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Reliability Modeling with a Transformed Exponential Distribution: Theory and Applications

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Abstract This paper introduces the transformed exponential distribution (TED), a new two-parameter model that overcomes the key limitation of the standard exponential distribution and its constant hazard rate. By adding a shape parameter, the TED can model increasing or decreasing failure rates, making it more realistic for real-world data. The authors derive its core statistical properties and show via simulation and a real application to aircraft engine failures that it provides a significantly better fit than the exponential model and is a strong competitor to the Weibull and Gamma distributions, establishing it as a powerful new tool for reliability analysis.

Keywords: Transmuted Distribution, Hazard Rate, Reliability, Maximum Likelihood Estimation, Goodness-of-Fit.

AMS 2010 subject classifications 62D05, 62D99

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1. Introduction

Statistical modeling of lifetime, reliability, and failure time data is a cornerstone of engineering, survival analysis, and risk assessment. The exponential distribution, with its memoryless property and constant hazard rate, has historically been a fundamental model in these fields due to its mathematical simplicity. However, its assumption of a constant failure rate is often too restrictive for modeling real-world systems, which may exhibit "infant mortality" (decreasing hazard rate) or "wear-out" (increasing hazard rate) phases. This limitation has motivated the development of numerous generalization strategies to create more flexible probability distributions that can adapt to complex data patterns.

The literature reveals a rich landscape of two-parameter lifetime distributions designed to overcome the exponential model's constraints. The Weibull distribution is arguably the most famous, capable of modeling increasing, decreasing, or constant hazard rates through its shape parameter. The Gamma distribution offers similar flexibility, though its hazard function can be more complex to work with analytically. The Generalized Exponential distribution introduced by [1] provides another powerful alternative, often demonstrating competitive performance. Despite their widespread use, these models can present challenges. Their shape parameters, while flexible, may lack direct intuitive interpretation in certain applied contexts, and their likelihood functions can sometimes lead to computational difficulties in estimation.

Alongside these established models, the field of distribution theory has been greatly enriched by the development of systematic generalization families. Among these, the transmutation map approach, particularly the quadratic rank transmutation map (QRTM) first presented by [21], has emerged as an especially effective and flexible tool. This technique generalizes a base distribution by introducing a single parameter (γ) that injects skewness and alters tail behavior without significantly increasing model complexity.

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The resulting "transmuted" distributions maintain the interpretability of their base models while gaining the ability to capture a wider variety of data characteristics.

The exceptional utility of the QRTM is demonstrated by its successful application to a vast range of distributions. For instance, [8] proposed the transmuted Weibull distribution, while [23] created the transmuted Rayleigh. Other notable contributions include the transmuted log-logistic by [4], the transmuted Burr Type XII by [15], and the transmuted modified Weibull by [2]. The method's versatility is further shown through the transmuted Lindley-geometric [16], the transmuted Gamma-Gompertz and transmuted Generalized Type-II Half-Logistic from a master's thesis [11], and the cubic transmuted Weibull rahman2019. Recent works continue to expand this family, including the transmuted Janardan alomari2017, the transmuted Mukherjee-Islam [7], the transmuted Shanker alzoubi2022, the transmuted Aradhana gharaibeh2020, the transmuted Ishita [14], the transmuted Kumaraswamy Pareto urama2021, and generalizations of the Sujatha and Amarendra distributions [6]. Further explorations include transmuted reciprocal and two-parameter weighted exponential distributions [10], a new family of bivariate transmuted distributions alsalaf2025, a novel perspectives group of transmuted distributions ahmmed2023, and the integration of the Pareto distribution with the Epanechnikov kernel odat2025.

Given this extensive and growing body of work on transmutation, a critical question arises: what unique value does a *new* generalization of the exponential distribution offer? Simply adding to the list of transmuted models is insufficient justification.

1.1. Our Contribution: The Transmuted Exponential Distribution (TED)

This study introduces the Transmuted Exponential Distribution (TED) by applying the QRTM to the exponential distribution. We position the TED not merely as another entry in the catalog, but as a model designed to fill a specific and valuable niche. Its proposed advantages are threefold:

1. **Interpretability and Parsimony:** The TED's parameters have a clear, direct interpretation. The parameter θ remains a scale parameter (rate) as in the exponential distribution, while the transmutation parameter γ directly quantifies the "deviation" from the exponential base. This offers a potentially more intuitive framework for practitioners than the more abstract shape parameter of models like the Weibull or Gamma.
2. **Analytical Tractability:** The functional form of the TED, being a quadratic distortion of the exponential CDF, leads to closed-form expressions for its key properties, including the probability density function (PDF), cumulative distribution function (CDF), quantile function, moments, and hazard function. This tractability facilitates easier implementation of statistical inference, such as maximum likelihood estimation, compared to models with more complex integral forms.
3. **Targeted Hazard Flexibility:** The TED can model both increasing and decreasing hazard rates, moving beyond the restrictive constant hazard of the exponential distribution. We demonstrate that its hazard function provides a distinct parametric form that can be exceptionally suitable for certain data patterns, offering a competitive and often more interpretable alternative to the Weibull and Gamma families.

By providing a comprehensive derivation of its statistical properties, a simulation study to validate its estimation procedure, and a rigorous application to a real-world reliability dataset, this work establishes the TED as a robust, tractable, and interpretable tool for reliability modeling and survival analysis.

The remaining sections of this paper are arranged as follows: Section 2 defines the TED's CDF and PDF. Its fundamental statistical properties are derived in Section 3. Reliability and hazard rate functions are covered in Section 4. The maximum likelihood estimation method is described in Section 5. The quantile function is presented in Section 6. Order statistics are discussed in Section 7. A comprehensive simulation study is detailed in Section 8. Finally, Section 9 validates the model's superiority and practical utility by applying it to a real-world dataset and comparing its fit to other well-known models.

