; hence y^* lies in the convex hull $[\min y_k, \max y_k]$. For (i), the centroid of a nonnegative fuzzy set supported in $[\min Y, \max Y]$ must lie in that interval. □

Corollary 3.41 (Reduction to classical fuzzy control after a Mode 0-only pass)
If a full De-Plithogenication pass is executed using Mode 0 only until termination, then μ's remain unchanged and *all* contradictions that could activate are reset to zero. Consequently, the resulting PFCS has $pCF_i \equiv 0$ and $pCF_y \equiv 0$ on the activated pairs, and its input–output map coincides with the classical fuzzy controller obtained by removing pCF entirely (cf. Theorem 3.5).

Proof
Mode 0 does not flip any membership, so the family $\{\mu_{i,\cdot}\}$ is preserved. By Theorem 3.34, all contradictions that exceed the thresholds are driven below them; in particular, all entries used by the controller become 0 on the activated pairs. Theorem 3.5 then applies. □

Proposition 3.42 (Monotonicity of the activation set in the thresholds)
For each variable i the activation set

$$A_{\tau_i}(b_i) := \{\, a \in Pv_i : \ c_i(a, b_i) \geq \tau_i \,\}$$

is antitone in τ_i: if $\tau_i' > \tau_i$ then $A_{\tau_i'}(b_i) \subseteq A_{\tau_i}(b_i)$. The same holds for the output side. Consequently, increasing thresholds can only (weakly) reduce the number of updates performed by any De-Plithogenication pass.

Proof
Immediate from the definition of A_{τ_i} as a superlevel set of $c_i(\cdot, b_i)$. □

Theorem 3.43 (Global step bound and complexity)
If a single pass processes all currently activated pairs (any modes), then the number of flips is at most

$$F_{\max} \leq \sum_{i=1}^{n} \#A_{\tau_i}(b_i) + \#A_{\tau_y}(b_y),$$

and the number of contradiction resets equals the same bound. In particular, for fixed anchors there are no more than

$$\sum_{i=1}^{n} \big(|Pv_i| - 1\big) + \big(|Pv_y| - 1\big)$$

resets in the worst case (when all non-anchor values activate).

Proof
Each activation involves exactly one unordered pair $\{a_i, b_i\}$ (or $\{b, b_y\}$) and can occur at most once because after it is processed the contradiction on that pair is set to $0 < \tau$ permanently (Theorem 3.35). Thus the maximum number of updates equals the number of activated pairs available at the beginning of the pass. The crude bound follows by observing that for fixed anchors there are exactly $|Pv_i| - 1$ possible pairs $\{a, b_i\}$ on variable i. □

4. Results: Algorithms for Upside-Down Logic in Plithogenic Fuzzy Control System

We present implementation-ready procedures for Upside–Down (UD) Logic on a Plithogenic Fuzzy Control System (PFCS). Throughout, n is the number of inputs, $\mathcal{R}$ the rule base of size $K := |\mathcal{R}|$, Pv_i the set of linguistic values of input i, and Pv_y that of the output. Anchors and thresholds are $(b_1, \ldots, b_n, b_y)$ and $(\tau_1, \ldots, \tau_n, \tau_y)$, respectively. We assume T (t-norm), S (s-norm), $\otimes$ (shaper), and a zero–order Sugeno defuzzifier unless noted.

Algorithm 1: UD-PREPROCESS for PFCS (flip + reset with caching)

Input: PFCS components; anchors $b_1, \ldots, b_n, b_y$; thresholds $\tau_1, \ldots, \tau_n, \tau_y$.
Output: Flip flags flip$[i, a] \in \{0, 1\}$, flipY$[b] \in \{0, 1\}$; cached attenuations β_r^U, γ_r^U for all $r \in \mathcal{R}$; inverted-pair index lists $\mathcal{L}(a)$, $\mathcal{L}_y(b)$.

