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The sensitivity analysis of multistate pension projections based on a vec-permutation approach

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Abstract The cohort component method in the context of pension projections can be translated into a multistate matrix model, in which beneficiaries of a pension scheme are classified jointly by their age and status (active contributor, invalid, retiree, widows/widowers), using the vec permutation matrix. The projection results depend on the mortality, retirement, disability, marital percentage and remarriage rates as well as the number of new entrants into the scheme on which the projections are based. Any change to a parameter will result in a corresponding change in the projection outcomes. Our objective is to systematically examine the relationships between various key projection outcomes—such as status-specific population sizes, dependency ratio, total cash flows, and PAYG cost rate—and the underlying age- and sex-specific projection parameters. To achieve this, we present the set of equations required to perform sensitivity and elasticity (i.e., proportional sensitivity) analyses of multistate projections, utilizing matrix calculus. We apply our methodology to a projection of the Moroccan pension system, which estimates population and cash flows disaggregated by age, sex, and status over the period 2020 to 2080.

Keywords Cohort component method, Pension projections, Multi-state model, vec-permutation matrix, Sensitivity analysis, Matrix calculus.

JEL classification C60, C61, C63, D61, g22, H55, J26, J32.

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1. Introduction

Half a century ago, in the journal Demography, Nathan Keyfitz introduced the concept of ”population projection as a matrix operator” [24]. He demonstrated that population projections based on the cohort-component method could be expressed through matrix models. Keyfitz [24] highlighted the importance of this approach, as it drew attention to the mathematical framework underlying the projections, encouraging more in-depth analysis of their properties using advanced mathematical tools. In the field of social security, pension actuaries commonly execute demographic and financial projections via computer algorithms, the specifics of which may be complex and not fully transparent, yet they enable nearly infinite adjustments to the relationships. Still, the advantages of treating projections as matrix operators continue to hold substantial value. In this paper, we continue in this tradition, applying matrix calculus techniques to conduct a comprehensive perturbation analysis of pension projections.

The first attempt to represent pension projections as a matrix operator dates back to the work of Crescentini and Spandonaro [16]. He showed that the demographic projection procedure can be regarded as the iteration of a matrix multiplication operation. This process, however, does not operate at the level of overall totals within the sub-populations of the pension schemes (active, invalids, retirees, widows/widowers). Each sub-population is

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subdivided at least by sex and age, and the matrix operation is repeated for each subdivision. In this paper, we consider a multi-state version of this model, which classify individuals jointly by their status(e.g., active, retired, disabled, widowed) and age. The model is based on the vec-permutation approach. This systematic method to the build and study of multi-state models, was introduced by Hunter and Caswell [21]. The approach to multistate models has been applied to models in which individuals are classified by stage and location [4, 18, 31, 37, 39], age and developmental stage [9, 34] , age and frailty [10], stage and infection status [25, 29] and combinations of stage and time or environmental state [8, 20, 32].

Multi-state projection results depend on a set of age-, sex- and time-specific rates as well as the rates of transition between states on which the projections are based. In our multi-state construct for pension projections, projection matrices are parameterized by schedules of mortality, retirement, disability, marital percentage and remarriage rates as well as new entrants into the scheme. If any parameter is changed, the projection results will differ. Understanding how projection parameters influence projection outcomes in the context of pension projections is essential for at least two key reasons :It makes it possible to determine which parameters have the greatest effect on the projection outcomes, and thus should be prioritized for careful modeling or estimation and It helps us better understand how the parameters we enter into projection software work together to shape the outcomes we care about [17].

Sensitivity and elasticity analyses are valuable techniques for systematically assessing how projection outcomes respond to individual parameters. Considering population projections of pensions as matrix operators makes possible the systematic calculation of the sensitivity of any population projection output to changes in any set of parameters, using matrix calculus, e.g.,[5–7, 13, 14]. In this paper, we derive the complete set of equations required to perform systematic sensitivity and elasticity analyses of multi-state pension projections, employing the vec-permutation matrix method. We illustrate the application and interpretive value of sensitivity and elasticity analyses in multi-state pension projections with an age-, sex-, and status-specific projection of the Moroccan Pension Scheme.

1.1. Notation

For reasons of consistency and clarity, we define here the notation and terminology used throughout the paper. Matrices are denoted by uppercase bold letters (e.g., $\mathbf{U}$), and vectors by lowercase bold letters (e.g., $\mathbf{n}$). Block-diagonal matrices and vectors that classify individuals simultaneously by age and status are denoted with a bar (e.g., $\bar{\mathbf{U}}$, $\bar{\mathbf{F}}$). The transpose of a vector $\mathbf{x}$ is written $\mathbf{x}^\top$. The unit vector $\mathbf{e}_i$ has a 1 in the i-th position and zeros elsewhere, $\mathbf{1}_s$ denotes a column vector of ones, and $\mathbf{I}_s$ is the $s \times s$ identity matrix. The Kronecker product is denoted by $\otimes$, and the Hadamard (elementwise) product by $\circ$. The vec operator stacks the columns of a matrix $\mathbf{X} \in \mathbb{R}^{m\times n}$ into a vector of size $mn \times 1$. The vec-permutation matrix $\mathbf{K}_{s,\omega}$ reorders vectors between "status-within-age" and "age-within-status" formats.

The main matrices used in the demographic and financial models are collected in Table 1.

Table 1. Summary of matrices used in the demographic and financial models.

Symbol	Expression	Size	Description
Matrix describing the full population			
$\bar{\mathbf{A}}$	$\bar{\mathbf{U}} + \bar{\mathbf{F}}$	$4\omega \times 4\omega$	Projection matrix for the full population
Matrices describing status survival			
$\bar{\mathbf{U}}$	$\mathbf{K}_{4,\omega}^{\top}\, \mathbb{D}\, \mathbf{K}_{4,\omega}\, \mathbb{U}$	$4\omega \times 4\omega$	Age–status survival and transition matrix
$\mathbb{U}$	$\sum_{i=1}^{\omega}(\mathbf{E}_{ii} \otimes \mathbf{U}_i)$	$4\omega \times 4\omega$	Block-diagonal survival matrix across ages
$\mathbf{U}_i$	$\mathbf{U}_i$	4×4	Survival matrix for age class i
$\mathbb{D}$	$\sum_{j=1}^{4}(\mathbf{E}_{jj} \otimes \mathbf{D}_j)$	$4\omega \times 4\omega$	Block-diagonal age-progression matrix
$\mathbf{D}_j$	$\mathbf{D}_j$	$\omega \times \omega$	Age-progression matrix for status j
Matrices describing status transitions			
$\bar{\mathbf{F}}$	$\mathbf{K}_{4,\omega}^{\top}\, \mathbb{H}\, \mathbf{K}_{4,\omega}\, \mathbb{F}$	$4\omega \times 4\omega$	Age–status flow matrix
$\mathbb{F}$	$\sum_{i=1}^{\omega}(\mathbf{E}_{ii} \otimes \mathbf{F}_i)$	$4\omega \times 4\omega$	Block-diagonal flow matrix across ages
$\mathbf{F}_i$	$\mathbf{F}_i$	4×4	Flow matrix for age class i
$\mathbb{H}$	$\sum_{j=1}^{4}(\mathbf{E}_{jj} \otimes \mathbf{H}_j)$	$4\omega \times 4\omega$	Block-diagonal flow-assignment matrix
$\mathbf{H}_j$	$\mathbf{H}_j$	$\omega \times \omega$	Flow-assignment matrix for status j
Matrices describing survival and economic valuation			
$\bar{\mathbf{U}}^{\ell}$	$\mathbf{K}_{4,\omega}^{\top}\, \mathbb{D}\, \mathbf{K}_{4,\omega}\, \mathbb{U}^{\ell}$	$4\omega \times 4\omega$	Survival matrix including economic valuation
$\mathbb{U}^{\ell}$	$\sum_{i=1}^{\omega}(\mathbf{E}_{ii} \otimes \mathbf{U}_i^{\ell})$	$4\omega \times 4\omega$	Block-diagonal financial survival matrix
$\mathbf{U}_i^{\ell}$	$\mathbf{U}_i \circ \pi_i^U$	4×4	Financial survival matrix for age class i
π_i^U	π_i^U	4×4	Salary and pension escalation factors
Matrices describing financial flows			
$\bar{\mathbf{F}}^{\ell}$	$\mathbf{K}_{4,\omega}^{\top}\, \mathbb{H}\, \mathbf{K}_{4,\omega}\, \mathbb{F}^{\ell}$	$4\omega \times 4\omega$	Flow matrix including economic valuation
$\mathbb{F}^{\ell}$	$\sum_{i=1}^{\omega}(\mathbf{E}_{ii} \otimes \mathbf{F}_i^{\ell})$	$4\omega \times 4\omega$	Block-diagonal financial flow matrix
$\mathbf{F}_i^{\ell}$	$\mathbf{F}_i \circ \pi_i^F$	4×4	Financial flow matrix for age class i
π_i^F	π_i^F	4×4	Pension benefit and reversion factors

2. The multistate pension model

2.1. The multistate population model

The multistate population model classifies members of a pension scheme jointly by:

- **Status of the member**, with four statuses:
  - Status 1: Active employee
  - Status 2: Disabled benefit recipients
  - Status 3: Retired benefit recipients
  - Status 4: Dead with a beneficiary "Widow/widower"
- **Age of the member**.

Each member of the pension scheme is classified according to both age and status, and the population at any moment can be described using this classification as

$$N = \begin{bmatrix} n_{0,1} & \cdots & n_{w-1,1} \\ n_{0,2} & \cdots & n_{w-1,1} \\ n_{0,3} & \cdots & n_{w-1,1} \\ n_{0,4} & \cdots & n_{w-1,1} \end{bmatrix} \tag{1}$$

The rows represent stages $(1, 2, 3, 4)$, and the columns represent age classes $(0, \cdots, w-1)$. The population vector $\bar{\mathbf{n}}$ is derived from the matrix N as follows.

$$\bar{\mathbf{n}} = vec(N) = \begin{bmatrix} n_{0,1} \\ n_{0,2} \\ n_{0,3} \\ n_{0,4} \\ - \\ \vdots \\ - \\ n_{w-1,1} \\ n_{w-1,2} \\ n_{w-1,3} \\ n_{w-1,4} \end{bmatrix} \tag{2}$$

In $\bar{\mathbf{n}}$, stages are organized within each age class. The vector can be rearranged so that age classes are instead grouped within stages,

$$vec(N^T) = \mathbf{K}_{4,w} vec(N) \tag{3}$$

where $\mathbf{K}_{4,w}$ is the vec-permutation matrix [19, 23]. This transformation plays a key role in analyzing the model, as will be demonstrated later. A detailed expression for $\mathbf{K}_{4,w}$ is provided in Box 1.

We consider a two-sex projection model, to account for differences between the two sexes in terms of mortality, retirement, invalidity, marriage, and remarriage. The male and female matrix models are constructed similarly and incorporate the corresponding sex-specific rates.

Note: The status 4 refers to widows in the male model, and to widowers in the female model.

We now introduce the matrices that characterize demographic rates dependent on age and status. Namely:

- $\mathbf{U}_i$: status survival matrix for age class i, where $i = 0, \ldots, w-1$
- $\mathbf{F}_i$: status-specific flow matrix for age class i, where $i = 0, \ldots, w-1$
- $\mathbf{D}_j$: age progression matrix for status j, where $j = 1, 2, 3, 4$
- $\mathbf{H}_j$: flows assignment matrix for status j, where $j = 1, 2, 3, 4$

The matrices $\mathbf{U}_i$ and $\mathbf{F}_i$ are of size 4×4, while the matrices $\mathbf{D}_j$ and $\mathbf{H}_j$ are of size $w \times w$.

The matrix $\mathbf{U}_i$ includes probabilities of remaining in the same status k, denoted by $p_i^{(kk)}$. The matrix $\mathbf{F}_i$ contains probabilities of transitioning from status k to status l, denoted $q_i^{(kl)}$. We give here the explicit formulas for these transition probabilities at age i(see [22] for more details):

$$\begin{aligned}
\text{Active to active:} \quad & p_i^{(aa)} = (1 - q_i^a)(1 - i_i^a)(1 - r_i) \\
\text{Active to retiree:} \quad & q_i^{(ar)} = (1 - q_i^a)(1 - i_i^a) r_{i+1} \\
\text{Active to invalid:} \quad & q_i^{(ai)} = i_i^a (1 - 0.5\, q_i^i) \\
\text{Active to widow/widower:} \quad & q_i^{(aw)} = q_i^a\, w_i \left[1 - 0.5(q_{y_i}^w + h_{y_i})\right] \\
\text{Retiree to retiree:} \quad & p_i^{(rr)} = 1 - q_i^r \\
\text{Retiree to widow/widower:} \quad & q_i^{(rw)} = q_i^r\, w_i \left[1 - 0.5(q_{y_i}^w + h_{y_i})\right] \\
\text{Invalid to invalid:} \quad & p_i^{(ii)} = 1 - q_i^i \\
\text{Invalid to widow/widower:} \quad & q_i^{(iw)} = q_i^i\, w_i \left[1 - 0.5(q_{y_i}^w + h_{y_i})\right] \\
\text{Widow/widower to widow/widower:} \quad & p_i^{(ww)} = 1 - q_i^w - h_i^w
\end{aligned}$$

where:

- $q_i^a, q_i^i, q_i^r, q_i^w$ are the mortality rates of actives, retirees, invalids, and widows/widowers respectively at age i,
- r_i is the retirement rate at age i,
- i_i^a is the invalidity rate for actives at age i,
- h_i is the remarriage rate at age i,
- w_i is the proportion married at age i,
- y_i is the mean age of the surviving spouse of a deceased person aged i.