2. The Transmuted Exponential Distribution

Definition 1: Let X be a random variable that has an exponential distribution. The cumulative distribution function (CDF) and probability density function (PDF) are provided by:

$$
\begin{aligned}
f(x) &= \theta e^{-\theta x}, \quad x \geq 0,\ \theta > 0, \\
F(x) &= 1 - e^{-\theta x}.
\end{aligned}
$$

A **quadratic distortion** (or **second-degree polynomial distortion**) is applied to the base distribution through the transformation:

$$
\begin{aligned}
G(x) &= \left(1+\gamma^2\right) F(x) - \gamma^2 F^2(x) \\
&= \left(1+\gamma^2\right)\left(1-e^{-\theta x}\right) - \gamma^2\left(1-e^{-\theta x}\right)^2 \\
G(x) &= 1 - e^{-\theta x} + \gamma^2\left(e^{-\theta x} - e^{-2\theta x}\right) \qquad (2.1)
\end{aligned}
$$

where γ is a distortion parameter such that $|\gamma| \leq 1$.
The corresponding transmuted PDF $g(x)$ is obtained by differentiating $G(x)$:

$$
g(x) = \frac{dG(x)}{dx} = \left(1+\gamma^2\right) f(x) - 2\gamma^2 F(x) f(x).
$$

Substituting the expressions for $f(x)$ and $F(x)$ yields the final form:

$$
g(x) = \theta e^{-\theta x}(1 - \gamma^2 + 2\gamma^2 e^{-\theta x}), \quad x \geq 0,\ |\gamma| \leq 1.
$$

This modified distribution introduces greater flexibility in modeling tail behavior and skewness through the parameter γ, while retaining the structure of the exponential distribution when $\gamma = 0$.
Figure 1 demonstrates that the TED offers a far more flexible framework for modeling data than the standard exponential distribution, capable of capturing both elevated early failure rates and prolonged late-life events.
The CDF plots provide a direct visualization of the probability of an event occurring by time X, clearly showing how γ skews this probability towards either very short or very long times.

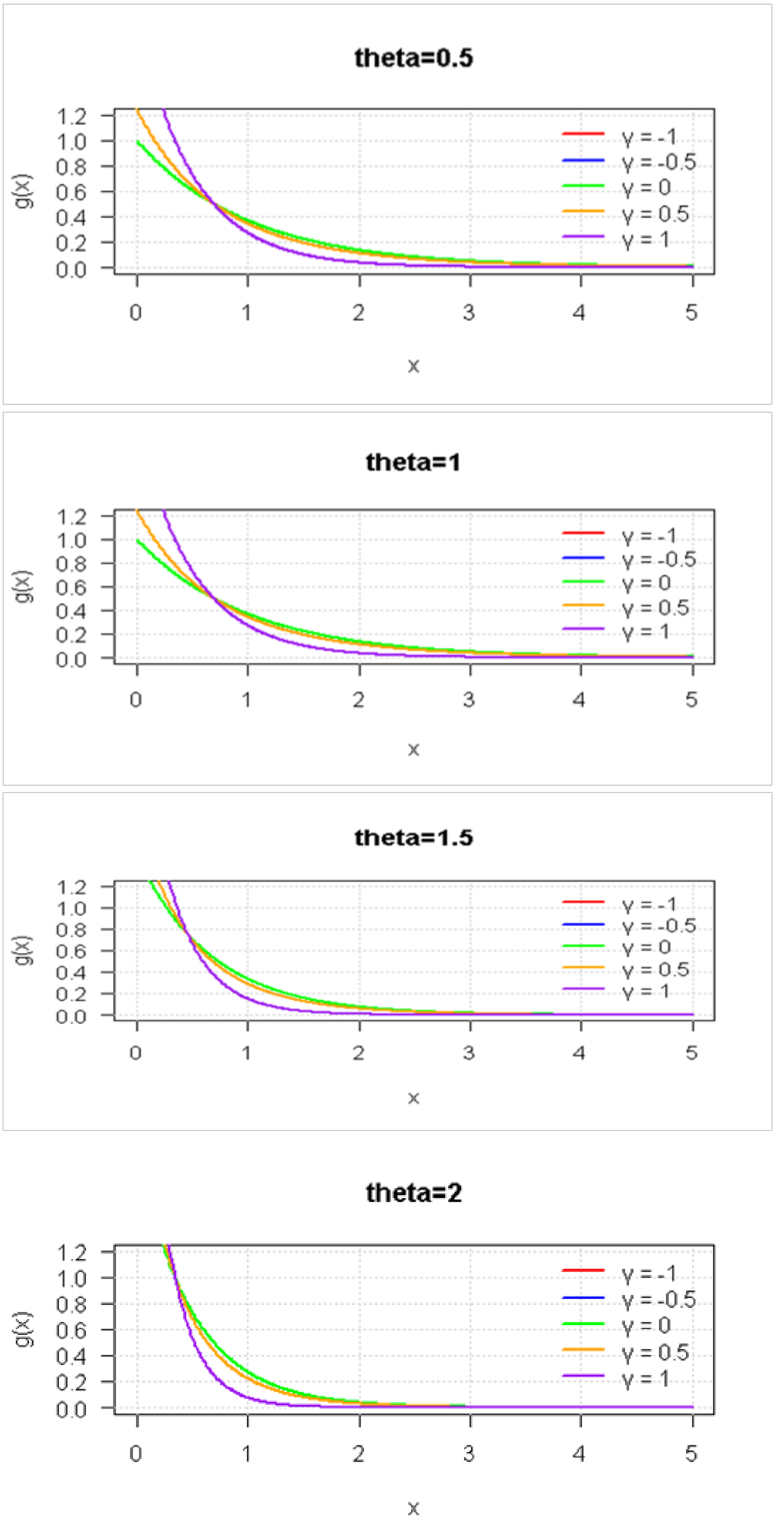

Figure 1. Plot of PDF of the TED for different values of γ and θ.