for $i = 1$ **to** n **do**
  for *each* $a \in Pv_i$ **do**
    $c \leftarrow pCF_i(a, b_i)$
    flip$[i, a] \leftarrow \mathbf{1}[c \geq \tau_i]$
    initialize $\mathcal{L}(a) \leftarrow \varnothing$
for *each* $b \in Pv_y$ **do**
  $c \leftarrow pCF_y(b, b_y)$; flipY$[b] \leftarrow \mathbf{1}[c \geq \tau_y]$; initialize $\mathcal{L}_y(b) \leftarrow \varnothing$
for *each rule* $r = (a_1, \ldots, a_n \Rightarrow b) \in \mathcal{R}$ **do**
  `// Build inverted-pair indices for incremental updates`
  for $i = 1$ **to** n **do**
    add r to $\mathcal{L}(a_i)$
  add r to $\mathcal{L}_y(b)$
  `// Cache post-UD contradiction attenuations`
  $\beta_r^U \leftarrow 1$
  for $i = 1$ **to** n **do**
    if flip$[i, a_i] = 1$ **then**
      factor $\leftarrow 1$ // reset $pCF_i(a_i, b_i)$ to 0
    else
      factor $\leftarrow 1 - pCF_i(a_i, b_i)$
    $\beta_r^U \leftarrow \beta_r^U \times$ factor
  if flipY$[b] = 1$ **then**
    $\gamma_r^U \leftarrow 1$ // reset $pCF_y(b, b_y)$ to 0
  else
    $\gamma_r^U \leftarrow 1 - pCF_y(b, b_y)$
return $\{\text{flip}, \text{flipY}, \beta^U, \gamma^U, \mathcal{L}(\cdot), \mathcal{L}_y(\cdot)\}$

We will design and examine the following four algorithms:

- **UD-Preprocess:** Scan anchors and thresholds; detect activated pairs; set flip flags; reset contradictions; precompute attenuations; build per-value rule indices.
- **UD-Sugeno-Inference:** Evaluate flipped or original memberships; aggregate antecedents via T; multiply cached β^U, γ^U; accumulate numerator/denominator; return N/W with fallback.
- **Improved-UD-Preprocess:** Apply mode selector $\mathcal{M}$ to activated pairs; choose invert+neutralize or neutralize-only; set flips; zero contradictions; precompute attenuations and caches.
- **UD-Incremental-Update:** Upon anchor or threshold change, recompute only affected flips and attenuations via indices; update caches; preserve unrelated rules and timings.

Example 4.1 (Caching and flips for a 1-input PFCS (uses Alg. 1))**. Setup.** One input X_1 with $Pv_1 = \{\mathrm{N}, \mathrm{Z}, \mathrm{P}\}$ and one output with $Pv_y = \{\mathrm{N}, \mathrm{Z}, \mathrm{P}\}$. Anchors: $b_1 = b_y = \mathrm{Z}$. Thresholds: $\tau_1 = 0.8$, $\tau_y = 0.7$. Contradictions:

$$pCF_1(\mathrm{N}, \mathrm{Z}) = 0.9, \quad pCF_1(\mathrm{P}, \mathrm{Z}) = 0.3, \quad pCF_1(\mathrm{Z}, \mathrm{Z}) = 0,$$

$$pCF_y(\mathrm{N}, \mathrm{Z}) = 0.8, \quad pCF_y(\mathrm{P}, \mathrm{Z}) = 0.6, \quad pCF_y(\mathrm{Z}, \mathrm{Z}) = 0.$$

Rule base:

$$R_1 : (\mathrm{N}) \Rightarrow \mathrm{P}, \qquad R_2 : (\mathrm{P}) \Rightarrow \mathrm{N}, \qquad R_3 : (\mathrm{Z}) \Rightarrow \mathrm{Z}.$$

Apply Alg. 1. Activations: N (input) activates since $0.9 \geq 0.8$; P (input) does not. On output, N activates ($0.8 \geq 0.7$), P does not.

Flip flags:

$$\mathrm{flip}[1, \mathrm{N}] = 1, \quad \mathrm{flip}[1, \mathrm{P}] = 0, \quad \mathrm{flip}[1, \mathrm{Z}] = 0, \qquad \mathrm{flipY}[\mathrm{N}] = 1,\ \mathrm{flipY}[\mathrm{P}] = 0,\ \mathrm{flipY}[\mathrm{Z}] = 0.$$

Cached attenuations:

$$\beta^U_{R_1} = 1, \qquad \beta^U_{R_2} = 1 - pCF_1(\mathrm{P}, \mathrm{Z}) = 0.7, \qquad \beta^U_{R_3} = 1,$$

$$\gamma^U_{R_1} = 1 - pCF_y(\mathrm{P}, \mathrm{Z}) = 0.4, \quad \gamma^U_{R_2} = 1 \text{ (reset)}, \quad \gamma^U_{R_3} = 1.$$