Then we write

$$\mathbf{U}_i = \begin{bmatrix} p_i^{(aa)} & 0 & 0 & 0 \\ 0 & p_i^{(ii)} & 0 & 0 \\ 0 & 0 & p_i^{(rr)} & 0 \\ 0 & 0 & 0 & p_i^{(ww)} \end{bmatrix} \tag{4}$$

$$\mathbf{F}_i = \begin{bmatrix} 0 & 0 & 0 & 0 \\ q_i^{(ai)} & 0 & 0 & 0 \\ q_i^{(ar)} & 0 & 0 & 0 \\ q_i^{(aw)} & q_i^{(iw)} & q_i^{(rw)} & 0 \end{bmatrix} \tag{5}$$

The matrix $\mathbf{D}_j$ moves individuals in status j from one age class to the next. It is a $w \times w$ subdiagonal matrix of the form:

$$\mathbf{D}_j = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & 0 \end{bmatrix} \tag{6}$$

The matrix $\mathbf{H}_j$ assigns incoming flows to status j into the appropriate age class. For the statuses of invalids ($j = 2$) and retirees ($j = 3$), we define the assignment matrices as

$$\mathbf{H}_2(i+1, i) = \mathbf{H}_3(i+1, i) = 1 \quad \text{for} \quad i = 0, \ldots, x_r - 1, \tag{7}$$

where x_r denotes the mandatory retirement age.
Finally, we define the assignment matrix for the flows into status 4 (widows/widowers), based on the assumption of an average age y_i of the surviving spouse of a deceased individual of age i. Then:

$$\mathbf{H}_4(y_i, i) = 1 \quad \text{for } i = 0, \ldots, w-1 \tag{8}$$

To build the age-by-status model via the vec-permutation matrix approach, we begin by forming block-diagonal matrices $\mathbb{U}$, $\mathbb{F}$, $\mathbb{D}$, and $\mathbb{H}$. These matrices place $\mathbf{U}_i$, $\mathbf{F}_i$, $\mathbf{D}_j$, and $\mathbf{H}_j$, respectively, along their main diagonals. That is:

$$\mathbb{U} = \sum_{i=0}^{w-1} \mathbf{E}_{ii} \otimes \mathbf{U}_i \tag{9}$$

$$\mathbb{F} = \sum_{i=0}^{w-1} \mathbf{E}_{ii} \otimes \mathbf{F}_i \tag{10}$$

$$\mathbb{D} = \sum_{j=1}^{4} \mathbf{E}_{jj} \otimes \mathbf{D}_j \tag{11}$$

$$\mathbb{H} = \sum_{j=1}^{4} \mathbf{E}_{jj} \otimes \mathbf{H}_j \tag{12}$$

Here, $\mathbf{E}_{ii}$ is a $w \times w$ matrix with a 1 at position (i, i) and zeros elsewhere, and similarly for $\mathbf{E}_{jj}$ of dimension 4×4.

To illustrate the block-diagonal structure introduced in Equations (9)–(12), consider a simplified case with $w = 2$ age classes (ages 0 and 1). Let U_0 and U_1 be the 4×4 status survival matrices for age classes 0 and 1, respectively.

Define E_{00} and E_{11} as 2×2 matrices with a 1 at positions (0,0) and (1,1), respectively, and zeros elsewhere. Then, the Kronecker products become:

$$E_{00} \otimes U_0 = \begin{bmatrix} U_0 & 0 \\ 0 & 0 \end{bmatrix}, \qquad E_{11} \otimes U_1 = \begin{bmatrix} 0 & 0 \\ 0 & U_1 \end{bmatrix}$$

Adding these gives the block-diagonal matrix:

$$\mathbb{U} = (E_{00} \otimes U_0) + (E_{11} \otimes U_1) = \begin{bmatrix} U_0 & 0 \\ 0 & U_1 \end{bmatrix}$$

This block-diagonal structure allows the survival matrix $\mathbb{U}$ to apply age-specific status transitions independently within each age class. Similar constructions apply to the matrices $\mathbb{F}$, $\mathbb{D}$, and $\mathbb{H}$, using the same Kronecker-based approach.

Reordering Age-Status Vectors via $K_{4,w}$

To apply demographic operations such as age progression or assignment of flows correctly, it is essential to reorder the population vector $vec(N)$. Originally, the vector $vec(N)$ arranges entries by *status within age class*. However, some operations (such as block-diagonal age progression matrices) are defined by *status*, and expect the data to be organized as *age within each status*. This reordering is performed by the *vec-permutation matrix* $\mathbf{K}_{4,w}$, which rearranges the order of elements to match the expectations of the operators.

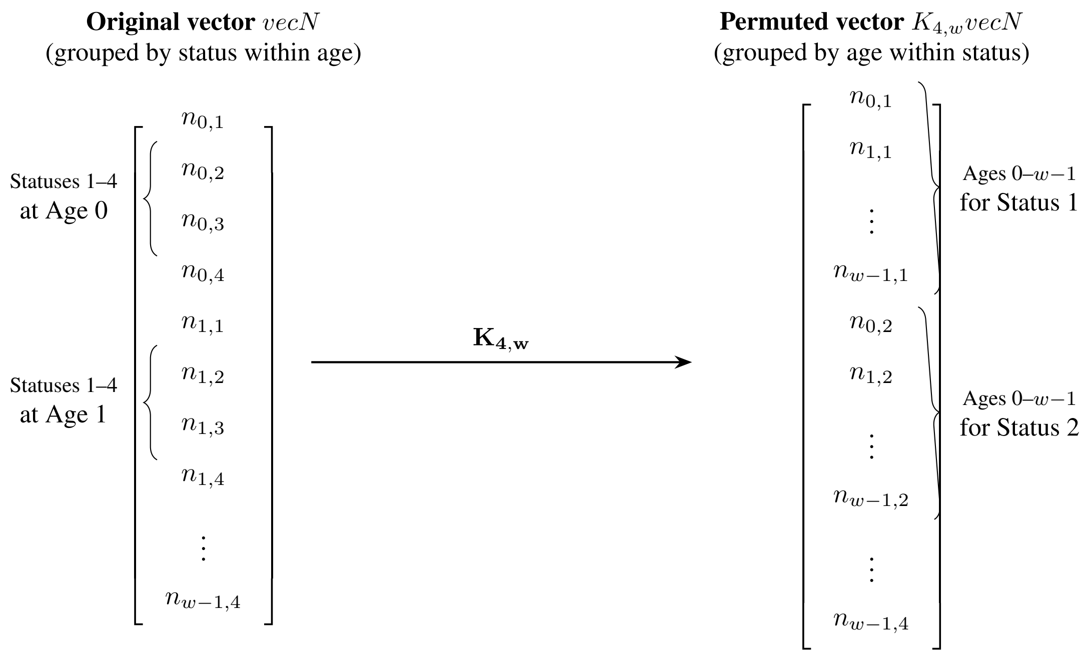


Figure 1. Reordering of the population vector $vecN$ using the vec-permutation matrix $K_{4,w}$, converting from status-within-age to age-within-status.

Numerical Example: Vec-Permutation with $K_{2,2}$ Let

$$N = \begin{bmatrix} n_{0,1} & n_{1,1} \\ n_{0,2} & n_{1,2} \end{bmatrix} \quad \Rightarrow \quad vec(N) = \begin{bmatrix} n_{0,1} \\ n_{0,2} \\ n_{1,1} \\ n_{1,2} \end{bmatrix}$$

Then applying $K_{2,2}$, we get:

$$K_{2,2} \cdot vec(N) = \begin{bmatrix} n_{0,1} \\ n_{1,1} \\ n_{0,2} \\ n_{1,2} \end{bmatrix} = vec(N^\top)$$

The general matrix $K_{s,a}$ reorders $vec(N)$ so that operations expecting row-major or column-major structure can be performed correctly.

The projection of the population vector accounting for survival and age progression involves several sequential steps. First, the status survival matrix $\mathbb{U}$ models transitions within each age class. Since the age progression matrix $\mathbb{D}$ acts on vectors organized by age within each status, the vec-permutation matrix $\mathbf{K}_{4,w}$ is used to reorder the population vector accordingly. After applying age progression, the inverse permutation $\mathbf{K}_{4,w}^T$ restores the original ordering. This combined operation is expressed as:

$$\bar{\mathbb{U}} = \mathbf{K}_{4,w}^T \mathbb{D} \mathbf{K}_{4,w} \mathbb{U} \tag{13}$$

Figure 2 illustrates this process, showing how survival, permutation, age progression, and inverse permutation combine to produce the updated population vector.

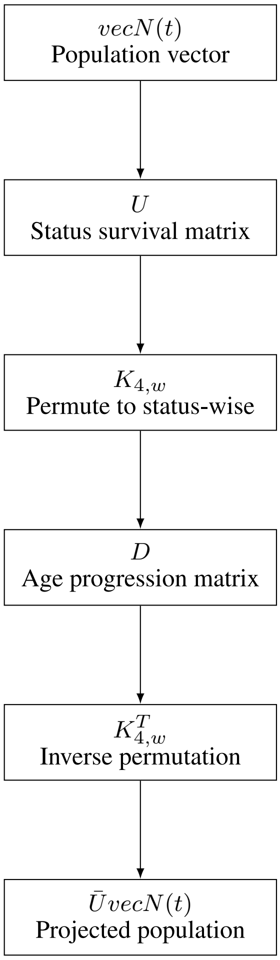


Figure 2. Operational flow of Equation (13) status survival is followed by permutation, age progression, and inverse permutation to return to original layout.

In a similar manner, modeling status-specific flows requires reordering the population vector to align with flow assignment operations. The flow matrix $\mathbb{F}$ is first applied to capture transitions between statuses within each age class. The vec-permutation matrix $\mathbf{K}_{4,w}$ then rearranges the vector to group ages within each status, enabling the flow assignment matrix $\mathbb{H}$ to allocate incoming flows appropriately by age. Finally, the inverse permutation $\mathbf{K}_{4,w}^T$ returns the vector to its original ordering. This operation is summarized as:

$$\bar{\mathbf{F}} = \mathbf{K}_{4,w}^T \mathbb{H} \mathbf{K}_{4,w} \mathbb{F} \tag{14}$$

Figure 3 depicts this sequence, clarifying how status-specific transitions are computed and assigned across age classes.

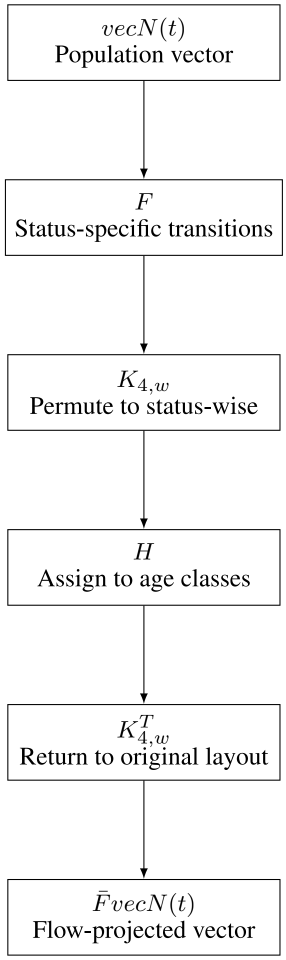


Figure 3. Operational flow of Equation (14) status-specific flows are computed with F, then reassigned to age using permutation and assignment matrices.

Combining the survival-age progression component and the status-specific flow component yields the full projection matrix:

$$\bar{\mathbf{A}} = \bar{\mathbf{U}} + \bar{\mathbf{F}} \tag{15}$$

Here, $\bar{\mathbf{U}} = \mathbf{K}^T_{4,w}\mathbb{D}\mathbf{K}_{4,w}\mathbb{U}$ captures survival and age progression, and $\bar{\mathbf{F}} = \mathbf{K}^T_{4,w}\mathbb{H}\mathbf{K}_{4,w}\mathbb{F}$ models status-specific flows. The matrix $\mathbf{K}_{4,w}$ is the **vec-permutation matrix** appropriate for 4 statuses and w age classes.