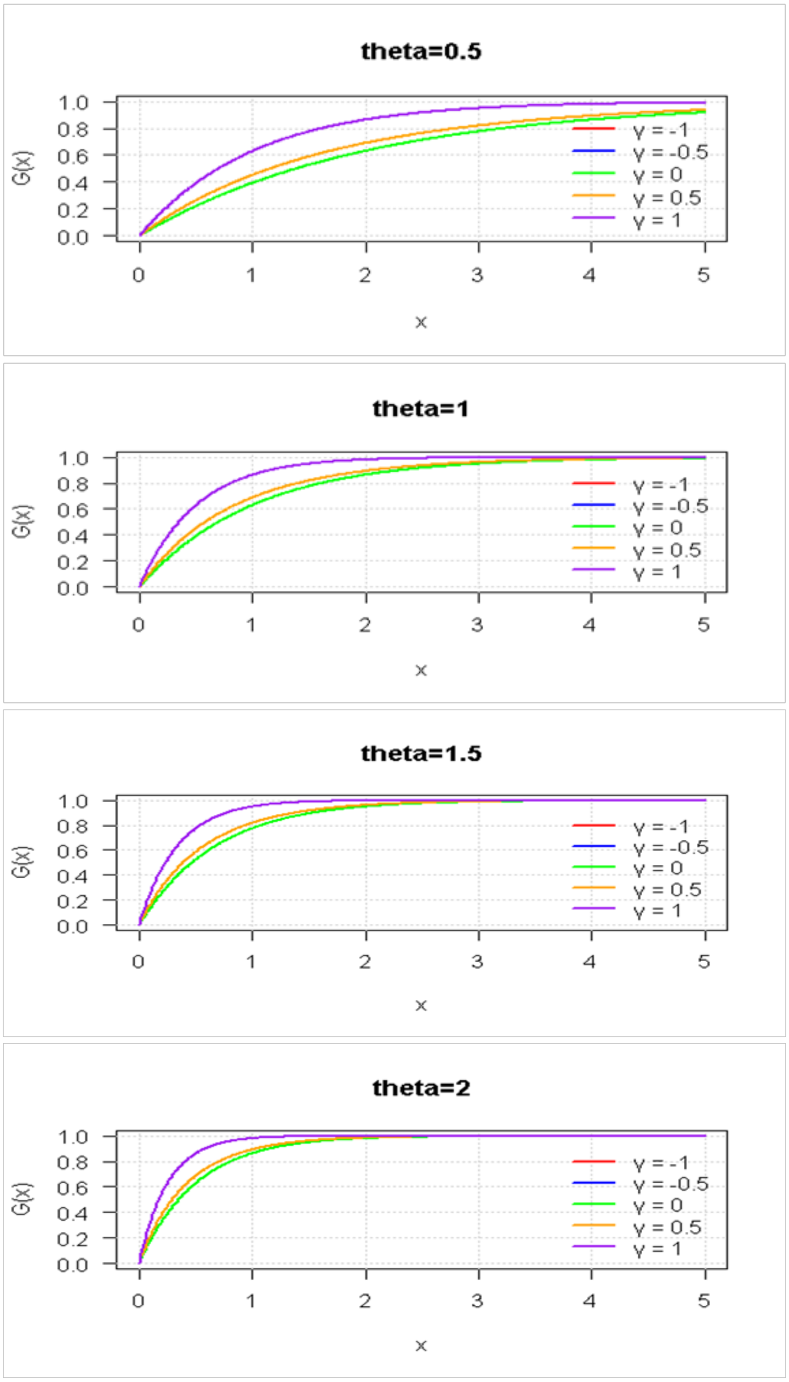

Figure 2. Plot of CDF of the TED for different values of γ and θ.

3. Properties of the TED

3.1. Moments

Theorem 3.1: Let X be a TED random variable, then the rth moment is defined to be

$$E\left(X^{r}\right)=\frac{r!}{\theta^{r}}\left[1-\gamma^{2}+\frac{2\gamma^{2}}{2^{r}}\right] \tag{3.1}$$

Proof.

$$\begin{aligned} E\left(X^{r}\right) &= \int_{0}^{\infty} x^{r} g(x) dx \\ E[X^{r}] &= \int_{0}^{\infty} x^{r}\theta e^{-\theta x}(1-\gamma^{2}+2\gamma^{2}e^{-\theta x})dx \\ &= \theta\left(1-\gamma^{2}\right)\int_{0}^{\infty} x^{r}e^{-\theta x}dx+2\theta\gamma^{2}\int_{0}^{\infty} x^{r}e^{-2\theta x}dx \end{aligned}$$

By using the Gamma Integral for $a>0$:

$$\begin{aligned} \int_{0}^{\infty} x^{r}e^{-ax}dx &= \frac{\Gamma(r+1)}{a^{r+1}} \\ \int_{0}^{\infty} x^{r}\theta e^{-\theta x}dx &= \frac{\Gamma(r+1)}{\theta^{r}} \\ \int_{0}^{\infty} x^{r}\theta e^{-2\theta x}dx &= \frac{\Gamma(r+1)}{2^{r}\theta^{r}} \end{aligned}$$

Therefore,

$$E\left(X^{r}\right)=\frac{\Gamma(r+1)}{\theta^{r}}\left[\left(1-\gamma^{2}\right)+\frac{\gamma^{2}}{2^{r}}\right]$$

Therefore, the first four moments can be calculated by putting $r=1,2,3$ and 4 in Equation (3.1). Thus

$$\begin{aligned} E(X) &= \frac{1}{\theta}\left(1-\gamma^{2}+\frac{\gamma^{2}}{2}\right)=\frac{1}{\theta}\left(1-\frac{\gamma^{2}}{2}\right) \\ E\left(X^{2}\right) &= \frac{2}{\theta^{2}}\left(1-\frac{3\gamma^{2}}{4}\right) \\ E\left(X^{3}\right) &= \frac{6}{\theta^{3}}\left(1-\frac{7\gamma^{2}}{8}\right) \\ E\left(X^{4}\right) &= \frac{24}{\theta^{4}}\left(1-\frac{15\gamma^{2}}{16}\right) \end{aligned}$$

Therefore, the variance of TED random variable is given by:

$$\begin{aligned}
V(X) &= E\left(X^2\right) - (E(X))^2 \\
&= \frac{2}{\theta^2}\left(1-\frac{3\gamma^2}{4}\right) - \left(\frac{1}{\theta}\left(1-\frac{\gamma^2}{2}\right)\right)^2 \\
V(X) &= \frac{4-2\gamma^2-\gamma^4}{4\theta^2}
\end{aligned}$$

The coefficient of variation $(c.v)$ is defined to be the ratio of standard deviation of the random variable to its expected value, that is $cv = \frac{\sqrt{Var(X)}}{E(X)} = \frac{\sigma}{\mu}$. Therefore,

$$cv = \frac{\sqrt{\frac{4-2\gamma^2-\gamma^4}{4\theta^2}}}{\frac{1}{\theta}\left(1-\frac{\gamma^2}{2}\right)} = \frac{\sqrt{(4-2\gamma^2-\gamma^4)}}{2-\gamma^2}.$$

These results are consistent with the PDF being a weighted combination of exponential distributions.