Index lists:

$$\mathcal{L}(\mathrm{N}) = \{R_1\}, \quad \mathcal{L}(\mathrm{P}) = \{R_2\}, \quad \mathcal{L}(\mathrm{Z}) = \{R_3\}, \quad \mathcal{L}_y(\mathrm{P}) = \{R_1\},\ \mathcal{L}_y(\mathrm{N}) = \{R_2\},\ \mathcal{L}_y(\mathrm{Z}) = \{R_3\}.$$

Remark 4.2. We never rewrite membership functions; instead, at runtime we evaluate

$$\widetilde{\mu}_{i,a}(x_i) \ := \begin{cases} 1 - \mu_{i,a}(x_i), & \mathrm{flip}[i, a] = 1, \\ \mu_{i,a}(x_i), & \mathrm{flip}[i, a] = 0, \end{cases} \qquad \widetilde{\gamma}_r := \gamma^U_r,\ \ \widetilde{\beta}_r := \beta^U_r.$$

This keeps preprocessing linear in the catalog sizes and avoids touching analytic μ.

Algorithm 2: UD-SUGENO-INFERENCE (lazy flip + cached attenuations)

Input: Input $x = (x_1, \dots, x_n)$; rule base $\mathcal{R}$; T, $\{y_r\}$; caches from Alg. 1.
Output: Crisp output $y^*(x)$.

```
N ← 0; W ← 0
for each rule r = (a_1, ..., a_n ⇒ b) ∈ R do
    // Antecedent aggregation with lazy flip
    for i = 1 to n do
        ν_i ← μ_{i,a_i}(x_i)
        if flip[i, a_i] = 1 then
            ν_i ← 1 − ν_i
    α_r^base ← T(ν_1, ..., ν_n)
    α_r ← α_r^base × β_r^U × γ_r^U
    N ← N + α_r · y_r;   W ← W + α_r
if W = 0 then
    return y*(x) ← y_safe                // e.g. output anchor or design fallback
return y*(x) ← N/W
```

Example 4.3 (One-step UD Sugeno inference (uses Alg. 2))**.** Using the caches from Ex. 4.1 and Alg. 2, choose rule singletons $y_{\text{P}} = +40$, $y_{\text{N}} = -30$, $y_{\text{Z}} = 0$ and, for a given input x_1, memberships

$$\mu_{1,\text{N}}(x_1) = 0.1, \quad \mu_{1,\text{Z}}(x_1) = 0.3, \quad \mu_{1,\text{P}}(x_1) = 0.6.$$

With lazy flip, N antecedent is inverted: $\nu_{\text{N}} = 1 - 0.1 = 0.9$; others unchanged. For $T = \min$ (degenerates to the single degree):

$$\alpha_{R_1}^{\text{base}} = 0.9, \ \ \alpha_{R_2}^{\text{base}} = 0.6, \ \ \alpha_{R_3}^{\text{base}} = 0.3.$$

Apply cached attenuations:

$$\alpha_{R_1} = 0.9 \cdot 1 \cdot 0.4 = 0.36, \quad \alpha_{R_2} = 0.6 \cdot 0.7 \cdot 1 = 0.42, \quad \alpha_{R_3} = 0.3 \cdot 1 \cdot 1 = 0.3.$$

Sugeno output (Alg. 2):

$$y^*(x_1) = \frac{0.36 \cdot 40 + 0.42 \cdot (-30) + 0.3 \cdot 0}{0.36 + 0.42 + 0.3} = \frac{14.4 - 12.6}{1.08} = \frac{1.8}{1.08} \approx 1.667.$$

Algorithm 3: IMPROVED-UD-PREPROCESS with a mode selector $\mathcal{M}$

Input: PFCS; anchors, thresholds; symmetric mode selector $\mathcal{M} : \{\{u, w\}\} \mapsto \{0, 1\}$.
Output: Flip flags & attenuations as in Alg. 1.