The complete equation for projecting the pension scheme population is then given by [14]:

$$\bar{\mathbf{n}}(t+1) = \bar{\mathbf{A}}(t)\,\bar{\mathbf{n}}(t) + \bar{\mathbf{b}}(t) \tag{16}$$

In this equation:

- $\bar{\mathbf{n}}(t)$ is the population vector at time t, classified by age within status;
- $\bar{\mathbf{A}}(t)$ is the full projection matrix combining demographic and flow dynamics;
- $\bar{\mathbf{b}}(t)$ is the recruitment vector, typically nonzero only in status 1 and at early age classes.

Box 1. The vec-permutation matrix

Tracy and Dwyer [38] introduced a matrix that was later given the name "permuted identity matrix" by MacRae [26]. A key feature of $\mathbf{K}_{m,n}$ is its ability to transform $vec\mathbf{A}$ into $vec\mathbf{A}^T$, where $\mathbf{A}$ is any matrix of size (m,n). This commutation matrix, as demonstrated by Macrae, is particularly useful for reversing the order in Kronecker products-a property that proves valuable when computing matrix derivatives. Barnett [3] and Conlisk [15] independently rediscovered this matrix. Balestra [2] extending Macrae's paper, gives a great number of important results on $\mathbf{K}_{m,n}$ in matrix differentiation.
Jan and Neudecker [23] and Henderson and Searle [19] reviewed the properties and derivation of the vec-permutation matrix. It has dimension $(mn \times mn)$ and is given by

$$\mathbf{K}_{m,n} = \sum_{i=1}^{m}\sum_{j=1}^{n}(\mathbf{E}_{ij} \otimes \mathbf{E}_{ij}^T) \tag{17}$$

where $\mathbf{E}_{ij}$ refers to a $m \times n$ matrix containing a 1 in the (i,j) position and zeros elsewhere and $\otimes$ denotes the Kronecker matrix product. Like all permutation matrices, the transpose of $\mathbf{K}_{m,n}$ is equal to its inverse, that is, $\mathbf{K}_{m,n}^T = \mathbf{K}_{m,n}^{-1}$.

2.1.1. Computational Implementation While the theoretical framework of the multistate population model is well-established, its practical implementation involves large-scale, structured matrix operations that require careful attention to computational efficiency.
We developed the model in `Python 3.10`, using libraries optimized for sparse matrix computation and numerical performance. Specifically, `NumPy` was used for general array operations, `SciPy.sparse` for efficient construction and manipulation of large block-diagonal matrices, and `Numba` to accelerate performance-critical components through just-in-time (JIT) compilation.

The matrices $\mathbb{U}$, $\mathbb{F}$, $\mathbb{D}$, and $\mathbb{H}$, each of size $4w \times 4w$, were constructed as sparse block-diagonal matrices using `scipy.sparse.block_diag`. This approach avoids the memory overhead associated with dense matrix representations, allowing the model to scale efficiently to realistic age ranges (e.g., $w = 100$).

The vec-permutation matrix $K_{4,w}$ and its transpose were not explicitly instantiated. Instead, their effect was implemented through efficient reshaping and transposition of vectors, based on the known structure of the permutation. This indexing-based approach allows the rearrangement of vector elements without incurring the computational cost of large matrix multiplications.

The projection matrices $\bar{U}$ and $\bar{F}$ were computed using a sequence of sparse matrix operations combined with these permutation functions. This stepwise implementation makes it possible to evolve the population vector over time with low memory usage and high computational speed. On a standard desktop machine, simulations involving $w = 100$ age classes and four member statuses run in minutes with peak memory usage under 2 GB.

To further clarify the implementation, a step-by-step pseudocode is provided in Appendix C, and the full source code is available in the supplementary materials.

2.2. The multistate financial model

We denote respectively by $\mathbf{S}_i$, $\mathbf{B}_i^{\text{I}}$, $\mathbf{B}_i^{\text{R}}$, and $\mathbf{B}_i^{\text{W}}$ the average salary, average invalidity pension, average retirement pension, and average widows/widowers pension at age i, for $i = 0, \dots, w-1$.

The financial state of the pension scheme is given by the vector:

$$\bar{\mathbf{I}} = \begin{bmatrix} S_0 \\ B_0^I \\ B_0^R \\ B_0^W \\ - \\ \vdots \\ - \\ S_{w-1} \\ B_{w-1}^I \\ B_{w-1}^R \\ B_{w-1}^W \end{bmatrix} \tag{18}$$

To compute the financial projections, we introduce the matrices $\bar{\mathbf{U}}^l$ and $\bar{\mathbf{F}}^l$:

$$\bar{\mathbf{U}}^l = \boldsymbol{K}_{4,w}^{\mathrm{T}}\, \mathbb{D}\, \boldsymbol{K}_{4,w}\, \mathbb{U}^l \tag{19}$$
$$\bar{\boldsymbol{F}}^l = \boldsymbol{K}_{4,w}^{\mathrm{T}}\, \mathbb{H}\, \boldsymbol{K}_{4,w}\, \mathbb{F}^l \tag{20}$$

where the matrices $\mathbb{U}^l$ and $\mathbb{F}^l$ are defined by:

$$\mathbb{U}^l = \sum_{i=0}^{w-1} \mathbf{E}_{ii} \otimes \mathbf{U}_i^l \tag{21}$$
$$\mathbb{F}^l = \sum_{i=0}^{w-1} \mathbf{E}_{ii} \otimes \mathbf{F}_i^l \tag{22}$$

The matrices $\mathbf{U}_i^l$ and $\mathbf{F}_i^l$ combine two factors:

- The probabilities of transitioning from one status to another.
- The escalation rates of ongoing salaries and pensions, as well as benefit rates and reversion rates necessary to compute new pensions.

We define:

$$\mathbf{U}_i^l = \mathbf{U}_i \circ \pi_i^U \tag{23}$$
$$\mathbf{F}_i^l = \mathbf{F}_i \circ \pi_i^F \tag{24}$$

where: - π_i^U captures the escalation rates of ongoing salaries and pensions; - π_i^F contains the rates necessary for computing new retirees, invalids, and widows/widowers pensions. They are given by:

$$\pi_i^U = \begin{bmatrix} \frac{s_{i+1}}{s_i}(1+\gamma) & 0 & 0 & 0 \\ 0 & (1+\beta) & 0 & 0 \\ 0 & 0 & (1+\beta) & 0 \\ 0 & 0 & 0 & (1+\beta) \end{bmatrix}, \quad i = 0, \ldots, w-1 \tag{25}$$

$$\pi_i^F = \begin{bmatrix} 0 & 0 & 0 & 0 \\ b_i & 0 & 0 & 0 \\ b_i & 0 & 0 & 0 \\ vb_i & v & v & 0 \end{bmatrix}, \quad i = 0, \ldots, w-1 \tag{26}$$

where: - s_i is the salary scale function at age i; - γ is the annual salary growth rate; - β is the annual pension indexation rate; - b_i is the benefit rate at age i; - v is the reversion rate applicable.

The complete equation for the financial projection of the pension scheme is thus given by:

$$\bar{\mathbf{n}}(t+1) \circ \bar{l}(t+1) = \left(\bar{\mathbf{U}}^l(t) + \bar{\mathbf{F}}^l(t)\right) \left(\bar{\mathbf{n}}(t) \circ \bar{l}(t)\right) + \bar{\mathbf{b}}(t) \circ \bar{\mathbf{g}} \tag{27}$$

where $\bar{\mathbf{g}}$ is a $(4w \times 1)$ vector containing the salaries of new entrants into the scheme, placed in the appropriate positions corresponding to status 1 (active members).
We define the financial projection matrix as:

$$\bar{\mathbf{A}}^l(t) = \bar{\mathbf{U}}^l(t) + \bar{\mathbf{F}}^l(t) \tag{28}$$

Replacing the term $\bar{\mathbf{n}} \circ \bar{l}$ by a shorthand notation $\bar{\mathbf{cash}}$, we obtain:

$$\bar{\mathbf{cash}}(t+1) = \bar{\mathbf{A}}^l(t) \cdot \bar{\mathbf{cash}}(t) + \bar{\mathbf{b}}(t) \circ \bar{\mathbf{g}} \tag{29}$$

3. Perturbation analysis of pension projections

The multistate pension projection depends on the age- and sex-specific mortality, invalidity, retirement, marital, and remarriage rates, as well as the number of new entrants fixed for each projection year.

To evaluate how a change in any of these input parameters impacts projection results, we use differential analysis. If the projection result y is a function of parameter x, the sensitivity and elasticity (proportional sensitivity) of y with respect to changes in x are defined as follows:

$$\text{Sensitivity} = \frac{dy}{dx} \quad , \quad \text{Elasticity} = \frac{\epsilon y}{\epsilon x} = x\frac{dy}{dx}\frac{1}{y}$$

The projection results in our case are the population vector $\bar{\mathbf{n}}(t)$ and the financial vector $\bar{\mathbf{cash}}(t)$ or any outcome derived from them.

In order to evaluate the sensitivity and elasticity of these vectors at any projection year t with respect to a modification of the parameter vector θ in any projection year s, we calculate for the population vector:

$$\text{Sensitivity} : \quad \frac{d\,\bar{\mathbf{n}}(t)}{d\,\theta^T(s)}$$

$$\text{Elasticity} : \quad \frac{\epsilon\,\bar{\mathbf{n}}(t)}{\epsilon\,\theta^T(s)} = \text{diag}(\bar{\mathbf{n}}(t)) \left(\frac{d\,\bar{\mathbf{n}}(t)}{d\,\theta^T(s)}\right) \text{diag}(\theta(s))^{-1}$$

and similarly for the financial vector $\bar{\mathbf{cash}}$.

The parameter vector $\theta(s)$ may have dimension p and can refer to age- and sex–specific mortality rates q, retirement rates r, invalidity rates i, marital rates w, remarriage rates h, and numbers of new entrants b.

Details of all the derivations presented in this section are given in Appendix A.

3.1. Sensitivity and elasticity of pension projections

Our methodology here is based on the work of Gassen and Caswell [12], who derived sensitivity and elasticity of projection results for the case of a population classified by age and sex. This is a direct generalization of those results to multistate pension projections using the vec-permutation model. As in [12], sensitivity and elasticity are calculated iteratively over the entire projection period, in parallel with the projection itself.

Population vector

We write the sensitivity of the population vector $\bar{\mathbf{n}}(t)$ with respect to $\theta(s)$ as:

$$\frac{d\bar{\mathbf{n}}(t+1)}{d\theta^T(s)} = \bar{\mathbf{A}}(t)\frac{d\bar{\mathbf{n}}(t)}{d\theta^T(s)} + \left(\bar{\mathbf{n}}^T(t)\otimes \mathbf{I}_{4w}\right)\frac{d\,\mathrm{vec}(\bar{\mathbf{A}}(t))}{d\theta^T(s)} + \frac{d\bar{\mathbf{b}}(t)}{d\theta^T(s)} \tag{30}$$

with initial condition:

$$\frac{d\bar{\mathbf{n}}(t)}{d\theta^T(s)} = \mathbf{0}_{4w\times p} \tag{31}$$

The elasticity of $\bar{\mathbf{n}}(t)$ with respect to $\theta(s)$ is given by:

$$\frac{\varepsilon\bar{\mathbf{n}}(t)}{\varepsilon\theta^T(s)} = \mathrm{diag}(\bar{\mathbf{n}}(t))^{-1}\cdot\frac{d\bar{\mathbf{n}}(t)}{d\theta^T(s)}\cdot\mathrm{diag}(\theta(s)) \tag{32}$$

Financial vector

For the financial vector $\bar{\mathbf{cash}}(t)$, we have:

$$\frac{d\bar{\mathbf{cash}}(t+1)}{d\theta^T(s)} = \bar{\mathbf{A}}^l(t)\frac{d\bar{\mathbf{cash}}(t)}{d\theta^T(s)} + \left(\bar{\mathbf{cash}}^T(t)\otimes \mathbf{I}_{4w}\right)\frac{d\,\mathrm{vec}(\bar{\mathbf{A}}^l(t))}{d\theta^T(s)} + \left(\mathbf{1}_p^T\otimes\bar{g}\right)\frac{d\bar{\mathbf{b}}(t)}{d\theta^T(s)} \tag{33}$$

with initial condition:

$$\frac{d\bar{\mathbf{cash}}(t)}{d\theta^T(s)} = \mathbf{0}_{4w\times p} \tag{34}$$

The elasticity of $\bar{\mathbf{cash}}(t)$ with respect to $\theta^T(s)$ is:

$$\frac{\varepsilon\bar{\mathbf{cash}}(t)}{\varepsilon\theta^T(s)} = \mathrm{diag}(\bar{\mathbf{cash}}(t))^{-1}\cdot\frac{d\bar{\mathbf{cash}}(t)}{d\theta^T(s)}\cdot\mathrm{diag}(\theta(s)) \tag{35}$$

Equations (30) and (33) reflect a general pattern seen across all the sensitivity analyses: they include components that capture the sensitivity at a given time t, along with terms that describe how this sensitivity evolves due to changes in the projection matrix $\bar{\mathbf{A}}$ (respectively $\bar{\mathbf{A}}^l$) and the vector $\bar{\mathbf{b}}$ (respectively $\bar{\mathbf{b}}\circ\bar{\mathbf{g}}$).

