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Using Wavelet Estimation of the Weighted Exponential Regression Model

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Abstract This paper aims to estimate the parameters of the weighted exponential regression model based on the methods of Maximum Likelihood, Jackknife, Wavelet, and Modified Jackknife method by Haar matrix to predict monthly mortality rates for leukemia patients in Iraq. We compared these methods by collecting samples from July 1, 2020, to February 18, 2023. Simulation was also used to generate three random samples with sizes of 8, 16, and 32 to represent the approved data using MATLAB program. The results indicated the following: The Modified Jackknife method by Haar matrix showed the smallest Mean Absolute Percentage Error compared to Maximum Likelihood, Jackknife, and Wavelet methods for the weighted exponential regression model. The results also showed the stability of the Modified Jackknife method by Haar matrix and the Wavelet method, whether by increasing or decreasing the sample size. That is, the specificity of both methods is not affected by the sample size. Regarding the real data results, the mortality rate from leukemia was forecast for eight months, which showed very low mortality rates that were close to zero.

Keywords Weighted exponential regression, Wavelet, Jackknife, Harr matrix, Growth curve, Leukemia

AMS 2010 subject classifications 26A25, 26A35

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1. Introduction

Over the past three decades, wavelet theory has emerged as one of the most powerful and versatile tools for signal and data analysis. Originally developed in the context of applied mathematics and electrical engineering, wavelets provide a flexible framework for representing data at multiple levels of resolution. Unlike classical Fourier analysis-which decomposes a signal into infinite-duration sinusoidal components-wavelet analysis uses localized basis functions that can simultaneously capture both time (or spatial) and frequency information. This dual localization makes wavelets particularly effective for analyzing nonstationary or transient data, which are common in real-world scientific and statistical problems [1].

At its core, the wavelet transform involves decomposing a signal or dataset into a family of functions generated through dilation (scaling) and translation (shifting) of a prototype function known as the mother wavelet. The resulting coefficients describe the signal's behavior across different frequency bands and time intervals, enabling researchers to identify localized patterns, abrupt changes, and hidden structures that may be invisible to traditional methods. This property has made wavelets a key part of multiresolution analysis (MRA) because they naturally connect the time and frequency domains.

From a statistical perspective, wavelets have demonstrated substantial advantages in noise reduction, data compression, feature extraction, and parameter estimation. In regression and time series analysis, wavelet-based

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models can efficiently handle heterogeneous, irregular, or high-dimensional data by adapting to local features of the signal. The wavelet shrinkage and thresholding techniques introduced by Donoho and Johnstone (1995) revolutionized nonparametric function estimation by offering near-optimal denoising performance under minimal assumptions. Recently, wavelets have also been integrated into machine learning and deep neural architectures, particularly convolutional neural networks (CNNs) and wavelet scattering transforms, to enhance the robustness and interpretability of feature extraction [2].

In applied sciences, wavelet methods have become indispensable across a broad spectrum of disciplines: In biostatistics and epidemiology, wavelet regression and decomposition are used to model temporal trends in mortality, disease incidence, and environmental exposure data. In engineering and physics, wavelet transforms aid in vibration analysis, fault detection, and signal compression. In image processing, discrete wavelet transforms (DWT) underpin modern standards such as JPEG2000, providing efficient multi-scale image representation. Also, in finance and economics, wavelets are employed to analyze volatility clustering and multiscale dependencies in financial time series. As well as in climate and environmental sciences, wavelet coherence and spectral analysis allow for the detection of nonstationary relationships between climatic variables.

The integration of wavelet theory with modern statistical and computational methods has led to the development of hybrid frameworks, such as wavelet-based regression, wavelet neural networks, and wavelet-enhanced resampling techniques (e.g., Jackknife-Wavelet combinations). These hybrid models combine the interpretability of classical statistical inference with the adaptability of modern machine learning, enabling improved forecasting accuracy and more robust estimation of complex nonlinear systems [3].