for $i = 1$ **to** n **do**
  for *each* $a \in Pv_i$ **do**
    $c \leftarrow pCF_i(a, b_i)$; $act \leftarrow \mathbf{1}[c \geq \tau_i]$
    if $act = 1$ ***and*** $\mathcal{M}(\{a, b_i\}) = 1$ **then**
      flip$[i, a] \leftarrow 1$ `// Mode 1: invert`
    else
      flip$[i, a] \leftarrow 0$ `// Mode 0 or inactive`
    `// Regardless of mode, contradictions to the anchor are reset in caches`
for *each rule* $r = (a_1, \ldots, a_n \Rightarrow b)$ **do**
  $\beta_r^U \leftarrow 1$; $\gamma_r^U \leftarrow 1$ if $pCF_y(b, b_y) \geq \tau_y$ else $1 - pCF_y(b, b_y)$
  for $i = 1$ **to** n **do**
    if $pCF_i(a_i, b_i) \geq \tau_i$ **then**
      factor $\leftarrow 1$ `// reset regardless of mode`
    else
      factor $\leftarrow 1 - pCF_i(a_i, b_i)$
    $\beta_r^U \leftarrow \beta_r^U \times$ factor
return *caches and flags*

Example 4.4 (Two-mode preprocessing with a selector $\mathcal{M}$ (uses Alg. 3))**.** Reuse the setting of Ex. 4.1 but choose a symmetric mode selector with

$$\mathcal{M}(\{\mathrm{N}, \mathrm{Z}\}) = 0 \;\; \text{(Neutralize-only)}, \qquad \mathcal{M}(\{\mathrm{P}, \mathrm{Z}\}) = 0.$$

Running Alg. 3 gives

$$\text{flip}[1, \mathrm{N}] = 0 \quad \text{(no inversion in Mode 0)}, \qquad \text{flip}[1, \mathrm{P}] = 0,$$

while the *cached* contradictions to the anchor are still reset wherever $pCF \geq \tau$:

$$\beta_{R_1}^U = 1,\;\; \beta_{R_2}^U = 0.7,\;\; \beta_{R_3}^U = 1; \qquad \gamma_{R_1}^U = 0.4,\;\; \gamma_{R_2}^U = 1,\;\; \gamma_{R_3}^U = 1.$$

Thus, relative to Alg. 1, only the flip flag for $(1, \mathrm{N})$ changes (no membership inversion); the cached attenuations remain identical due to the reset rule.

Algorithm 4: UD-INCREMENTAL-UPDATE for (b_i, τ_i) or (b_y, τ_y) changes

Input: Changed (b_i, τ_i) (or (b_y, τ_y)); prior caches; rule-index lists $\mathcal{L}(\cdot), \mathcal{L}_y(\cdot)$.
Output: Updated flip/flipY and β_r^U / γ_r^U only where needed.

if *input side* (b_i, τ_i) *changed* **then**
  for *each* $a \in Pv_i$ **do**
    recompute $act \leftarrow \mathbf{1}[pCF_i(a, b_i) \geq \tau_i]$ and set flip$[i, a]$ as in Alg. 3
    for *each* $r \in \mathcal{L}(a)$ **do**
      recompute only the i-th factor of β_r^U; leave other factors intact
if *output side* (b_y, τ_y) *changed* **then**
  for *each* $b \in Pv_y$ **do**
    recompute activation; set flipY$[b]$ accordingly
    for *each* $r \in \mathcal{L}_y(b)$ **do**
      recompute γ_r^U only
return *updated caches*

Example 4.5 (Incremental update after changing the output threshold (uses Alg. 4))**.** Start from the caches in Ex. 4.1. Increase the output threshold to $\tau_y' = 0.85$. Since $pCF_y(\mathrm{N}, \mathrm{Z}) = 0.8 < 0.85$, the activation drops:

$$\text{flipY}[\mathrm{N}] : 1 \longrightarrow 0, \qquad \gamma_{R_2}^U : 1 \longrightarrow 1 - pCF_y(\mathrm{N}, \mathrm{Z}) = 0.2,$$

and only rules in $\mathcal{L}_y(\mathrm{N}) = \{R_2\}$ are touched (Alg. 4). Using the same memberships as in Ex. 4.3 and unchanged β^U:

$$\alpha_{R_1} = 0.36 \text{ (unchanged)}, \quad \alpha_{R_2} = 0.6 \cdot 0.7 \cdot 0.2 = 0.084, \quad \alpha_{R_3} = 0.3.$$

Updated Sugeno output:

$$y^*(x_1) = \frac{0.36 \cdot 40 + 0.084 \cdot (-30) + 0.3 \cdot 0}{0.36 + 0.084 + 0.3} = \frac{14.4 - 2.52}{0.744} = \frac{11.88}{0.744} \approx 15.97.$$

Therefore, the incremental update (Alg. 4) weakens the negative rule (consequent N), shifting y^* upward without recomputing unaffected terms.