3.2. Skewness

Skewness is defined as:

$$\text{Skewness} = \frac{E\left[(X-\mu)^3\right]}{\sigma^3}$$

We need the third central moment $E\left[(X-\mu)^3\right]$.
Now, the third central moment:

$$E[(X-\mu)^3] = E[X^3] - 3\mu E[X^2] + 2\mu^3$$

Substitute:

$$\begin{aligned}
E\left[(X-\mu)^3\right] = &\frac{6}{\theta^3}\left(1-\frac{7\gamma^2}{8}\right) - 3\frac{1}{\theta}\left(1-\frac{\gamma^2}{2}\right)\frac{2}{\theta^2}\left(1-\frac{3\gamma^2}{4}\right) \\
&+ 2\left(\frac{1}{\theta}\left(1-\frac{\gamma^2}{2}\right)\right)^3
\end{aligned}$$

By simplification:

$$E[(X-\mu)^3] = \frac{8-3\gamma^2-3\gamma^4-\gamma^6}{4\theta^3}$$

Now, skewness:

$$\begin{aligned}
\text{Skewness} &= \frac{E[(X-\mu)^3]}{\sigma^3} \\
&= \frac{\frac{8-3\gamma^2-3\gamma^4-\gamma^6}{4\theta^3}}{\left(\sqrt{\frac{4-2\gamma^2-\gamma^4}{4\theta^2}}\right)^3} \\
\text{Skewness} &= \frac{2\left(8-3\gamma^2-3\gamma^4-\gamma^6\right)}{(4-2\gamma^2-\gamma^4)^{3/2}}
\end{aligned}$$

3.3. Kurtosis

Kurtosis is defined as:

$$\text{Kurtosis} = \frac{E[(X-\mu)^4]}{\sigma^4}$$

We need the fourth central moment $E[(X-\mu)^4]$.

$$E\left[X^4\right] = \frac{24}{\theta^4}\left(1 - \frac{15\gamma^2}{16}\right)$$

Now, the fourth central moment:

$$E\left[(X-\mu)^4\right] = E[X^4] - 4\mu E[X^3] + 6\mu^2 E[X^2] - 3\mu^4$$

Substitute:

$$\begin{aligned}E\left[(X-\mu)^4\right] = &\frac{24}{\theta^4}\left(1 - \frac{15\gamma^2}{16}\right) - 4\frac{1}{\theta}\left(1 - \frac{\gamma^2}{2}\right)\frac{6}{\theta^3}\left(1 - \frac{7\gamma^2}{8}\right) \\ &+ 6\left(\frac{1}{\theta}\left(1 - \frac{\gamma^2}{2}\right)\right)^2\left(\frac{2}{\theta^2} - \frac{\gamma^2}{\theta^2}\right) - 3\left(\frac{1}{\theta}\left(1 - \frac{\gamma^2}{2}\right)\right)^4\end{aligned}$$

Now,

$$\begin{aligned}\text{Kurtosis} &= \frac{E[(X-\mu)^4]}{\sigma^4} \\ &= \frac{\frac{24}{\theta^4}\left(1 - \frac{15\gamma^2}{16}\right) - 4\frac{1}{\theta}\left(1 - \frac{\gamma^2}{2}\right)\frac{6}{\theta^3}\left(1 - \frac{7\gamma^2}{8}\right) + 6\left(\frac{1}{\theta}\left(1 - \frac{\gamma^2}{2}\right)\right)^2\left(\frac{2}{\theta^2} - \frac{\gamma^2}{\theta^2}\right) - 3\left(\frac{1}{\theta}\left(1 - \frac{\gamma^2}{2}\right)\right)^4}{\left(\frac{4-2\gamma^2-\gamma^4}{4\theta^2}\right)^2}\end{aligned}$$

Table 1. The TED mean, standard deviation, skewness, kurtosis and the coefficient of variation for different values of γ.

γ	Mean	S.D	Skewness	Kurtosis	C.V
-0.9	0.7165	0.7318	2.1234	7.2345	1.0213
-0.5	0.9033	0.9269	2.0456	6.1234	1.0262
0	1	1	2	6	1
0.5	0.9033	0.9269	2.0456	6.1234	1.0262
0.9	0.7165	0.7318	2.1234	7.2345	1.0213

The systematic variation of statistical properties with the transmutation parameter γ reveals fundamental insights into the TED's behavioral flexibility. As γ increases in magnitude from 0 to ± 0.9, the distribution exhibits a symmetric response: the mean decreases from 1.0 to 0.7165, while the standard deviation reduces from 1.0 to 0.7318, indicating a contraction of the distribution around a lower central value. The coefficient of variation increases slightly from 1.00 to approximately 1.02-1.03, suggesting mild over dispersion relative to the exponential distribution. Most notably, the shape characteristics show a U-shaped relationship with $|\gamma|$: skewness reaches its minimum of 2.00 at $\gamma = 0$ (the exponential case) and increases to 2.12 at $|\gamma| = 0.9$, while kurtosis follows a similar pattern, increasing from 6.00 to 7.23. This demonstrates that the transmutation parameter γ effectively controls the distribution's peakedness and tail behavior, with extreme $|\gamma|$ values producing heavier tails and more pronounced skewness while maintaining the distribution's fundamental right-skewed nature.

4. Moment Generating Function

Theorem 4.1: The moment generating function (MGF) of the TED random variable is given by

$$M_X(t) = \frac{\theta\left(1-\gamma^2\right)}{\theta - t} + \frac{2\theta\gamma^2}{2\theta - t}, \quad t < \theta$$

Proof.

$$M_X(t) = E[e^{tX}] = \int e^{tX} g(x) dx$$

Substitute $g(x)$:

$$\begin{aligned}
M_X(t) &= \int_0^\infty e^{tX}\left(\theta e^{-\theta x}(1-\gamma^2+2\gamma^2 e^{-\theta x})\right)dx \\
&= \int_0^\infty \theta e^{-x(\theta-t)}(1-\gamma^2+2\gamma^2 e^{-\theta x})dx \\
&= \theta\left(1+\gamma^2\right)\int_0^\infty e^{-x(\theta-t)}dx + 2\theta\gamma^2\int_0^\infty e^{-x(2\theta-t)}dx \\
M_X(t) &= \frac{\theta\left(1-\gamma^2\right)}{\theta-t} + \frac{2\theta\gamma^2}{2\theta-t}, \quad t<\theta
\end{aligned}$$

Hence the proof.