In summary, wavelet analysis provides a unifying mathematical language for exploring structure in data across scales. Its strength lies in its ability to adaptively decompose signals, denoise information, and reveal patterns hidden across time and frequency domains. As data in contemporary research become increasingly complex, high-dimensional, and dynamic, the wavelet framework continues to evolve, bridging the gap between theory and application across diverse scientific fields. However, when the model dimension is high, or the sample size is small, or the parameter estimates are highly correlated, the resulting covariance matrix may become ill-conditioned or nearly singular - meaning its inverse is unstable. By incorporating the Haar matrix (a simple orthonormal wavelet transform), we can transform the parameter space into another domain where the signal is separated into coarse (low-frequency) and detail (high-frequency) components. Then, by smoothing or thresholding the high-frequency parts and transforming back, we effectively fill the "gap" (reduce singularity) and obtain a well-conditioned, stable covariance matrix [4].

The Jackknife method was also used, which is one of the modern statistical tools that contributed to improving the accuracy of estimation and modeling efficiency in complex data It is mainly used for re-estimation, reducing bias, and estimating variance through re-sampling, making it a powerful tool in cases where the sample size is small or the model is complex [5].

The following is a review of some previous studies on the subject of Jackknife: Lipsitz and Parzen (1996) discussed how Jackknife can be used to estimate variance in the Cox model when the survival data are correlated rather than as if the samples are independent. The researchers concluded that a "one-step" Jackknife estimator of variance is asymptotically equivalent to the variance estimator for Wei et al. (1989) and Lee et al. (1992) and proposed a robust variance estimate that is consistent for the asymptotic variance [6, 7, 8]. Klein and Andersen (2005) used Jackknife to generate pseudo-values from the cumulative incidence function of competitive events, and then these values are used in generalized estimating equations to model factors affecting mortality in the context of bone marrow transplantation. They study the properties of this estimator and apply the technique to a study of the effect of alternative donors on relapse for patients given a bone marrow transplant for leukemia [9]. Galaska et al. (2008) applied the Jackknife procedure to estimate error bars for ensemble wavelet spectra derived from the WTMM method. By successively leaving out one observation or one segment, the authors evaluated the stability of their wavelet-based estimates. This demonstrates how Jackknife resampling can be used to quantify variability in wavelet-transform outputs. The results showed that the MDFA seems to be more sensitive as compared with the WTMM method in differentiating between multifractal properties of the heart rate in healthy subjects and patients with left ventricular systolic dysfunction [10]. Li, Wu, and Tu (2012) propose modification to use "random weighting" to improve the approximate distribution of capabilities in the Jackknife with Relative Risk Model (Cox

PH), especially when there is a large truncation or critical data. The distribution of the highest partial likelihood estimate for the regression coefficient in the Cox model was to be approximated using the random weighting approach. Simulation experiments revealed that this approach was less susceptible to severe censoring than the bootstrap method, but it might not be as second-order accurate as the bootstrap approximation [11]. Parner et al. (2023) models time-to-event data with ”censoring” and ”left-truncation” using Jackknife-generated ”pseudo-observations” and with the so-called ”infinitesimal jackknife” to speed up computation. They present a modification of the infinitesimal jackknife pseudo-observations that provide unbiased estimates in a left-truncated cohort [12].