3.2. The derivatives of matrices

We present here the derivatives of the matrices $\bar{\mathbf{A}}$ (respectively $\bar{\mathbf{A}}^l$) and $\bar{\mathbf{b}}$ with respect to the set of parameters. Differentiating the matrices $\bar{\mathbf{A}}$ (respectively $\bar{\mathbf{A}}^l$) with respect to the parameters is especially challenging, as the parameters are embedded within $\mathbf{U}_i$ (respectively $\mathbf{U}_i^l$) and $\mathbf{F}_i$ (respectively $\mathbf{F}_i^l$), which are themselves intricately nested inside $\bar{\mathbf{U}}$ (respectively $\bar{\mathbf{U}}^l$), $\bar{\mathbf{F}}$ (respectively $\bar{\mathbf{F}}^l$), and $\bar{\mathbf{A}}$ (respectively $\bar{\mathbf{A}}^l$). As in [11], the derivative of the matrix $\bar{\mathbf{A}}(t)$ with respect to $\theta(s)$ is given by

$$\begin{aligned}\frac{d\,\mathrm{vec}(\bar{\mathbf{A}}(t))}{d\theta^T(s)} &= (\mathbf{I}_{4w}\otimes\mathbf{Q}_{\mathbb{U}})\sum_{i=0}^{w-1}(\mathbf{E}_{ii}\otimes\mathbf{K}\otimes\mathbf{I}_4)(\mathrm{vec}(\mathbf{I}_w)\otimes\mathbf{I}_{4^2})\frac{d\,\mathrm{vec}(\mathbf{U}_i(t))}{d\theta^T(s)}\\ &+ (\mathbf{I}_{4w}\otimes\mathbf{Q}_{\mathbb{F}})\sum_{i=0}^{w-1}(\mathbf{E}_{ii}\otimes\mathbf{K}\otimes\mathbf{I}_4)(\mathrm{vec}(\mathbf{I}_w)\otimes\mathbf{I}_{4^2})\frac{d\,\mathrm{vec}(\mathbf{F}_i(t))}{d\theta^T(s)}\end{aligned} \tag{36}$$

where $\mathbf{Q}_{\mathbb{U}} = \mathbf{K}_{4,w}^T\mathbb{D}\mathbf{K}_{4,w}$ and $\mathbf{Q}_{\mathbb{F}} = \mathbf{K}_{4,w}^T\mathbb{H}\mathbf{K}_{4,w}$ are the fixed matrix products involved in the definition of $\bar{\mathbf{U}}$ ($\bar{\mathbf{U}}^l$) and $\bar{\mathbf{F}}$ ($\bar{\mathbf{F}}^l$). and similarly for $\bar{\mathbf{A}}^l(t)$.

Until now, the parameter vector has been left unspecified, as the findings have been applicable regardless of its

particular values. We now turn to more specific scenarios where θ represents a vector of mortality rates, invalidity rates, retirement rates, marital rates, remarriage rates or numbers of new entrants. As an illustrative example, we express here the derivatives of $\mathbf{U}_i$($\mathbf{U}_i^l$) and $\mathbf{F}_i$ ($\mathbf{F}_i$) with respect to the parameters $\mathbf{r}$ and $\mathbf{b}$.
A change in the parameter vector at time s influences the projection matrices only when $t = s$. To formally express this temporal dependence, we employ the Kronecker delta function:

$$\delta(s,t) = \begin{cases} 1 & \text{if } s = t \\ 0 & \text{if } s \neq t \end{cases} \tag{37}$$

.

- **Retirement**: $\boldsymbol{\theta} = \mathbf{r}$

$$\frac{d\,\text{vec}(\mathbf{U}_i(t))}{d\mathbf{r}^T(s)} = \delta(s,t)\,\mathbf{e}_i^T \otimes \begin{bmatrix} -(1-q_i)(1-i_i) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T$$

$$\frac{d\,\text{vec}(\mathbf{U}_i^l(t))}{d\mathbf{r}^T(s)} = \delta(s,t)\,\mathbf{e}_i^T \otimes \begin{bmatrix} -(1-q_i)(1-i_i)s_{i+1}/s_i(1+\mu) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T$$

$$\frac{d\,\text{vec}(\mathbf{F}_i(t))}{d\mathbf{r}^T(s)} = \delta(s,t)\,\mathbf{e}_i^T \otimes \begin{bmatrix} 0 & 0 & (1-q_i)(1-i_i) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T$$

$$\frac{d\,\text{vec}(\mathbf{F}_i^l(t))}{d\mathbf{r}^T(s)} = \delta(s,t)\,\mathbf{e}_i^T \otimes \begin{bmatrix} 0 & 0 & (1-q_i)(1-i_i)b_i & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T$$

  where $\mathbf{e}_i$ is a $(w \times 1)$ vector with 1 in the i-th entry and 0 elsewhere. The derivative of $\bar{\mathbf{b}}$ with respect to $\mathbf{r}$ is 0.
- **New entrants**: $\boldsymbol{\theta} = \mathbf{b}$
  Recall that the vector $\bar{\mathbf{b}}$ inherits the same structure as $\bar{\mathbf{n}}$, but can show non-null entries for only status 1. We suppose that new entrants operate from age 0 to age m. The derivative of $\bar{\mathbf{b}}$ with respect to $\mathbf{b}$ is obtained as follows:

$$\frac{d\,\text{vec}(\bar{\mathbf{b}}(t))}{d\mathbf{b}^T(s)} = \delta(s,t) \left[\begin{array}{c} \mathbf{I}_m \\ \hline \mathbf{0}_{(w-m)\times m} \end{array} \right] \otimes \mathbf{e}_1$$

  where $\mathbf{e}_1$ is a (4×1) vector with 1 in the first entry and 0 elsewhere. The derivatives of $\mathbf{U}_i$ and $\mathbf{F}_i$ with respect to $\mathbf{b}$ are zero.

3.3. Sensitivity and elasticity of projection outcomes

Equations (30) and (33) yield the sensitivities of each age class, at every point in time from 0 to T, with respect to changes in mortality, invalidity, retirement, marital and remarriage rates, and new entrants numbers across all age classes and time points. However, this results in a high-dimensional data structure that contains far more detail than is typically required. To extract meaningful insights, this information must be distilled to focus on the sensitivities of specific projection outcomes of interest. Reports of actuarial valuation of pension schemes mainly present projections of (1) the total status-specific population sizes (active, invalids, retirees, widows/widowers)(2) the corresponding cash flows (3) key ratios such as the system dependency ratio defined as the ratio between total number of pensioners and total number of insured actives, and the PAYG cost rate calculated as the ratio of annual

expenditures over total insurable earnings.
In this section, we derive the sensitivity and elasticity of such outcomes from the derivatives of $\bar{\mathbf{n}}(t)$ and $\bar{\mathbf{cash}}(t)$ given in (30) and (33). Derivations are given in Appendix A.3.

We start by the status-specific population sizes which are derived from the population vector $\bar{\mathbf{n}}(t)$ as:

$$\begin{aligned}
\text{Population size of actives:} \quad & A(t) = \mathbf{1}_w^T(\mathbf{I}_w \otimes \mathbf{e}_1^T)\bar{\mathbf{n}}(t) \\
\text{Population size of invalids:} \quad & I(t) = \mathbf{1}_w^T(\mathbf{I}_w \otimes \mathbf{e}_2^T)\bar{\mathbf{n}}(t) \\
\text{Population size of retirees:} \quad & R(t) = \mathbf{1}_w^T(\mathbf{I}_w \otimes \mathbf{e}_3^T)\bar{\mathbf{n}}(t) \\
\text{Population size of widows/widowers:} \quad & W(t) = \mathbf{1}_w^T(\mathbf{I}_w \otimes \mathbf{e}_4^T)\bar{\mathbf{n}}(t)
\end{aligned}$$

The sensitivity and elasticity of the population size of actives, for example, to parameter changes at time s, are given by:

$$\frac{dA(t)}{d\theta^T(s)} = \mathbf{1}_w^T(\mathbf{I}_w \otimes \mathbf{e}_1^T)\frac{d\bar{\mathbf{n}}(t)}{d\theta^T(s)} \tag{38}$$

$$\frac{\varepsilon A(t)}{\varepsilon\theta^T(s)} = \frac{1}{A(t)}\frac{dA(t)}{d\theta^T(s)}\,\mathrm{diag}(\theta(s)) \tag{39}$$

Similarly, we calculate the sensitivity and elasticity of $I(t)$, $R(t)$, and $W(t)$.
We move now to the corresponding cash flows which are expressed as:

$$\begin{aligned}
\text{Aggregate insurable earnings for year } t: \quad & A^l(t) = \mathbf{1}_w^T(\mathbf{I}_w \otimes \mathbf{e}_1^T)\bar{\mathbf{cash}}(t) \\
\text{Total invalidity benefit payments in year } t: \quad & I^l(t) = \mathbf{1}_w^T(\mathbf{I}_w \otimes \mathbf{e}_2^T)\bar{\mathbf{cash}}(t) \\
\text{Total retirement benefit payments in year } t: \quad & R^l(t) = \mathbf{1}_w^T(\mathbf{I}_w \otimes \mathbf{e}_3^T)\bar{\mathbf{cash}}(t) \\
\text{Total survivor benefit payments in year } t: \quad & W^l(t) = \mathbf{1}_w^T(\mathbf{I}_w \otimes \mathbf{e}_4^T)\bar{\mathbf{cash}}(t)
\end{aligned}$$

Again, the sensitivity and elasticity of the insurable earnings, for example, to parameter changes at time s, are given by:

$$\frac{dA^l(t)}{d\theta^T(s)} = \mathbf{1}_w^T(\mathbf{I}_w \otimes \mathbf{e}_1^T)\frac{d\bar{\mathbf{cash}}(t)}{d\theta^T(s)} \tag{40}$$

$$\frac{\varepsilon A^l(t)}{\varepsilon\theta^T(s)} = \frac{1}{A^l(t)}\frac{dA^l(t)}{d\theta^T(s)}\,\mathrm{diag}(\theta(s)) \tag{41}$$

In the same way, we calculate the sensitivity and elasticity of $I_l(t)$, $R_l(t)$, and $W_l(t)$.
Finally, we show how to calculate the sensitivity and elasticity of the two key ratios: the dependency ratio and the PAYG cost rate, which are obtained as:

$$\begin{aligned}
\text{Dependency ratio at year } t: \quad & c(t) = \frac{a\bar{\mathbf{n}}(t)}{b\bar{\mathbf{n}}(t)} \\
\text{PAYG cost rate at year } t: \quad & d(t) = \frac{a\bar{\mathbf{cash}}(t)}{b\bar{\mathbf{cash}}(t)}
\end{aligned}$$

where

$$b = \mathbf{1}_w^T(\mathbf{I}_w \otimes \mathbf{e}_1^T), \quad a = \mathbf{1}_w^T\left(\mathbf{I}_w \otimes (\mathbf{e}_2^T + \mathbf{e}_3^T + \mathbf{e}_4^T)\right)$$

As in [5], the sensitivity and elasticity of $c(t)$ are:

$$\frac{dc(t)}{d\theta^T(s)} = \frac{b\bar{\mathbf{n}}(t)\cdot a\frac{d\bar{\mathbf{n}}(t)}{d\theta^T(s)} - a\bar{\mathbf{n}}(t)\cdot b\frac{d\bar{\mathbf{n}}(t)}{d\theta^T(s)}}{[b\bar{\mathbf{n}}(t)]^2} \tag{42}$$

$$\frac{\varepsilon c(t)}{\varepsilon\theta^T(s)} = \frac{1}{c(t)}\frac{dc(t)}{d\theta^T(s)}\,\mathrm{diag}(\theta(s)) \tag{43}$$

We proceed in the same way to derive the sensitivity and elasticity of the PAYG cost rate $d(t)$ at year t.

—Aggregating perturbations over time—Our focus here is on the sensitivity of the population vector at time t to a modification in $\theta(s)$ that is applied uniformly across the entire time horizon, from $s = 0$ to $s = T$. A key example of this involves assessing how the population vector at the final projection time, $\bar{\mathbf{n}}(T)$, responds to changes in age-specific rates applied consistently across all ages and time points throughout the projection period.