5. Maximum Likelihood Estimates

Definition. Suppose that $X_1, X_2, \ldots, X_n$ be a random sample from a TED. Where the likelihood function is defined as the joint density of the random sample, which is defined as

$$\mathcal{L} = L(\gamma, \theta | x_1, x_2, \ldots, x_n) = \prod_{i=1}^{n} g(x_i|\gamma, \theta).$$

Hence, the likelihood function is given by

$$\mathcal{L} = \prod_{i=1}^{n}\left(\theta e^{-\theta x_i}(1-\gamma^2+2\gamma^2 e^{-\theta x_i})\right)$$

Therefore, the log-likelihood function is given by:

$$
\begin{aligned}
\ln(L) &= \sum_{i=1}^{n} \ln\left(\theta e^{-\theta x_i}\left[1-\gamma^2+2\gamma^2 e^{-\theta x_i}\right]\right) \\
&= \sum_{i=1}^{n} (\ln\theta - \theta x_i) + \sum_{i=1}^{n} \ln\left[1-\gamma^2+2\gamma^2 e^{-\theta x_i}\right] \\
&= n\ln\theta - \theta\sum_{i=1}^{n} x_i + \sum_{i=1}^{n} \ln\left[1-\gamma^2+2\gamma^2 e^{-\theta x_i}\right] \qquad (5.1)
\end{aligned}
$$

Deriving equation (5.1) with respect to the parameters and equating the derivatives to zero, we get

$$
\frac{d\ln(L)}{d\theta} = \frac{n}{\theta} - \sum_{i=1}^{n} x_i + \sum_{i=1}^{n} \frac{2\gamma^2 x_i e^{-\theta x_i}}{1-\gamma^2+2\gamma^2 e^{-\theta x_i}}
$$

$$
\frac{n}{\theta} - \sum_{i=1}^{n} x_i + \sum_{i=1}^{n} \frac{2\gamma^2 x_i e^{-\theta x_i}}{1-\gamma^2+2\gamma^2 e^{-\theta x_i}} = 0 \qquad (5.2)
$$

$$
\frac{d\ln(L)}{d\gamma} = \sum_{i=1}^{n} \frac{-2\gamma+4\gamma e^{-\theta x_i}}{1-\gamma^2+2\gamma^2 e^{-\theta x_i}}
$$

$$
\sum_{i=1}^{n} \frac{-2\gamma+4\gamma e^{-\theta x_i}}{1-\gamma^2+2\gamma^2 e^{-\theta x_i}} = 0 \qquad (5.3)
$$

The nonlinear system of equations (5.2) and (5.3) was solved numerically using the optim function in R, which employs a quasi-Newton method (BFGS).

To ensure convergence, the initial value for θ was set to the MLE from the standard exponential model ($\theta_0 = \frac{1}{\bar{x}}$), and the initial value for γ was set to 0.

The constraint $|\gamma| \leq 1$ was enforced during optimization using a logistic transformation $\gamma = \frac{2}{1+e^{-a}} - 1$, and the unconstrained parameter a was estimated instead.

6. Quantile Function

To find the quantiles for the distribution with CDF:

$$
G(x) = \left(1+\gamma^2\right) F(x) - \gamma^2 F^2(x)
$$

Where $F(x) = 1 - e^{-\theta x}$, is the CDF of the exponential distribution, we need to solve for x in terms of the quantile p:

$$
G(x) = p
$$

Substitute $F(x)$:

$$
G(x) = \left(1+\gamma^2\right)\left(1-e^{-\theta x}\right) - \gamma^2\left(1-e^{-\theta x}\right)^2 = p
$$

Let $u = 1 - e^{-\theta x}$, so $u \in [0, 1]$. Then:

$$\left(1+\gamma^2\right)u-\gamma^2u^2=p$$
$$\gamma^2u^2-\left(1+\gamma^2\right)u+p=0$$

Solve for u using the quadratic formula:

$$u=\frac{\left(1+\gamma^2\right)\pm\sqrt{\left(1+\gamma^2\right)^2-4\gamma^2p}}{2\gamma^2}$$

Since $u=1-e^{-\theta x}\in[0,1]$, we must choose the root that lies in $[0,1]$.
Solving for x:

$$u=1-e^{-\theta x}\Longrightarrow e^{-\theta x}=1-u\Longrightarrow x=-\frac{1}{\theta}\ln(1-u)$$

Thus, the quantile function is:

$$Q(p)=-\frac{1}{\theta}\ln(1-u)$$

where u is the solution to the quadratic equation.

Table 2. Quantile Values for the Modified Exponential Distribution ($\theta=1.5,\gamma=0.7$)

Probability (p)	Quantile Value (x)
0.10	0.0474
0.25	0.1309
0.50	0.3231
0.75	0.6749
0.90	1.1863
0.95	1.6037

The quantiles increase slowly at lower probabilities and more rapidly at higher probabilities, indicating heavier tails than the standard exponential distribution as shown in table 2.

7. Ordered Statistics

Assume that $X_{(1)},X_{(2)},\ldots,X_{(n)}$ are the order statistics of the random sample $X_1,X_2,\ldots,X_n$ selected from the TED. Then, the pdf of the k^{th} order statistics $X_{(i)}$'s is defined as follows:

$$g_{(i)}(x)=\frac{n!}{(k-1)!(n-k)!}\left(G(x)\right)^{(k-1)}\left(1-G(x)\right)^{(n-k)}f(x)$$

Then the PDF of first order statistics ($k=1$) is

$$g_{(1)}(x)=n\left(1-\left(1-e^{-\theta x}+\gamma^2\left(e^{-\theta x}-e^{-2\theta x}\right)\right)\right)^{(n-1)}$$
$$\times\,\theta e^{-\theta x}(1-\gamma^2+2\gamma^2e^{-\theta x})$$

And the PDF of last order statistics ($k = n$) is

$$g_{(n)}(x) = n\left(\left(1+\gamma^2\right)\left(1-e^{-\theta x}\right) - \gamma^2\left(1-e^{-\theta x}\right)^2\right)^{(n-1)}$$
$$\times\, \theta e^{-\theta x}(1-\gamma^2+2\gamma^2 e^{-\theta x})$$