The following is a review of some previous studies on the subject of wavelet: Konar and Chattopadhyay (2015) employed a genetic algorithm and a wavelet transformation technique to diagnose induction motor failures. The vibration signal was utilized and transformed into the time-frequency domain to acquire the distinctive attributes of the malfunction and investigate the potential for an automated fault classification procedure [13]. Benkedjouh et. al. (2018) adopted a new method for monitoring the wear state of the tool based on continuous wavelet transform (CWT) and blind source separation (BSS), which is one of the most powerful signal processing methods. Prognostics and Health Management (PHM) of status monitoring systems were proposed to predict errors and estimate the remaining useful life of the components. The results showed that the proposed method, CWT-BSS, can effectively reflect the deterioration of the performance of cutting tools for the milling process [14]. Hsueh et. al. (2019) presented the functioning state of a three-phase induction motor is detected and its status as a defective or healthy motor is classified using a unique approach. Five different types of induction motor faults and one typical operating state are recorded using electrical current signal data. The empirical wavelet transform is used in the methodology’s first section to show a pattern recognition technique that converts the raw current signal into two-dimensional (2-D) grayscale pictures that include information about the defects. It is suggested to use a deep CNN (Convolutional Neural Network) model to automatically extract reliable characteristics from the grayscale photos in order to identify induction motor problems. The outcomes of the tests demonstrate that the suggested approach achieves competitive accuracy in the failure diagnosis of the induction motors and that it performed better than other deep learning techniques and conventional statistical approaches [15]. Hamilton et. al. (2022) state that numerous methods have been used to analyse self-similarity statistically, and many of them are based on multiscale ideas. Real data traces, which are frequently characterized by significant temporal or geographic mean level movements, missing values, or extreme observations, frequently break basic distributional assumptions that are relied upon by most. For estimating self-similarity in two-dimensional data, a brand-new, reliable method based on weighted Theil regression is suggested (images). The approach is contrasted with two well-known wavelet-based estimate techniques: the Abry-Veitch (AV) bias-correcting estimator and ordinary least squares (OLS). As an example, the robust approach’s self-similarity estimate is used to classify digitized mammography pictures as malignant or non-cancerous, demonstrating the applicability of this estimate as a predictive feature. The diagnosis utilized in this instance is based on the characteristics of picture backdrops, which is normally a modality that is not used in breast cancer screening. Using a variety of wavelet bases and multiresolution levels, the classification results reveal an accuracy of almost 68% [16].

The current research completed what preceded it by employing the Wavelet method in estimating the weighted exponential regression model by comparing four methods (maximum Likelihood, Wavelet, Jackknife, and Modified Jackknife by Haar matrix) to estimate the two parameters of the model. The Jackknife method was modified by employing the Haar matrix, which is part of the Wavelet method, to form the proposed method. To achieve this, the study reviewed the weighted exponential regression model with the four methods and then used the simulation method to compare the estimates of those methods. It also devoted a special section to applying the findings of the research to study mortality rates due to leukemia in Iraq.

2. Materials and Methods

2.1. Weighted exponential regression model

Consider the weighted exponential regression model, which has a probability density function for the random variable x [17]:

$$y_i = \frac{\alpha_1 + 1}{\alpha_1} \alpha_2 e^{(-\alpha_2 x_i)} (1 - e^{-\alpha_1 \alpha_2 x_i}) + \varepsilon_i \tag{1}$$

Where x is the predictor variable, ε is the error term vector distributing normal with 0 mean and σ^2 variance, which is independently identically distributed and uncorrelated. Also, α_1 is the shape parameter ($\alpha_1 > 0$), and α_2 is the scale parameter ($\alpha_2 > 0$).

The transformed form of the exponential model was used in this research according to the following formula [18]:

$$\phi(x, \tau) = e^{-\alpha_1 x}(1 - e^{-\alpha_2 x}) + \varepsilon \tag{2}$$

substituting (2) yields:

$$Y(x) = \phi(x, \theta) + \varepsilon \tag{3}$$

Where $\theta = (\alpha_1, \alpha_2)$ and $\phi(x, \tau) = e^{-\alpha_1 x}(1 - e^{-\alpha_2 x}) + \varepsilon$ is the nonlinear weighted exponential regression function with respect to the parameters [18].

$\phi(x, \theta)$ is transformed into a linear form using Taylor's series, so that Model (3) appears as follows [19, 20]:

$$y_i^* = \sum_{j=1}^{2} x_{ij}^* \beta_j + \varepsilon_i \quad \forall \quad i = 1, 2, \cdots, n \tag{4}$$

where $Y_i^* = Y_i - \phi(x_i, \theta_0)$, $\beta_j = (\theta_j - \theta_{j0})$, $x_{ij}^* = \left(\frac{\partial \phi(x_i, \theta)}{\partial \theta_j} | \theta_j = \theta_{j0}\right)$, and θ_0 represent the initial values of θ.

Therefore, the linear model will be as follows:

$$y^* = x^* \beta + \varepsilon \tag{5}$$

Where (y^*) is a vector of rank $(n \times 1)$, (x^*) is a matrix of rank $(n \times 2)$, (β) is a vector of rank (2×1), and (ε) is a vector of rank $(n \times 1)$.