Following the formulation used by Gassen and Caswell [12], we represent the sensitivity of $\bar{\mathbf{n}}(T)$ to this type of cumulative perturbation as:

$$\frac{d\,\bar{\mathbf{n}}(T)}{d\,\theta^T(0,T)} = \sum_{s=0}^{T}\frac{d\,\bar{\mathbf{n}}(T)}{d\,\theta^T(s)} \tag{44}$$

The corresponding elasticity is:

$$\frac{\varepsilon\,\bar{\mathbf{n}}(T)}{\varepsilon\,\theta^T(0,T)} = \sum_{s=0}^{T}\frac{\varepsilon\,\bar{\mathbf{n}}(T)}{\varepsilon\,\theta^T(s)} \tag{45}$$

This also applies to the vector of cash flows. We write:

$$\frac{d\,\bar{\mathbf{cash}}(T)}{d\,\theta^T(0,T)} = \sum_{s=0}^{T}\frac{d\,\bar{\mathbf{cash}}(T)}{d\,\theta^T(s)} \tag{46}$$

The corresponding elasticity is:

$$\frac{\varepsilon\,\bar{\mathbf{cash}}(T)}{\varepsilon\,\theta^T(0,T)} = \sum_{s=0}^{T}\frac{\varepsilon\,\bar{\mathbf{cash}}(T)}{\varepsilon\,\theta^T(s)} \tag{47}$$

4. A projection of the CMR's civilian pension scheme

The Moroccan pension system is characterized by the coexistence of pension schemes multiple and diversified: the CMR [2] and the RCAR [3] for the public and semi-public sector, the CNSS [4] and CIMR [5] for the private sector as well as other internal regimes for some public establishments. This coexistence is accompanied by a partitioning which means that one does not know what the others are doing. In the context of the national pension debate regarding financial sustainability, the most pressing concern is the financial situation of the CMR's civilian pension scheme [6]. The scheme has recorded a technical deficit of 2.677 MDH in 2015 and 4.76 MDH in 2016. The date of total depletion of reserves was estimated at 2022 if no reform was introduced, making it possible to improve its resources and limit its commitments. In 2016, the government launched a parametric reform of the civilian pension scheme, structured around several key measures, including the gradual raising of the retirement age, increase in contribution rates, modification of the pension payment base, and the accrual rate. The parametric reform of the CMR in 2017

[2]Caisse Marocaine des Retraites
[3]Régime Collectif des Allocations de Retraite
[4]Caisse Nationale de Sécurité Sociale
[5]Caisse Interprofesionnelle Marocaines des Retraites
[6]The CMR manages two compulsory plans:
- the civil pensions scheme which covers civil servants from the State, employees of local authorities and those of public establishments;
- the military pensions scheme for members of the Royal Armed Forces and the Royal Gendarmerie and Auxiliary Forces personnel.

made it possible to balance the pricing of the scheme for the rights acquired after the reform, however, the weight of those acquired previously continues to weigh on its viability, according to the Insurance and Social Welfare Supervisory Authority (ACAPS).

Using the age $\times$ status matrix model, we undertake demographic and financial projections of the CMR's civilian pension scheme over the next 60 coming years, according to the assumptions listed below (See Appendix B). We use data and assumptions published by the Insurance and Social Security Supervisory Authority (ACAPS), in his yearly report "Sector of Social Security (2020)". The social security Database contains data on age-sex structure of beneficiaries and the corresponding average salaries and pensions at the end of 2020. Based on our projections, we highlight two key findings. First, the system dependency ratio, defined as the quotient between the population of pensioners and the population of insured active, is expected to increase from 0.53 in 2020 to 0.73 in 2034. Second, the reserve funds will likely be exhausted by 2028. These projection results serve as a foundation for governmental planning. Analyzing their sensitivity and elasticity to changes in underlying assumptions is therefore crucial—not only for the demographic research community but also for policymakers in Morocco.

4.1. Limited Uncertainty Analysis: Impacts of Key Demographic Assumptions

To complement the baseline projection and better account for real-world variability, we conduct a limited uncertainty analysis focusing on the impact of key demographic parameters on the financial trajectory of the CMR's civilian pension scheme. While the sensitivity analysis quantifies the mathematical responsiveness of outputs to small parameter changes, uncertainty analysis reflects plausible ranges in real input data, allowing us to explore how the model behaves under alternative demographic scenarios.

We analyze three critical parameters:

- Mortality rates
- Invalidity (disability) incidence rates
- Early retirement rates

Each parameter is varied independently, while all others are held at baseline levels. The financial indicators of interest are:

- The projected reserve fund depletion year, and
- The average PAYG (Pay-As-You-Go) cost rate between 2020 and 2080.

Scenario Details

Mortality: The baseline scenario uses the TD 88–90 mortality table, as applied in official actuarial projections. The alternative scenario replaces it with the TV 88–90 table, which reflects slightly lower mortality rates and thus longer life expectancy, increasing liabilities.
Invalidity: The baseline invalidity rate is set at 0.0001 (0.01%). We simulate two alternative rates: 0.00005 (-50%) and 0.0002 ($+100\%$) to reflect uncertainty in disability incidence.
Early Retirement: The baseline scenario uses observed early retirement rates from administrative data. In alternative scenarios, we simulate:

- A high-retirement scenario, where early retirement rates are doubled, and
- A low-retirement scenario, where early retirement rates are halved.

These changes influence the flow of contributors exiting the system earlier than the statutory retirement age.

Summary of results

Parameter Scenario Description	Reserve Depletion Year	Avg. PAYG Cost Rate (2020–2080)
Mortality TD 88–90 (Baseline)	2028	50.81%
TV 88–90 (Lower Mortality)	2028	61.46%
Invalidity 0.00005 (Low incidence)	2028	50.73%
0.0001 (Baseline)	2028	50.81%
0.0002 (High incidence)	2028	51.16%
Retirement Early retirement × 0.5 (Lower)	2028	48.24%
Baseline	2028	50.81%
Early retirement × 2 (Higher)	2028	55.69%

Table 2. Summary of limited uncertainty analysis results on the CMR civilian pension scheme.

Interpretation

This limited uncertainty analysis demonstrates that while the projected reserve fund depletion year remains fixed at 2028 across all tested scenarios, plausible variations in key demographic parameters can lead to substantial changes in the average PAYG cost rate between 2020 and 2080.

Mortality assumptions have a pronounced impact on cost rates: replacing the baseline TD 88–90 mortality table with the lower mortality TV 88–90 table increases the average PAYG cost rate significantly, from 50.81% to 61.46%, reflecting the higher liabilities associated with longer life expectancy.

Invalidity incidence also affects costs, though more moderately: halving the baseline invalidity rate slightly reduces the PAYG cost rate to 50.73%, while doubling it raises costs to 51.16%, indicating the financial burden of higher disability incidence.

Early retirement behavior exhibits a strong influence on financial sustainability. Halving early retirement rates lowers the PAYG cost rate to 48.24%, whereas doubling them increases it sharply to 55.69%, underscoring the sensitivity of pension costs to contributors exiting the system earlier than the statutory retirement age.

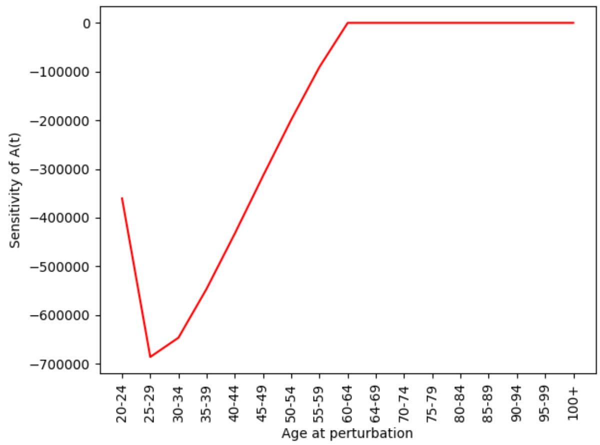
Overall, these findings highlight that even without changes in the depletion year, demographic uncertainties can substantially affect the pension scheme's ongoing cost structure. This underscores the critical importance of precise demographic modeling and robust scenario analysis for effective pension system management.

5. Sensitivity and elasticity of the pension projections of the CMR's civilian pension scheme

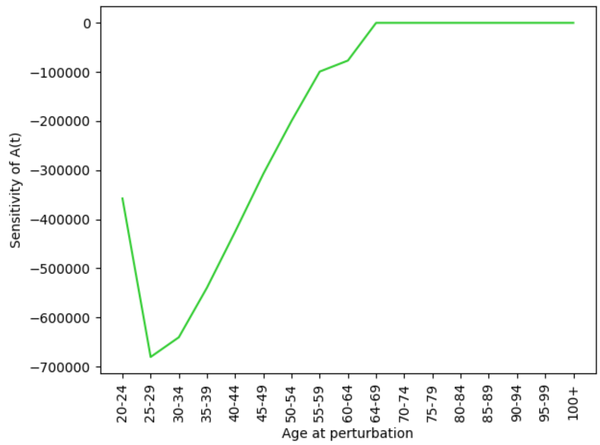
We examine the sensitivity and elasticity of the pension projections at the final time point T = 60, considering the combined population of males and females. For reasons of calculus simplification, we carry out the projections and the sensitivities for five-year age groups (17 from 20 to 105).

5.1. Sensitivity of satus-specific population sizes

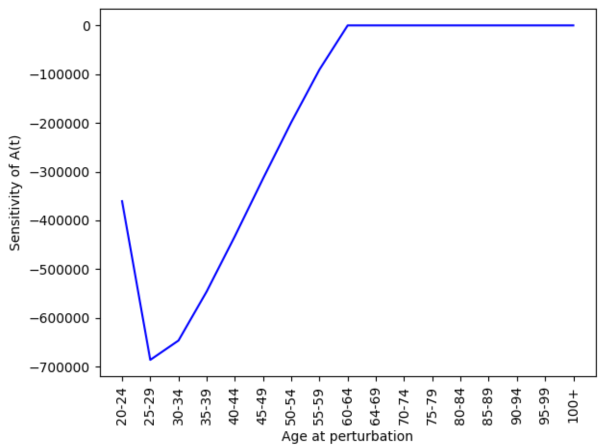
Figure 4, 5, 6 and 7 draw respectively the sensitivity of the total population sizes $A(T)$, $R(T)$, $I(T)$, and $W(T)$ at the final time point $T = 60$, to variations in age-specific mortality, retirement, invalidity, marital percentage, and remarriage rates, as well as the number of new entrants applied in every projection year, as a function of the age at which the rate is altered, as specified in equation 44.



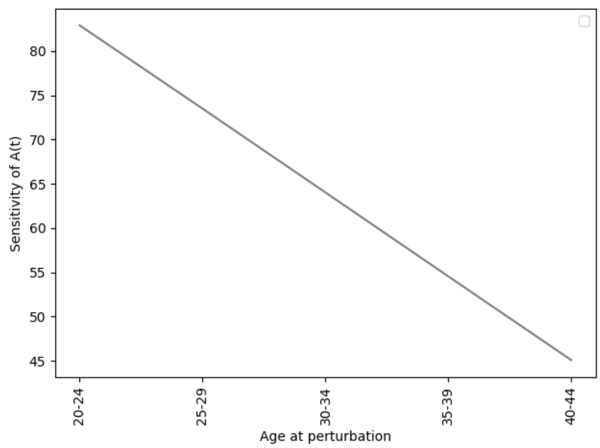
(4a) Sensitivity to mortality.



(4b) Sensitivity to retirement.



(4c) Sensitivity to invalidity.



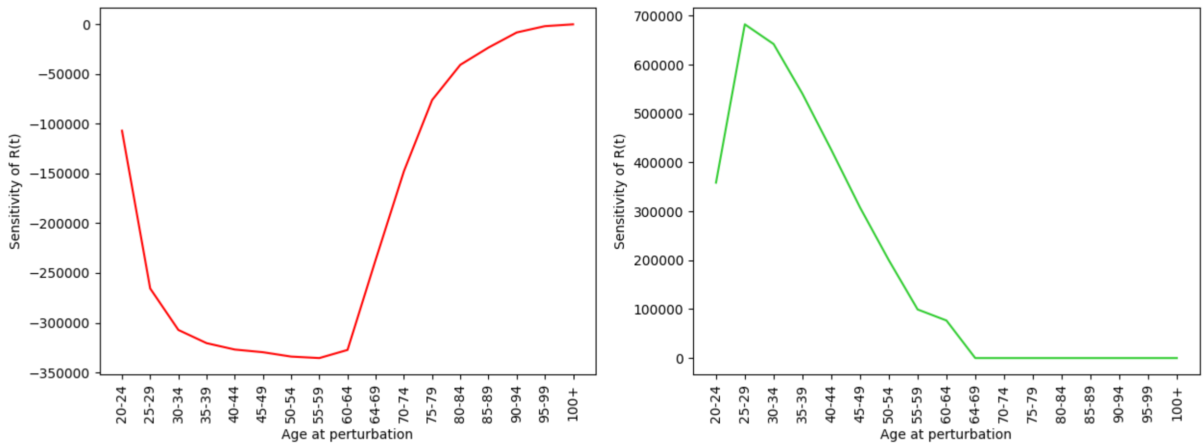
(4d) Sensitivity to new entrants.