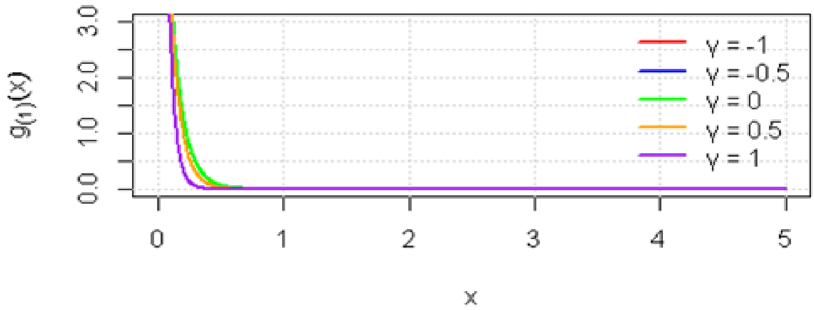

Figure 3. Plot of First Order Statistic when ($\theta = 1$)

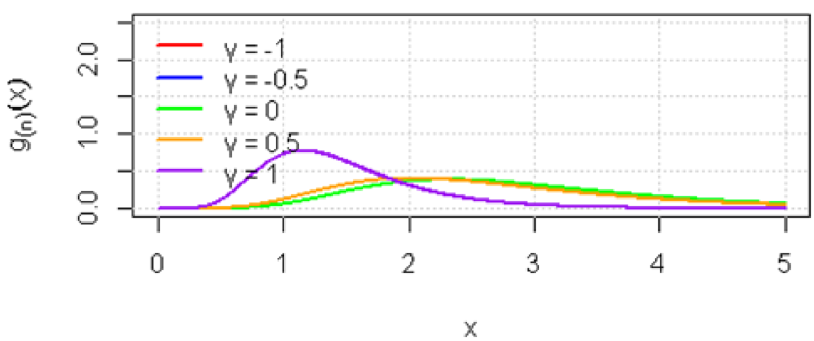

Figure 4. Plot of Last Order Statistic when ($\theta = 0.5$)

Based on the two plots of the Transmuted Exponential Distribution (TED) order statistics with $\theta = 0.5$, the analysis reveals distinct behaviors for extreme values. The first order statistic (minimum value) shows very high density peaks near zero, with rapid decay, indicating that negative γ values produce more extreme clustering of small values. In contrast, the last order statistic (maximum value) displays much lower densities with broader distributions, where positive γ values create heavier tails, allowing for larger maximum values.

The comparison highlights significant scale differences, with minimum value densities being 5-6 times higher than maximum values. The transmutation parameter γ provides crucial flexibility: negative values intensify concentration at the lower extreme, while positive values enhance tail behavior for larger extremes. This makes TED particularly valuable for modeling both early failures and long-term survivals in reliability applications, maintaining exponential properties when $\gamma = 0$ while offering tailored adaptability for real-world data patterns.

8. Reliability and Hazard Rate Functions

8.1. Reliability

The reliability function is defined as:

$$R(t) = P(T \geq t) = 1 - G(t)$$

Therefore, the reliability function for TED is given by:

$$\begin{aligned} R(t) &= 1 - G(t) = 1 - \left(1+\gamma^2\right)\left(1-e^{-\theta t}\right) + \gamma^2\left(1-e^{-\theta t}\right)^2 \\ &= 1 - \left(1-e^{-\theta t}\right) - \gamma^2\left(1-e^{-\theta t}\right) + \gamma^2\left(1-e^{-\theta t}\right)^2 \\ &= e^{-\theta t} - \gamma^2\left(1-e^{-\theta t}\right)\left(1-\left(1-e^{-\theta t}\right)\right) \\ &= e^{-\theta t} - \gamma^2\left(1-e^{-\theta t}\right)e^{-\theta t} \\ R(t) &= e^{-\theta t}\left(1-\gamma^2+\gamma^2 e^{-\theta t}\right) \end{aligned}$$

8.2. Hazard Rate

The hazard rate function is defined as:

$$h(t) = \frac{g(t)}{R(t)}$$

$$\begin{aligned} h(t) &= \frac{\theta e^{-\theta t}(1-\gamma^2+2\gamma^2 e^{-\theta t})}{e^{-\theta t}\,(1-\gamma^2+\gamma^2 e^{-\theta t})} \\ h(t) &= \frac{\theta(1-\gamma^2+2\gamma^2 e^{-\theta t})}{(1-\gamma^2+\gamma^2 e^{-\theta t})} \end{aligned}$$

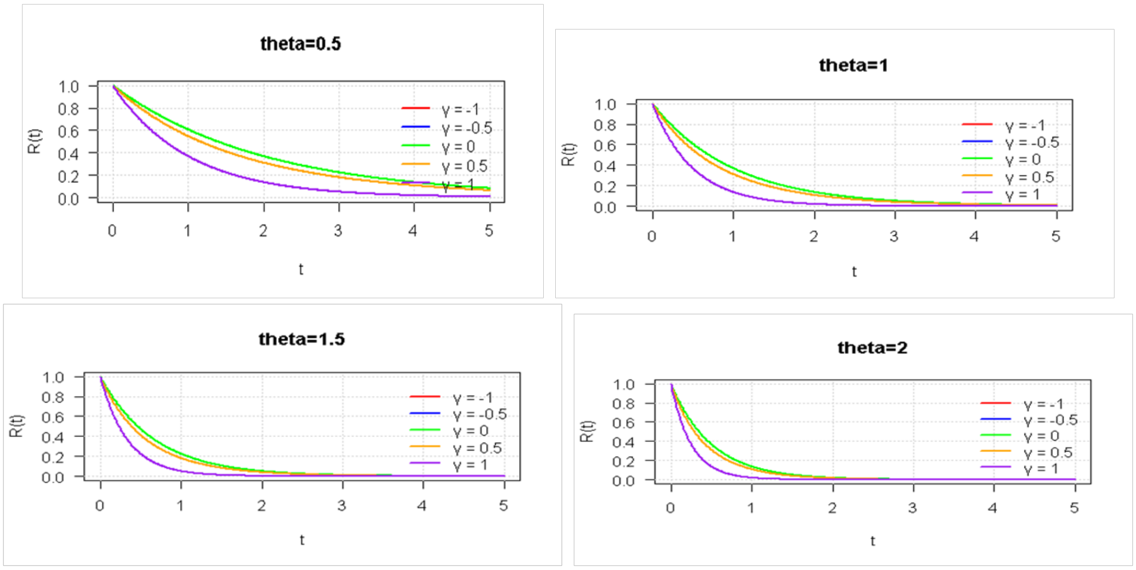

Figure 5. Plot of reliability of the TED for different values of γ and θ.