2.2. Maximum Likelihood method

First, we rely on the formula of the linear model (5) to estimate its parameters β according to the maximum likelihood method, where it is important and commonly used since it contains properties for a good estimate and has an invariant property [21]. Assuming initial values for the parameters of the weighted exponential model represented by (2). The estimation formula is as follows [22]:

$$\hat{\beta} = (x^{*\prime} x^*)^{-1} x^{*\prime} y^* \tag{6}$$

By adding this estimator to the initial value of the parameters in the weighted exponential regression model (θ_{10}), we obtain the estimator of the parameters $(\hat{\theta}_1)$. This process continues by adopting the value of $(\hat{\theta}_1)$ as the initial value and re-estimating the model (5) to obtain the second estimate $(\hat{\theta}_2)$.

This iterative estimation process continues until we reach convergent estimates of the parameters $(\hat{\theta})$, which represent the Maximum Likelihood estimator for the parameter of the weighted exponential regression model.

2.3. Wavelet method

The first to address wavelet was the mathematician Alfred Haar in 1909, who proposed Haar's wavelet. From research, it was concluded that waves are small waves whose amplitude starts at zero with a specific time period and zero mean, noting that there are several types of wavelets depending on the type of data in terms of being discrete or continuous.

The wavelet is generated from the scaling function $\phi(t)$, also known as the mother function, and the wavelet function $\psi(t)$, commonly referred to as the father function. These functions are expanded through scaling and translation processes applied to both $\phi(t)$ and $\psi(t)$, resulting in a set of two-dimensional functions representing the wavelet transform at advanced stages, as shown in the following formulas [23, 24].

There are many different basic functions in wavelet transformation, including the Haar wavelet, which is the simplest type of wavelet and takes the form of a step function. The Haar wavelet consists of two basic functions $\psi(t)$ and $\phi(t)$, which are given by the following formulas [25]:

$$\phi_{a,b}(t) = \frac{1}{\sqrt{a}}\phi\left(\frac{t-b}{a}\right) \tag{7}$$

$$\psi_{a,b}(t) = \frac{1}{\sqrt{a}}\psi\left(\frac{t-b}{a}\right) \tag{8}$$

Where: $a \in \Re_+$ represents the scaling parameter.
$b \in \Re$ represents the translation parameter.
$\psi_{a,b}(t)$ represents the detailed or wavelet function.
$\phi_{a,b}(t)$ approximation or scaling function.

There are many different basic functions in wavelet transformation, including the Haar wavelet, which is the simplest type of wavelet and takes the form of a step function.

According to Antoniadis (1997), the formula for the father function $\psi(t)$ and $\phi(t)$, which are given by the following formulas [25]:

$$\psi_{i,j}(t) = 2^{\frac{i}{2}}\psi(2^i t - j) \tag{9}$$

$$\phi_{i,j}(t) = 2^{i/2}\phi(2^i t - j), i = 0, \cdots, i\text{–}1 \quad and \quad j = 0, \cdots, 2^i\text{–}1 \tag{10}$$

where:

$$\psi(t) = \begin{cases} +1 & 0 \le t < \frac{1}{2} \\ -1 & \frac{1}{2} < t \le 1 \\ 0 & O.W. \end{cases} \tag{11}$$

$$\phi(t) = \begin{cases} 1 & 0 \le t \le 1 \\ 0 & O.W. \end{cases} \tag{12}$$

The discrete wavelet transform converts the original data or input vector from the spatial domain to the wavelet domain, producing an output vector of the same size. The discrete wavelet transform can be represented as a linear function and expressed as a $(n \times n)$ matrix.

The wavelet transform process is carried out in a number (J) of the stages using a process called Multi-Resolution Analysis (MRA), where the sample size must satisfy $(n = 2^J)$. The original data or signal is decomposed into two parts using filters, and the size is reduced (2^J) by a down sampling process of $(J-1)$. In the second stage, only the approximate coefficients resulting from the first (previous) stage are also decomposed into two parts using filters and the size is reduced, and so on with the remaining stages. The result of the process will include the detailed coefficients of the last stage and the last value of the approximate coefficients [3].