Figure 4. Sensitivity of the population size of active

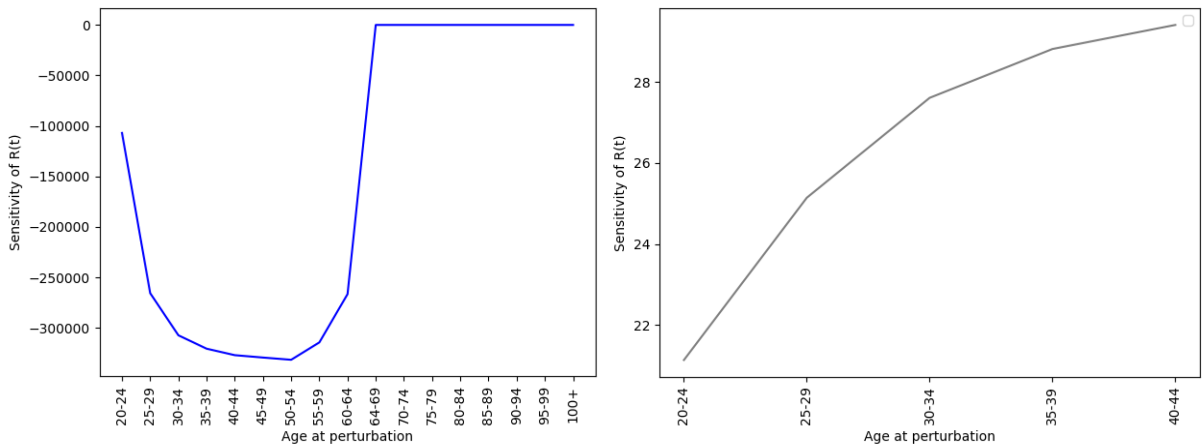
Figures (4a),(4b) and (4c) shows that changes in mortality, disability, and retirement rates have the greatest impact on the final size of the active population when they occur at younger ages. This is because individuals in younger age groups remain in the population for a longer time, allowing the impact of early changes to accumulate over the projection period.

For example, a sensitivity of -6.86×10^5 in the mortality rate at age 25–29 indicates that, in theoretical terms, an increase of one unit in the mortality rate (from 0 to 1) would reduce the size of the active population by approximately 686,000 individuals at the end of the projection horizon. In practical terms, the mortality rate at these ages is around 0.001 (0.1%). A 10% increase in this parameter (to 0.0011) would therefore reduce the active population by about 69 individuals, since $0.0001 \times (-6.86 \times 10^5) = -69$. *This represents a relatively small effect in absolute numbers, but it demonstrates that even limited changes in early-adulthood mortality can leave a measurable long-term impact on the working-age population.*

The sensitivity of total population size of active to perturbations in numbers of new entrants (Figure (4d)) decreases with age. At age class 20-24, adding one new active per year—one male and one female—raises the final population size by over 82 individuals. The sensitivity to changes in new entrants is vastly smaller than the sensitivity to changes in other vital rates. This difference arises because new entrants are counted in absolute numbers, whereas mortality, disability, and retirement are expressed as per capita rates. As explained below, elasticities provide a way to compare these by standardizing the values as proportional changes.



(5a) Sensitivity to mortality. (5b) Sensitivity to retirement.



(5c) Sensitivity to invalidity. (5d) Sensitivity to new entrants.

Figure 5. Sensitivity of the population size of retirees

Figures (5a), (5b) and (5c) suggest that perturbations in mortality, disability, retirement have the greatest impact on $R(T)$ between ages 25 and 55. At the start of the projection period, large cohorts of active are passing through theses ages, so any alterations in vital rates affect a substantial portion of the population.

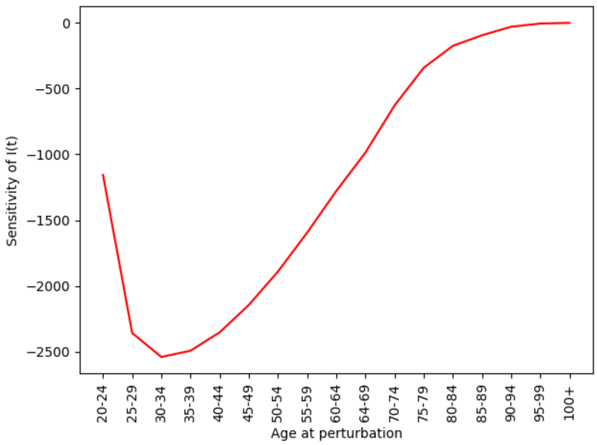
Mortality perturbations affect the number of retirees through two main mechanisms. First, changes in mortality at pre-retirement ages reduce the number of new retirees by decreasing the pool of active individuals who survive to retirement. For example, increasing mortality by one unit at age 55–59 reduces the final population size by approximately 3.35×10^5. Second, changes in mortality at post-retirement ages reduce the number of existing retirees directly, as more individuals die while already retired.

As for invalidity rates, increasing the yearly incidence systematically reduces the proportion of people entering retirement. At age 50–54, for example, a one-unit increase in invalidity rates results in a reduction of the final population size by approximately 3.31×10^5.

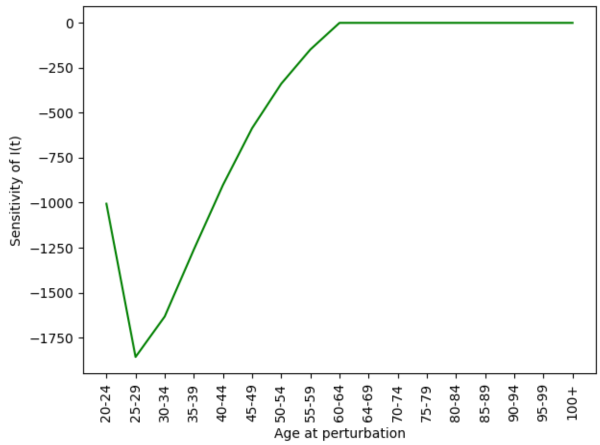
The sensitivity of the total population size to retirement rates declines progressively up to the mandatory retirement age class 60–64, due to a retroactive effect: increasing retirement at younger ages reduces the number of active individuals who can transition into retirement in the future. However, this effect is less pronounced at very young ages, where cohorts are smaller. Notably, a sensitivity of $+6.82 \times 10^5$ in the retirement rate at ages 25–29 implies that, under the theoretical case of a one-unit increase in the retirement rate , the retiree population at the end of the projection period would rise by approximately 682,000 individuals. In practice, retirement at these ages is rare, with baseline rates close to 0.01 (1%). An increase of one percentage point (from 1% to 2%) corresponds to $+0.01$, which would increase the number of retirees by about 6,820 individuals, since $0.01 \times (6.82 \times 10^5) = 6{,}820$. *This is a substantial effect, highlighting that even small changes in early retirement behavior can significantly amplify the long-run size of the retiree population.* Importantly, this increase is more than double the reduction that would result from an equivalent rise in mortality or invalidity at the same age (both around -2.65×10^5), underlining the much stronger leverage of early retirement compared to other demographic factors.

These sensitivity patterns highlight retirement behavior as the most powerful lever for influencing population structure and dependency. The substantially higher sensitivities to retirement rates—especially at working ages between 25 and 49—suggest that policies encouraging later retirement or reducing early exits from the labor force could significantly improve the long-term support ratio and reduce dependency burdens. While improving adult health and survival (i.e., lowering mortality) and reducing disability incidence remain important, their demographic effects are clearly smaller in magnitude. Therefore, pension reforms, incentives for delayed retirement, and active aging strategies should be prioritized in efforts to manage the demographic transition.

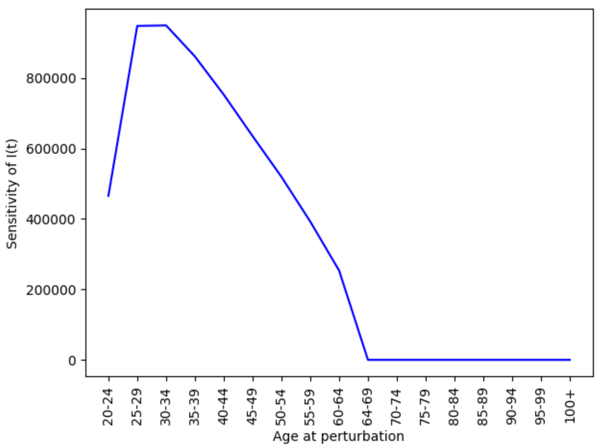
The sensitivity of total population size of retirees to perturbations in numbers of new entrants (Figure (5d)) increases with age. Cohorts that pass through young age groups spend many years in the active population and barely contribute to to the size of the population of retirees. At age class 20-24, adding one new active per year—one male and one female—raises the final population size by over 21 individuals.



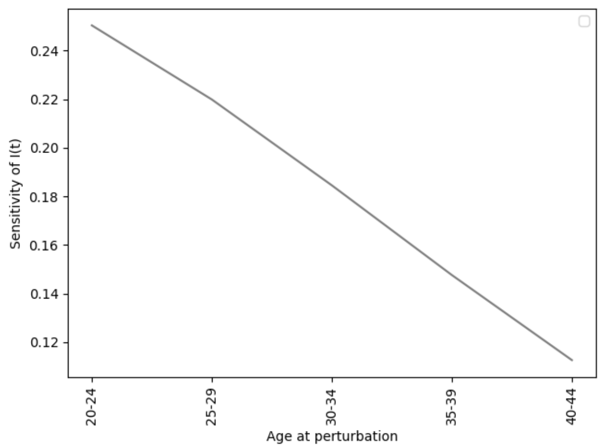
(6a) Sensitivity to mortality.



(6b) Sensitivity to retirement.



(6c) Sensitivity to invalidity.



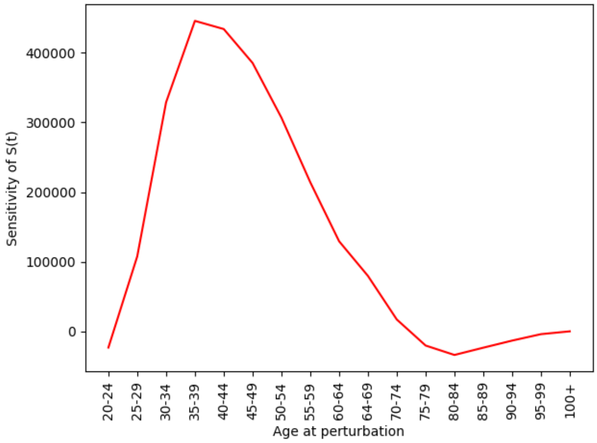
(6d) Sensitivity to new entrants.

Figure 6. Sensitivity of the population size of invalids

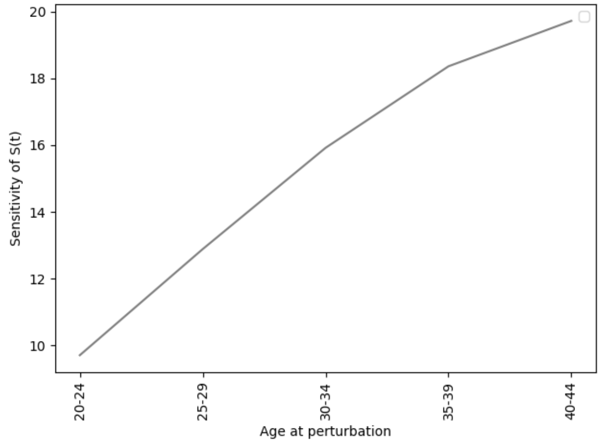
Figures (6a), (6b), and (6c) show that perturbations in mortality, disability, and retirement affect the final number of invalids, but the magnitudes differ substantially.

- **Mortality** reduces the invalid population by approximately -2.36×10^3 at age 25–29, acting both by lowering the number of active individuals who could become invalid and by decreasing existing invalids.
- **Disability** has a far stronger effect: a one-unit increase in invalidity rates at the same age raises the invalid population by about 9.48×10^5, nearly 400 times larger than the effect of mortality.
- **Retirement** reduces the invalid population by roughly -1.86×10^3, comparable to the effect of mortality but negligible relative to disability (over 500 times smaller).