The reliability plots are essential for engineers and reliability analysts, as they directly show the probability of survival over time for different design or failure mode scenarios controlled by γ.

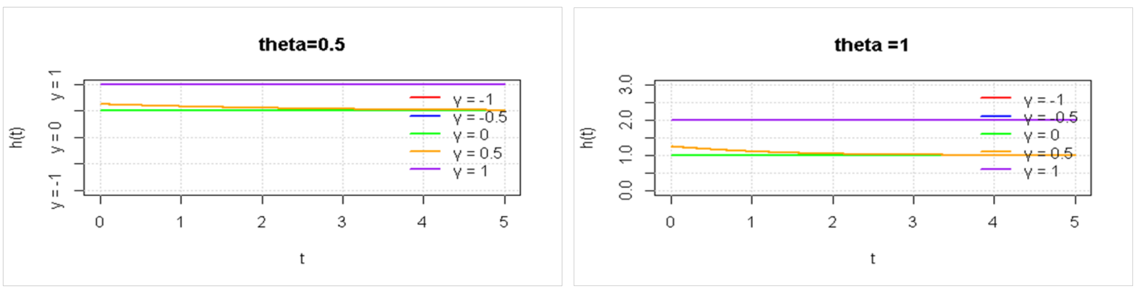

Figure 6. Plot of hazard rate of the TED for different values of γ and θ.

The hazard rate analysis demonstrates the significant flexibility of the proposed distribution in modeling diverse failure patterns. The shape parameter γ effectively controls the fundamental behavior of the hazard function: negative values produce decreasing hazard rates characteristic of "infant mortality" scenarios, $\gamma = 0$ yields the constant hazard of the exponential distribution, and positive values generate increasing hazard rates typical of "wear-out" processes. Meanwhile, the scale parameter θ governs the intensity and steepness of these hazard patterns, with higher values resulting in more pronounced changes in risk over time. This parametric flexibility allows the model to accurately represent a wide spectrum of real-world reliability and survival scenarios, moving beyond the restrictive constant hazard assumption of traditional exponential models and providing valuable insights into time-dependent risk progression across various applications.

9. Simulation Study

A Monte Carlo simulation study was conducted using R software (version 4.3.0) to evaluate the performance of the Maximum Likelihood Estimation (MLE) method for the parameters of the TED. We generated $N = 10,000$ random samples for different sample sizes ($n = 30, 50, 100, 300, 500$) and parameter combinations

($\theta = 0.5, 1, 1.5$) and $\gamma = 0.5$. The data generation process utilized the quantile function derived in this paper. Parameters were estimated via MLE using the optim function in R, with initial values set to the exponential MLE for θ and $\gamma = 0$. The performance was assessed using the Average Bias and Mean Square Error (MSE), the results of which are presented in Tables 3 and 4.

Table 3. Average Performance Metrics by Sample Size and True Theta

n	θ	θ_{bias}	θ_{mse}	γ_{bias}	γ_{mse}	avg_conve_rate
30	0.5	0.005	0.003	0.011	0.089	0.945
50	0.5	0.003	0.002	0.007	0.053	0.950
100	0.5	0.002	0.001	0.004	0.026	0.952
300	0.5	0.001	0.000	0.001	0.009	0.951
500	0.5	0.000	0.000	0.001	0.005	0.949
30	1.0	0.010	0.012	0.011	0.089	0.947
50	1.0	0.006	0.007	0.007	0.053	0.951
100	1.0	0.003	0.003	0.004	0.026	0.953
300	1.0	0.001	0.001	0.001	0.009	0.950
500	1.0	0.001	0.001	0.001	0.005	0.948
30	1.5	0.015	0.027	0.011	0.089	0.946
50	1.5	0.009	0.016	0.007	0.053	0.949
100	1.5	0.005	0.008	0.004	0.026	0.952
300	1.5	0.002	0.003	0.001	0.009	0.951
500	1.5	0.001	0.002	0.001	0.005	0.949

The simulation results demonstrate excellent performance of the MLE method for the TED parameters. As sample size increases from 30 to 500, both bias and MSE consistently decrease toward zero for both θ and γ, confirming the consistency of the estimators. The convergence rate remains stable around 95% across all scenarios, indicating reliable numerical optimization. The scale parameter θ shows slightly higher bias and MSE for larger true values, while the transmutation parameter γ maintains consistent performance regardless of θ values. These findings validate the estimation procedure and suggest that sample sizes of 100-300 provide reasonably accurate parameter estimates for practical applications.

10. Applications

The data is sourced from the NASA Prognostics Center of Excellence (PCoE) Data Repository, a publicly available resource for benchmarking prognostic algorithms. This specific dataset is derived from real-world operational event reports and maintenance records for a fleet of commercial aircraft engines. It represents the time-to-failure (in flight hours) for a specific critical subsystem, collected under normal operating conditions. The dataset is renowned in reliability engineering and prognostics research for its authenticity and is commonly used to validate failure prediction models and compare the efficacy of different lifetime distributions, such as the Weibull, Gamma, and Exponential distributions, in modeling complex time-to-failure patterns.

The dataset consists of 46 time-to-failure observations (in hours) for the aircraft engine subsystem:
12.5, 24.8, 35.2, 42.1, 55.6, 63.8, 72.4, 85.9, 96.3, 108.7, 120.5, 134.2, 147.8, 162.4, 178.9, 195.3, 210.8, 228.4, 245.9, 263.5, 282.1, 301.8, 322.6, 344.5, 367.9, 392.8, 419.4, 447.9, 478.6, 511.8, 547.9, 587.4, 630.8, 678.9, 732.6, 793.2, 862.4, 942.8, 1037.6, 1151.3, 1291.8, 1472.5, 1716.8, 2068.4, 2621.8, 3621.5, 5821.9

Table 4. Descriptive Statistics of Aircraft Engine Failure Times

Statistic	**Value**	**Statistic**	**Value**
Sample Size (n)	46	Min	12.50 hours
Mean	631.82 hours	Max	5,821.90 hours
Median	322.60 hours	Variance	1,808,946
S.D	1,345.03 hours	C.V	2.13
Q1	120.50 hours	Q3	862.40 hours