2.4. *Jackknife method*

The Jackknife is a resampling method whose aim is to estimate the bias and variance and produce an approach confidence interval for the parameters of different statistical estimators. It is called the delete-one Jackknife when a single observation is deleted, and the deleted Jackknife if multiple observations are deleted.

In this paper, the set of observations is the vectors $(z_1, z_2, \ldots, z_n)$ where $z_i = (y_i, x_i)'$ represent the value attached to $i^t h$ observation, and we will use the delete-one Jackknife algorithm for the regression. The Jackknife algorithm is set as [26]:
1. Draw a random sample of size n from the population and label the vector elements $z_i = (y_i, x_i)'$as the vector $y_i = (y_1, y_2, \ldots, y_n)'$ and the matrix $x_i = (x_1, x_2, \ldots, x_n)'$.
2. Delete the first row of the vector z_i and label $(n-1)$ from the remaining set of observations $y^{(1)} = (y_2, y_3, \ldots, y_n)'$, $x^{(1)} = (x_2, x_3, \ldots, x_n)'$ as delete-one Jackknife sample $z^{(1)}$ and estimate regression parameters $\hat{\theta}^1$ by MLE from $z^{(1)}$ which represent the parameters estimate of weighted exponential regression model, then delete the second row in the same way and estimate the regression coefficients $\hat{\theta}^2$ by MLE from $z^{(2)}$, so on by the same way, delete one case from set observations and estimate $\hat{\theta}^n$ by MLE from $z^{(n)}$, where $\hat{\theta}^n$ represent the regression parameters vector which are estimated by Jackknife after deleting the observation n from z [22].
3. Using the same iteration mechanism described in Section 2.2 to obtain the Jackknife estimate of the weighted exponential regression model parameters $\hat{\theta}$.
4. Calculate the regression coefficient of the Jackknife, which represents the arithmetic mean of $(\hat{\theta}^1, \hat{\theta}^2, \ldots, \hat{\theta}^n)$ as follow:

$$\bar{\hat{\theta}}_{(J)} = \frac{\sum_{j=1}^{n} \hat{\theta}^{(J)}}{n} \tag{13}$$

2.5. Modified Jackknife method by Haar matrix

The first step of this method is to estimate the parameters by using wavelet transform according to the regression model:

$$y^{\bullet} = \phi(x^{\bullet}, \theta) + \varepsilon^{\bullet} \tag{14}$$

Where $y^{\bullet} = wy$, $x^{\bullet} = wx$, θ are the vector parameters (α_1, α_2), $\varepsilon^{\bullet} = w\varepsilon$, and w is the Haar matrix [27].

The modified method's steps may be summarized as follows:
1. Draw a random sample of size n from the population and label the vector elements $d_i = (y_i^{\bullet}, x_i^{\bullet})'$ as the vector $y_i^{\bullet} = (y_1^{\bullet}, y_2^{\bullet}, \cdots, y_n^{\bullet})'$ and the matrix $x_{ij}^{\bullet} = (x_1^{\bullet}, x_2^{\bullet}, \cdots, x_n^{\bullet})'$.
2. Delete the first row of the vector d_i and label $(n-1)$ from the remaining set of observations $y^{\bullet^{(1)}} = (y_2, y_3, \cdots, y_n)'$, $x^{\bullet^{(1)}} = (x_2, x_3, \cdots, x_n)'$ as delete-one Jackknife sample $d^{(1)}$ and estimate regression parameters $\hat{\theta}^{(1)}$ by MLE from $d^{(1)}$ which represent the parameters estimate of weighted exponential regression model, then delete the second row in the same way and estimate the regression coefficients $\hat{\theta}^{(2)}$ by MLE from $d^{(2)}$, so on by the same way, delete one case from set observations and estimate $\hat{\theta}^n$ by MLE from $d^{(n)}$, where $\hat{\theta}^n$ represent the regression parameters vector which are estimated by the Modified Jackknife after deleting the observation n from d.
3. Using the same iteration mechanism described in Section 2.2 to obtain the Modified Jackknife estimator of the weighted exponential regression model parameters $(\hat{\theta})$.
4. Calculate the regression coefficient of the Jackknife, which represents the arithmetic mean of $(\hat{\theta}^1, \hat{\theta}^2, \cdots, \hat{\theta}^n,)$ as follow [28]:
$\bar{\hat{\theta}}_{(MJ)} = \frac{\sum_{j=1}^{n} \hat{\theta}^{(J)}}{n}$