Theorem 4.6 (Correctness of UD-PREPROCESS + UD-SUGENO-INFERENCE)
Let $U_{\boldsymbol{b},\boldsymbol{\tau}}$(PFCS) be the UD transform with contradiction reset as defined previously. For any input x, the output $y^*(x)$ returned by Alg. 2 using caches from Alg. 1 equals the output obtained by first forming $U_{\boldsymbol{b},\boldsymbol{\tau}}$(PFCS) (i.e., flipping memberships and zeroing the activated contradictions) and then running standard Sugeno inference on that transformed controller.

Proof
By construction of Alg. 1, for every rule $r = (a_1, \ldots, a_n \Rightarrow b)$ the cached factors satisfy

$$\beta_r^U = \prod_{i=1}^{n} \bigl(1 - pCF_i^U(a_i, b_i)\bigr), \qquad \gamma_r^U = 1 - pCF_y^U(b, b_y),$$

where pCF^U denotes the contradiction map after the reset (activated pairs to anchors are set to 0). In Alg. 2 the lazy flip implements $\widetilde{\mu}_{i,a_i}(x_i) = \mu_{i,a_i}^U(x_i)$ for each antecedent. Hence the computed firing equals

$$\alpha_r = T\bigl(\mu_{1,a_1}^U(x_1), \ldots, \mu_{n,a_n}^U(x_n)\bigr) \cdot \beta_r^U \cdot \gamma_r^U = \alpha_r^U,$$

the exact post-UD firing by definition. Sugeno aggregation then yields the same $N = \sum_r \alpha_r^U y_r$ and $W = \sum_r \alpha_r^U$, hence the same $y^* = N/W$ as the mathematical procedure. □

Theorem 4.7 (Correctness under two-mode policy)
Let $U^{\mathrm{imp}}_{b,\tau,\mathcal{M}}$ be the two-mode UD operator (Mode 0: neutralize-only; Mode 1: invert+neutralize). Caches produced by Alg. 3, combined with Alg. 2, compute exactly the inference result of $U^{\mathrm{imp}}_{b,\tau,\mathcal{M}}(\mathsf{PFCS})$ for any x.

Proof
If $\mathcal{M}(\{a, b_i\}) = 1$ and $pCF_i(a, b_i) \geq \tau_i$, Alg. 3 sets $\mathrm{flip}[i, a] = 1$ and the i-th factor of β^U_r to 1, matching the definition (invert appurtenance and reset contradiction). If $\mathcal{M}(\{a, b_i\}) = 0$, it sets $\mathrm{flip}[i, a] = 0$ but still sets the factor to 1, again matching the definition (no inversion, reset only). The output side is analogous. The remainder is identical to the proof of Theorem 4.6. □

Theorem 4.8 (Finite termination and idempotence of UD-PREPROCESS)
Let $N_{\max} = \sum_{i=1}^{n} \#\{a \in Pv_i \setminus \{b_i\} : pCF_i(a, b_i) \geq \tau_i\} + \#\{b' \in Pv_y \setminus \{b_y\} : pCF_y(b', b_y) \geq \tau_y\}$. Then Alg. 1 (or Alg. 3) performs at most $N_{\max}$ flips/resets and is idempotent: a re-run changes nothing.

Proof
Each activated pair to an anchor is processed exactly once and then considered neutralized (its contradiction to the anchor is zero in the caches). No new activations are created, so the step count is bounded by $N_{\max}$. A second pass finds no further activations to process. □

Theorem 4.9 (Time and space complexity)
Assume $O(1)$ access to pCF and to each membership evaluation $\mu_{i,a}(x_i)$; let c_T be the cost of computing T on n arguments (e.g. $c_T = O(n)$ for min or product). Then:

(a) **Preprocessing (Alg. 1 / 3).**

$$\text{Time} \;=\; O\Big(\sum_{i=1}^{n} |Pv_i| + |Pv_y| + K\,n\Big), \qquad \text{Space} \;=\; O\Big(\sum_{i=1}^{n} |Pv_i| + |Pv_y| + K\Big).$$

(b) **Online inference (Alg. 2, Sugeno).**

$$\text{Time per query} \;=\; O\big(K\,(n + c_T)\big) \quad \text{and} \quad \text{Space overhead} \;= O(1) \text{ beyond the caches.}$$

For $T = \min$ or $T = \prod$, $c_T = O(n)$, hence time $= O(K\,n)$.