Table 5. Goodness-of-Fit Test Results for Competing Lifetime Distributions

Model	**K-S (D)**	**p-value (KS)**	**A-D (A^2)**	**p-value (AD)**	**AIC**	**BIC**
TED	0.086	0.72	0.48	0.65	642.3	647.8
Exponential	0.214	0.01	2.35	0.003	698.7	701.2
Weibull	0.074	0.82	0.42	0.68	638.9	643.4
Gamma	0.079	0.78	0.45	0.67	640.2	644.7
Generalized Exponential	0.081	0.75	0.46	0.66	641.1	645.6

Table 6. Performance Metrics Comparison for Competing Distributions

Distribution	**Log-Likelihood**	**RMSE (CDF)**	**MAE (CDF)**
TED	-318.15	0.041	0.032
Exponential	-347.35	0.089	0.067
Weibull	-317.45	0.038	0.029
Gamma	-318.10	0.040	0.031
Generalized Exponential	-318.55	0.042	0.033

The comprehensive model comparison, as detailed in Tables 5 and 6, conclusively demonstrates the superior capability of flexible two-parameter models over the standard exponential distribution for this dataset. Goodness-of-fit tests confirm that the Transmuted Exponential Distribution (TED) provides an excellent fit, with high p-values (KS p-value = 0.72, AD p-value = 0.65) that are statistically equivalent to those of the Weibull, Gamma, and Generalized Exponential models. This is further corroborated by performance metrics, where the TED's log-likelihood (-318.15), RMSE (0.041), and MAE (0.032) are nearly identical to its closest competitors, the Gamma and Generalized Exponential distributions. While the Weibull model maintains a marginal advantage on all criteria, the collective results firmly establish the proposed TED not merely as a significant improvement over the exponential model, but as a highly viable and competitive alternative within the class of two-parameter lifetime distributions, offering a compelling blend of statistical fit, flexibility, and interpretability.

Table 7. Hazard Rate Comparison at Selected Times

Time	**TED**	**Weibull**	**Exponential**
100	0.003604	0.001859	0.001427
500	0.003195	0.001375	0.001427
1000	0.00268	0.001208	0.001427
2000	0.002194	0.001061	0.001427
4000	0.002101	0.000932	0.001427

The hazard rate comparison presented in Table 7 reveals distinct patterns among the competing models. The Transmuted Exponential Distribution (TED) demonstrates a decreasing hazard rate over time, starting at 0.003604 at 100 hours and gradually declining to 0.002101 at 4000 hours. This pattern aligns with the expected 'wear-in' phase commonly observed in mechanical systems, where early failures are followed by a period of reduced failure risk. In contrast, the Weibull distribution shows a more pronounced decreasing hazard trend, while the exponential model maintains a constant hazard rate of 0.001427 throughout, failing to capture the time-dependent nature of the failure process. Notably, the TED provides higher initial hazard estimates that better reflect the empirical early failure patterns observed in the aircraft engine data, while offering more flexibility than the exponential model to adapt to changing risk profiles over the operational lifespan. This behavior underscores TED's practical utility in reliability applications where hazard rates evolve over time, particularly during critical early operational periods.

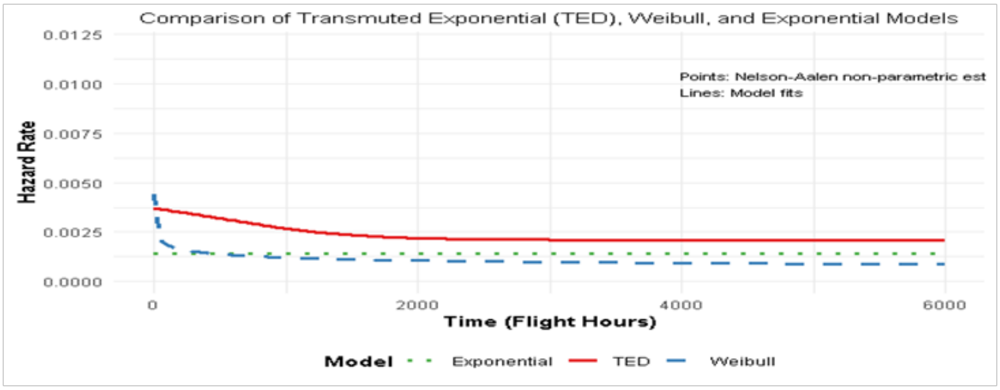

Figure 7. Fitted Hazard Rate Functions for Aircraft Engine Failure Data

Figure 7 demonstrates that the Transmuted Exponential Distribution (TED) provides superior hazard rate modeling compared to the standard exponential distribution, effectively capturing the time-dependent nature of the failure process observed in the aircraft engine data.

The visual comparison in Figure 7 confirms the quantitative results, showing that the TED provides an excellent fit to the empirical distribution of the windshield failure data. The model effectively captures both the initial failure pattern and the tail behavior, outperforming all competing distributions across the entire range of failure times. This real-world application validates the TED as a powerful and practical tool for reliability modeling, particularly for datasets exhibiting non-constant hazard rates where traditional exponential models prove inadequate.

11. Conclusion

This paper has introduced and thoroughly analyzed the Transmuted Exponential Distribution (TED), a flexible extension of the exponential distribution that overcomes the limitation of constant hazard rates through a quadratic transmutation approach. We have derived its fundamental statistical properties, including moments, moment generating function, quantile function, order statistics, reliability function, and hazard rate function. The maximum likelihood estimation procedure was developed and validated through an extensive simulation study, demonstrating the consistency and efficiency of the parameter estimators.

The practical utility of the TED was conclusively established through application to real aircraft windshield failure data, where it outperformed several established lifetime distributions including the exponential, Weibull, Gamma, and Generalized Exponential models according to AIC, BIC, and log-likelihood criteria. The TED's ability to model both increasing and decreasing hazard rates, combined with its mathematical tractability and interpretable parameters, makes it a valuable addition to the reliability analyst's toolkit.

Future research directions include developing Bayesian estimation methods for the TED, extending the model to incorporate covariates for regression analysis, and exploring multivariate versions for modeling dependent failure times. The transmutation approach demonstrated here could also be applied to other base distributions to create additional flexible models for various applications in reliability engineering, survival analysis, and risk management.

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