3. Simulation Study

Three random samples of size (8, 16, and 32) were generated to represent the data approved for comparison according to the initial values of the parameters of the weighted exponential regression model shown in (2) and shown in Table 1, and under the assumption of the presence of a random error term distributed normally with a zero mean and variance (0.5), as data on monthly death rates from various leukemias in Iraq were simulated by using MATLAB with repeated for 500 simulation runs.

Table 1. Initial values for the weighted exponential regression model

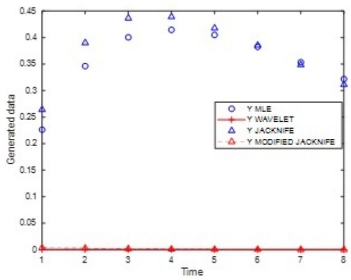
n	Parameters	
	α_1	α_2
8	0.3	0.05
16	0.2	0.03
32	0.1	0.01

The research adopted the criterion of the Mean Absolute Percentage Error (MAPE) to compare the methods used to estimate the model parameter, and the result appears in the following Table 2.

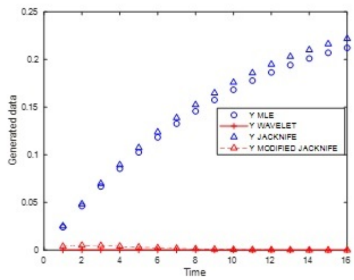
Table 2. Comparison of simulation results according to the MAPE criterion

n	MLE	Wavelet	Jackknife	Modified Jackknife
8	99.92	99.96	104.65	99.466
16	202.59	100	210.78	98.021
32	247.8	100	249.25	99.916

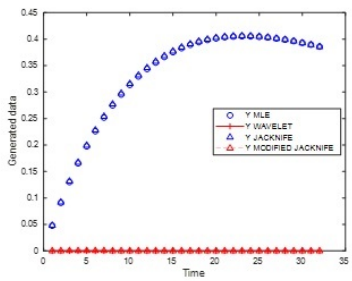
Based on what was presented in Table 2, it is clear that the results showed preference for the Modified Jackknife method by Haar matrix, as it gave the lowest MAPE value for all sample sizes. The wavelet method also gave the second-best results. The results also showed the stability of the Modified Jackknife method by Haar matrix and Wavelet method depending on the MAPE values, whether by increasing or decreasing the size of series, which gives the two methods specificity by not being affected by the sample size.



(a) Size of series 8



(b) Size of series 16



(c) Size of series 32

Figure 1. Predicting monthly infection ratios depending on the generation data.

Figure 1 shows the prediction results for eight values according to the generated time series, which show a clear decrease if the Modified Jackknife method by Haar matrix is used, while the results of the other methods came close to zero, which gives a clear indication of the accuracy and realism of the Modified Jackknife method by Haar matrix.

4. Real Data and Implementation

Table 3. Confirmed monthly death ratios of Leukemia diseases in Iraq for July 2020 to February 2023

Month	Monthly death rates
07/2020	0.050
08/2020	0.056
09/2020	0.036
10/2020	0.045
11/2020	0.038
12/2020	0.029
01/2021	0.038
02/2021	0.056
03/2021	0.040
04/2021	0.028
05/2021	0.038
06/2021	0.032
07/2021	0.026
08/2021	0.026
09/2021	0.038
10/2021	0.042
11/2021	0.033
12/2021	0.036
01/2022	0.033
02/2022	0.045
03/2022	0.027
04/2022	0.037
05/2022	0.027
06/2022	0.040
07/2022	0.042
08/2022	0.029
09/2022	0.034
10/2022	0.042
11/2022	0.029
12/2022	0.042
01/2023	0.043
02/2023	0.077

Source: Iraqi Ministry of Health, 2023: http://moh.gov.iq

The data issued by the Iraqi Ministry of Health regarding the number of daily deaths due to leukemia in Iraq was adopted for the period from July 1, 2020, until February 18, 2023, and then the monthly death rates were calculated by taking the daily mortality rates for the specified month and then calculating the monthly average of those rates as shown in Table 3 to be adopted as real data for the purpose of processing according to the regression model.