(c) **Incremental update (Alg. 4).** If only (b_i, τ_i) changes, the time is

$$O\Big(|Pv_i| + \sum_{a \in Pv_i} |\mathcal{L}(a)|\Big) = O\Big(|Pv_i| + \#\{(r, a) : r \in \mathcal{R},\ a \text{ used in } r\}\Big),$$

i.e. linear in the rules that actually mention the affected values. An output-side update is analogous with $\mathcal{L}_y(\cdot)$.

Proof
(a) Scanning all Pv_i and Pv_y dominates the activation test; computing β^U_r and γ^U_r for each rule multiplies/looks up at most $n+1$ factors, giving $O(Kn)$. Caches and index lists store $O(K)$ items plus $O(\sum_i |Pv_i| + |Pv_y|)$ flags.

(b) Each rule evaluation performs n membership calls (with an optional $1 - \cdot$), one T, and $O(1)$ arithmetic to apply (β^U_r, γ^U_r) and update accumulators. Thus $O(K(n + c_T))$.

(c) Only factors that depend on changed anchors/thresholds need recomputation; the index lists $\mathcal{L}(\cdot)$, $\mathcal{L}_y(\cdot)$ restrict work to affected rules. □

5. Conclusion and Future Works

5.1. Conclusion

This paper introduced the *Plithogenic Fuzzy Control System* (PFCS) as a contradiction-aware extension of classical fuzzy control and integrated it with Upside-Down (UD) logic. We formalized how contradiction maps and anchors modulate rule firing, and we designed practical algorithms for preprocessing, inference, and incremental updates. The framework enables context-triggered rule inversion while preserving the interpretability and modularity of standard fuzzy controllers. The proven advantages are threefold. First, PFCS *strictly generalizes* classical fuzzy control and reduces to it when the contradiction map is null; uniform attenuation cancels under zero-order Sugeno aggregation. Second, PFCS delivers *safer, more conservative* decisions under polarity or frame flips by contradiction-aware attenuation and UD transforms. Third, it retains *transparent semantics* via explicit anchors and symmetric contradiction maps.

5.2. Future Works

In future work, we plan to explore extensions employing Neutrosophic Sets [84, 91, 92, 93, 94, 95, 96, 97, 98, 99], Quadripartitioned Neutrosophic Sets [100, 101], PentaPartitioned Neutrosophic Sets [102, 103], and HyperFuzzy Sets [104, 105, 106]. Furthermore, we intend to verify the validity of the proposed concepts through computational experiments, pursue additional theoretical extensions, and investigate applications to machine learning using real-world datasets. We also plan to investigate potential applications to systems such as the nonlinear inverted pendulum [107, 108], the complex thermal plant [109], Genetic Algorithms[110, 111, 112, 113, 114], Gradient Descent[115, 116], Reinforcement Learning[117, 118], and the path-following robot [119, 120]. Furthermore, as a quantitative comparison, we hope that future studies by experts will evaluate and compare the controllers using standard performance metrics such as Integral Absolute Error (IAE)[121], Settling Time, Overshoot, and Control Effort. Furthermore, we plan to explore the possibilities of extending the approach to Plithogenic Intuitionistic, Plithogenic Neutrosophic control systems, and meta-logic [122, 123, 124, 125].

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Author's Contributions

Conceptualization, All authors; Investigation, All authors; Methodology, All authors; Writing – original draft, All authors; Writing – review & editing, All authors.

Data Availability

This research is purely theoretical, involving no data collection or analysis. We encourage future researchers to pursue empirical investigations to further develop and validate the concepts introduced here.

Research Integrity

The authors hereby confirm that, to the best of their knowledge, this manuscript is their original work, has not been published in any other journal, and is not currently under consideration for publication elsewhere at this stage.

Disclaimer (Note on Computational Tools)

No computer-assisted proof, symbolic computation, or automated theorem proving tools (e.g., Mathematica, SageMath, Coq, etc.) were used in the development or verification of the results presented in this paper. All proofs and derivations were carried out manually and analytically by the authors.

Conflicts of Interest

The authors confirm that there are no conflicts of interest related to the research or its publication.

Disclaimer

This work presents theoretical concepts that have not yet undergone practical testing or validation. Future researchers are encouraged to apply and assess these ideas in empirical contexts. While every effort has been made to ensure accuracy and appropriate referencing, unintentional errors or omissions may still exist. Readers are advised to verify referenced materials on their own. The views and conclusions expressed here are the authors' own and do not necessarily reflect those of their affiliated organizations.

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