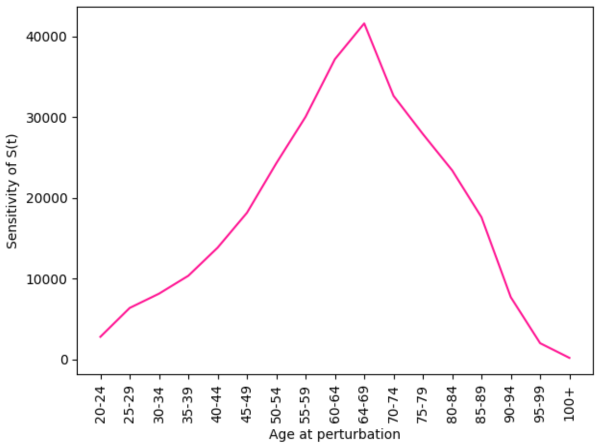
These benchmarks clearly indicate that the invalid population is overwhelmingly sensitive to changes in disability rates, while mortality and retirement have much smaller impacts.
The sensitivity of the total population size of invalids to perturbations in numbers of new entrants (Figure (6d)) decreases with age, as the population of invalids arises immediately from the population of actives. At age class 20–24, adding one new active per year—one male and one female—raises the final population size by over 0.25 individuals.



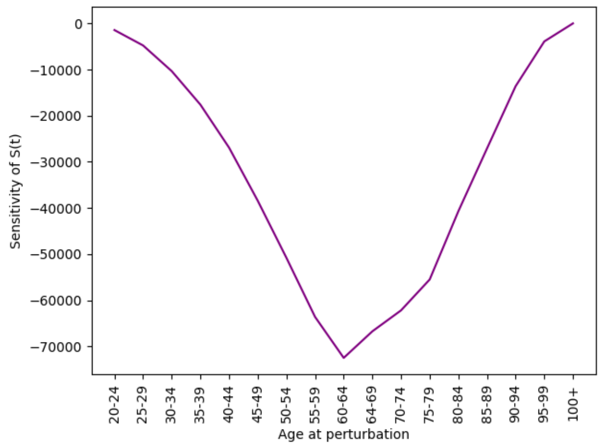
(7a) Sensitivity to mortality.



(7b) Sensitivity to new entrants.



(7c) Sensitivity to marital percentage.



(7d) Sensitivity to remarriage rates.

Figure 7. Sensitivity of the population size of widows/widowers

Figures (7a),(7b), (7c) and (7d) show the sensitivities of the total population size of widows and widowers, $W(T)$ at $T = 60$, to perturbations in mortality, marital percentage, remarriage rates, and the number of new entrants applied each projection year.

Mortality increases before the 75–79 age group tend to increase the widowed population by creating more new widows and widowers, while after this age, higher mortality reduces the number of existing widows and widowers. For example, increasing mortality by one unit at age 25–29 raises the widowed population by approximately 1.07×10^5, whereas the same increase at age 75–79 decreases it by about 2.02×10^4.

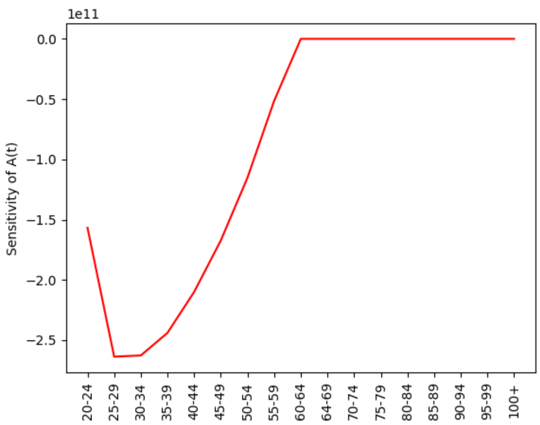
Perturbations in marital and remarriage rates have the strongest effects on the widowed population size between ages 50 and 90. The sensitivity results confirm that projection parameters related to marital and remarriage rates in this age range must be defined with particular care, as they significantly influence the final population size. This is especially important because higher marital stability increases the number of widows and widowers eligible for survivor pensions, potentially raising pension costs, while remarriage typically leads to the loss of survivor pension benefits, reducing this financial burden.

Managing mortality rates, especially before age 75, can influence the size of the widowed population by affecting the balance between new widows/widowers created and existing ones lost. Separately, marital stability and remarriage rates—particularly among older adults—have significant demographic and financial impacts. Higher marital stability increases the number of survivors eligible for pensions, potentially raising pension costs, while remarriage usually results in the loss of survivor pension rights, reducing this burden. Careful calibration of these parameters is essential for accurate population projections and the design of sustainable pension policies.

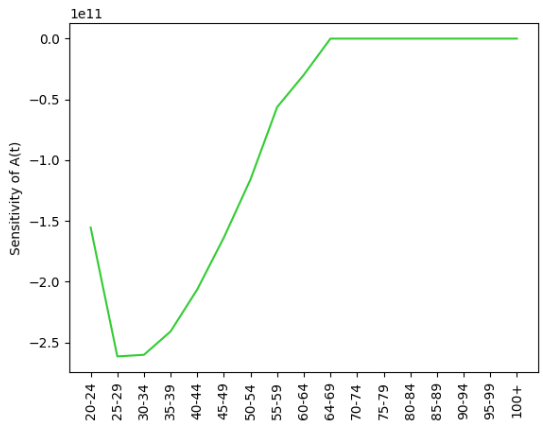
Regarding new entrants, sensitivity increases with age, largely due to higher mortality at older ages. For example, adding one new active individual per year at age 20–24 increases the final widowed population by about 9 individuals.

5.2. Sensitivity of satus-specific cash flows

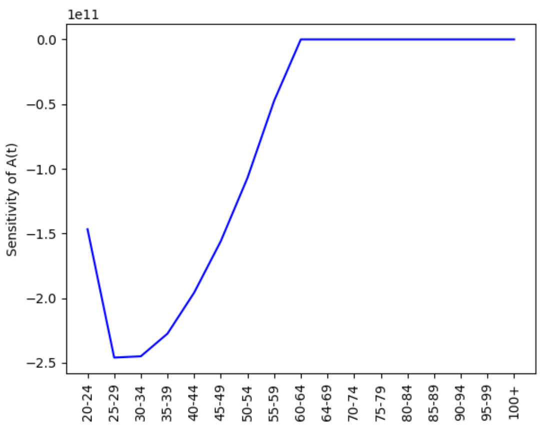
Figure 8, 9, 12 and 11 draw respectively the sensitivity of the cash flows $A^l(T)$, $R^l(T)$, $I^l(T)$, and $W^l(T)$ at the final time point $T = 60$, to variations in age-specific mortality, retirement, invalidity, marital percentage and remarriage rates, as well as number of new entrants applied in every projection year, as a function of the age at which the rate is altered, as specified in equation (46).



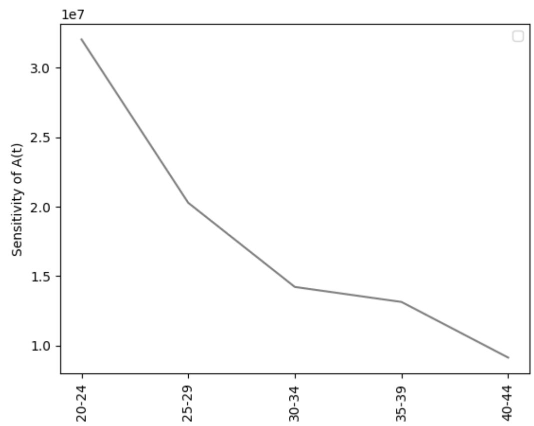
(8a) Sensitivity to mortality.



(8b) Sensitivity to retirement.

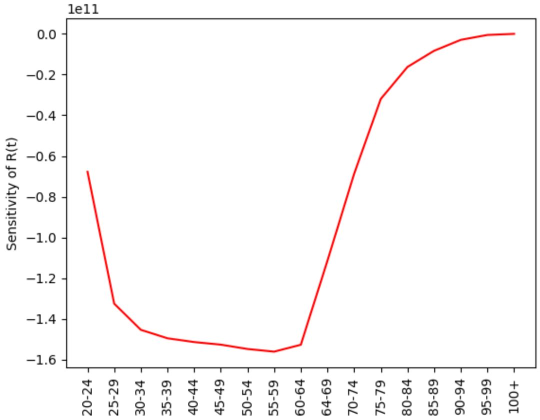


(8c) Sensitivity to invalidity.

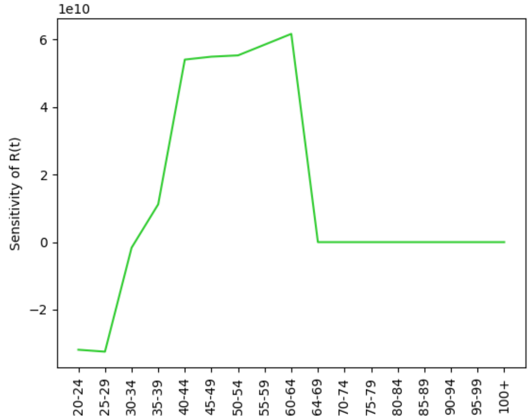


(8d) Sensitivity to new entrants.

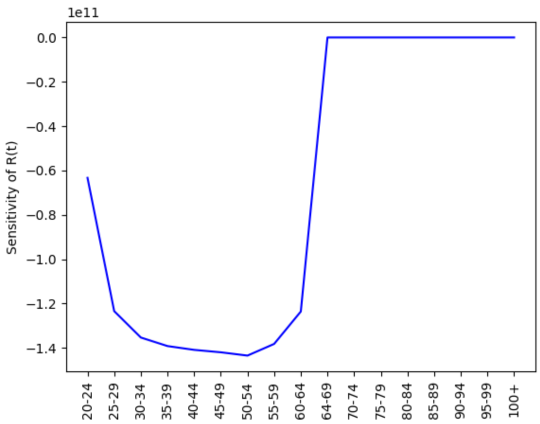
Figure 8. Sensitivity of the total insurable earnings



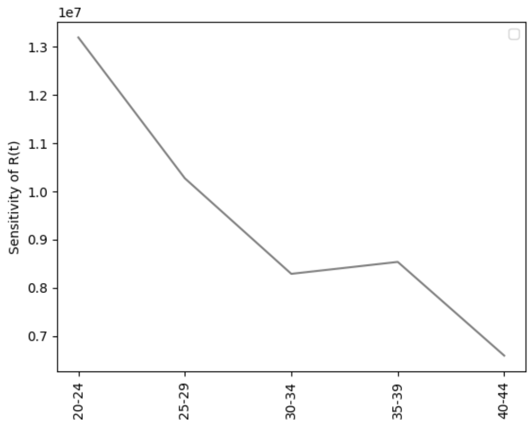
(9a) Sensitivity to mortality.



(9b) Sensitivity to retirement.

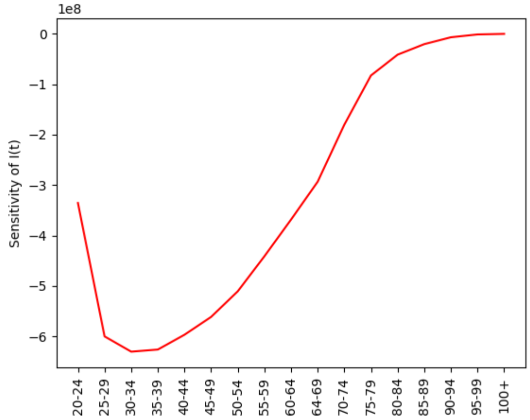


(9c) Sensitivity to invalidity.

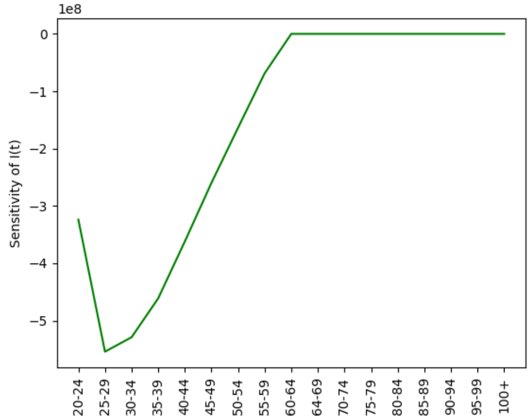


(9d) Sensitivity to new entrants.