The parameters of the weighted exponential regression model were also estimated by adopting the monthly death from leukemia shown in Table 3 as a dependent variable and assuming initial values for the two model parameters $(\alpha_1 = 0.05, \alpha_2 = 0.02)$.

Returning to the simulation results, the two best methods were adopted to find the results, as shown in Table 4.

Table 4. Estimation the parameters of weighted exponential regression model depend on the monthly ratios of Leukemia diseases

Method	α_1	α_2
Wavelet	0.0307	0.052
Modified Jackknife	0.0001	0.0004

As shown in Table 5, through the use of the $MAPE$ criterion, it can be seen that the Modified Jackknife method by Haar matrix was the best compared to the Wavelet method, as it gave the lowest MAPE of 0.052.

Table 5. MAPE of weighted exponential regression model depend on the monthly ratios of Leukemia diseases

Method	**MAPE**
Wavelet	97.549
Modified Jackknife	93.461

The current study also predicted the death rate as a result of leukemia for the following eight months, as shown in Table 6. The results showed very low death rates that reached 0.0022, which indicates good follow-up and treatment for patients.

Table 6. Prediction the monthly death ratios of Leukemia diseases

Month	**Monthly death ratios of Leukemia diseases**	
	Wavelet	**Modified Jackknife**
1	0.0012	0.0024
2	0.0012	0.0023
3	0.0012	0.0023
4	0.0012	0.0022
5	0.0012	0.0021
6	0.0012	0.0021
7	0.0012	0.0020
8	0.0012	0.0020
Mean	**0.0012**	**0.0022**

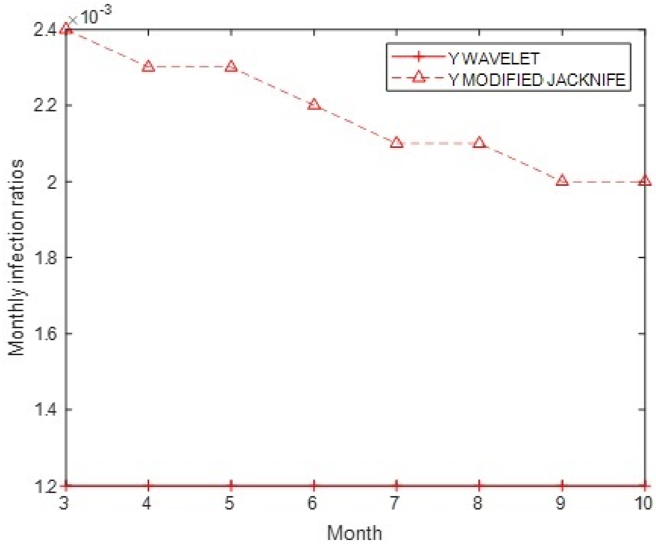


Figure 2. Predicting monthly ratios of Leukemia diseases depending on the real data.

From Figure 2, it can be seen that the death rates are very low based on the Modified Jackknife method by Haar matrix, and they are close to zero.

5. Conclusion

The simulation study indicated that the Modified Jackknife by Haar matrix showed the smallest Mean Absolute Percentage Error compared to Maximum Likelihood, Jackknife, and Wavelet methods for the weighted exponential regression model. The results also showed the stability of the Modified Jackknife method by Haar matrix and the Wavelet method, whether by increasing or decreasing the sample size. That is, the specificity of both methods is not affected by the sample size. Also, as for the real data results, the mortality rate from leukemia was forecasted for eight months according to the best method (Modified Jackknife method by Haar matrix), which showed very low mortality rates that were close to zero and indicates that patients receive appropriate monitoring and follow-up.

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