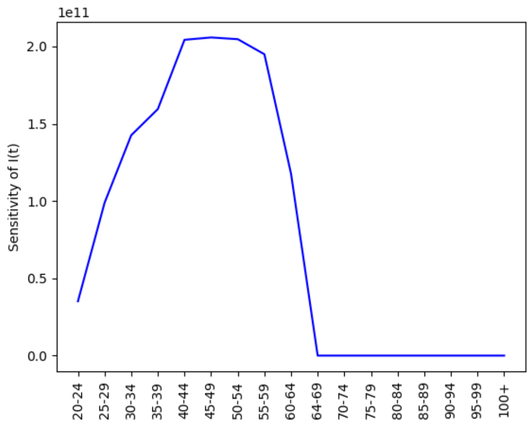
Figure 9. Sensitivity of the total retirement benefit payments



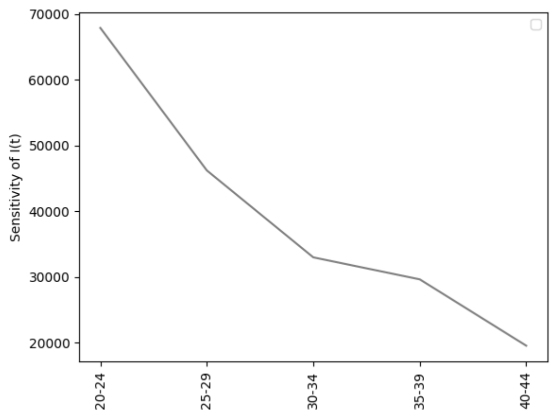
(10a) Sensitivity to mortality.



(10b) Sensitivity to retirement.

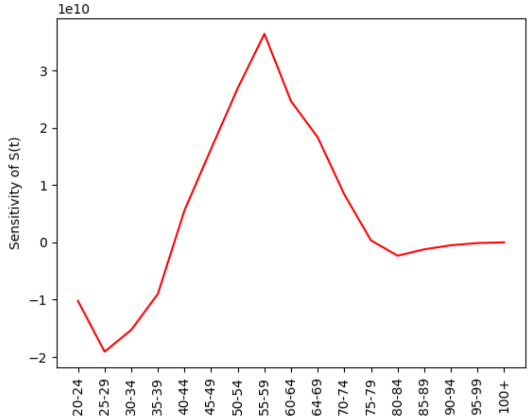


(10c) Sensitivity to invalidity.

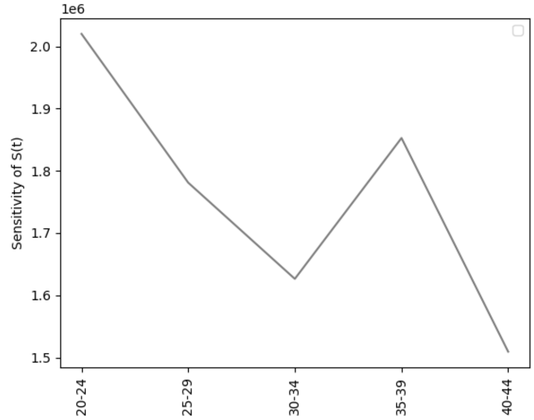


(10d) Sensitivity to new entrants.

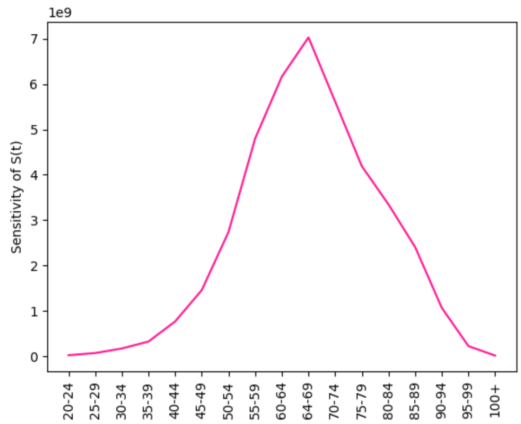
Figure 10. Sensitivity of the total invalidity benefit payments



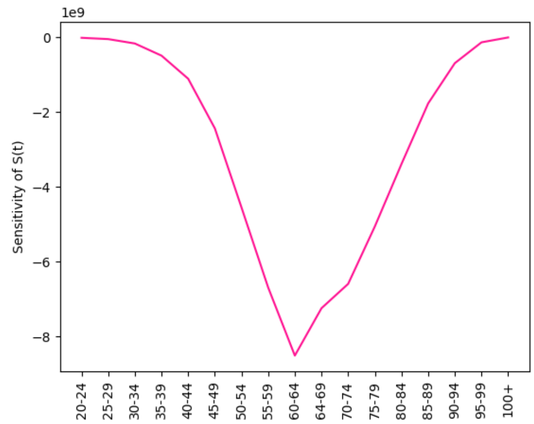
(11a) Sensitivity to mortality.



(11b) Sensitivity to retirement.



(11c) Sensitivity to invalidity.



(11d) Sensitivity to new entrants.

Figure 11. Sensitivity of the total survivor benefit payments

The graphs revealed two key insights that we highlighted.

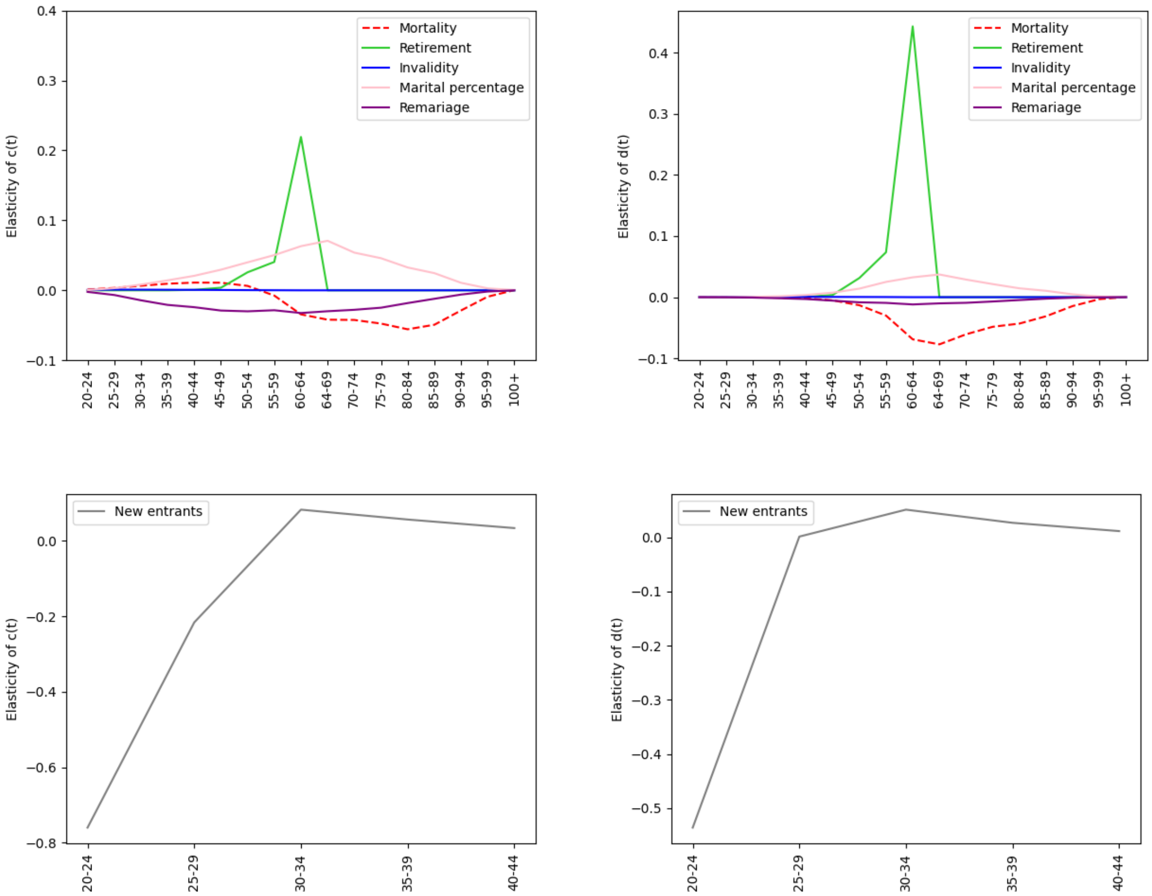
- First, the majority of cash flow sensitivity graphs show similar shapes as population sensitivity graphs. To avoid overloading the paper, the reader is invited to read the financial sensitivity results as we have already done for the population sensitivity results in section 5.1.
- Second, in some graphs, such as Figures 9.b and 11.a, negative sensitivities appear at younger ages. This outcome reflects a timing trade-off between the number of beneficiaries and the level of benefits. When retirement or survivorship occurs earlier, more individuals start receiving pensions immediately, but these pensions are based on lower salary histories, since additional years of wage growth are forgone. Conversely, when retirement is delayed, fewer individuals exit the workforce in the short run and contributions continue, but once they eventually retire their pensions are calculated on higher salary bases, resulting in larger benefit amounts. Over the projection horizon, the dominance of the "more beneficiaries with lower pensions" effect at younger exit ages explains the negative values observed in the cash flow sensitivities. These results highlight that the financial impact of retirement timing depends on the interaction between contribution duration, wage escalation, and pension indexation. For policymakers, the key implication is that raising

retirement ages can help contain short- and medium-term demographic pressures, but unless accompanied by adjustments to benefit formulas, it may also increase long-term expenditure commitments.

Over a long projection horizon, the balance between these two forces determines the sign of the sensitivity. At younger ages, the reduction in per-capita pension value from earlier exits dominates the increase in beneficiary numbers, which leads to negative sensitivities in the cash flow projections. This does not imply that delaying retirement always reduces expenditures; in fact, later retirement can increase future pension outlays by locking in higher benefit amounts. Instead, these results highlight that the financial impact of retirement timing depends critically on the interaction between contribution years, salary escalation, and pension indexation. For policymakers, the key message is that retirement age reforms may alleviate short- and medium-term demographic pressure, but unless accompanied by adjustments to benefit formulas, they can also increase long-term expenditure commitments.

5.3. *Elasticity of the dependency ratio and the PAYG cost rate*

Although previous figures illustrate how additive perturbations affect different age groups, direct comparisons between the parameters are challenging. This is because new entrants is expressed in absolute numbers, whereas mortality, disability, retirement, remarriage and marital percentage are represented as rates. To enable meaningful comparisons across these different types of inputs, elasticities are used. Figures (12a) and (12b) present the elasticity of the dependency ratio and the PAYG cost rate at time T = 60 in response to changes in mortality, disability, retirement, remarriage, marital percentage and new entrants, plotted by the age at which each perturbation is applied.



(12a) Elasticity of the dependency ratio.

(12b) Elasticity of the PAYG cost rate.

Figure 12. Elasticity of the dependency ratio and the PAYG cost rate

The dependency ratio is sensitive to demographic changes across all age groups, though the scale and direction of these effects vary considerably. Among the most influential factors are changes in the number of new labor market entrants at ages 20–24. Because this age group is large and contributes to the workforce over many years without significantly increasing the retired population, policies that support youth employment and smooth school-to-work transitions can have a strong, long-term effect in reducing the dependency ratio.

The analysis also reveals that the elasticity of the dependency ratio to mortality rates becomes negative after age 50 and peaks around age 80. This means that improvements in survival at older ages, while socially desirable, tend to increase the number of dependents and thus raise the dependency ratio. However, the overall impact of mortality changes is moderate when compared to other demographic factors.

Invalidity rates have the smallest proportional effect on the dependency ratio, suggesting that policy interventions in this area would yield limited returns in terms of improving demographic balance.

By contrast, retirement patterns have a particularly significant impact. An elasticity of -0.2 for the retirement rate at ages 60–64 means that a 1% reduction in the retirement rate at these ages leads to a 0.2% reduction in the dependency ratio at the end of the projection horizon. *This is a pronounced effect, underscoring that adjustments to the effective retirement age are among the most powerful levers available to mitigate upward pressure on the dependency ratio.*

Finally, changes in marital and remarriage rates among older adults moderately affect the dependency ratio, as they influence the size of the dependent population. A 1% increase in the marital rate for individuals aged 65–69, for instance, is associated with an increase in the dependency ratio of over 0.07%. However, these demographic behaviors are less directly influenced by policy compared to labor market and retirement-related measures.

The PAYG cost rate is more sensitive to mortality and retirement rates than the dependency ratio, because these parameters directly affect the number of beneficiaries and the duration of benefit payments, thus significantly impacting total expenditures. Lower mortality increases the length of pension payments, while a higher retirement rate raises the number of pensioners, both leading to greater financial costs. Conversely, parameters like disability, marital percentage, and remarriage primarily influence the demographic composition of dependents without strongly affecting individual benefit amounts or duration, making the dependency ratio more sensitive to these factors than the PAYG cost rate.

6. Discussion

Pension projections rely heavily on detailed demographic data. The projection of a typical pension scheme, in which 96 ages (from 15 to 110) are projected over 60 years on the basis of annual rates of mortality, retirement, disability, new entrants and marital and remarriage rates, includes over 34,560 pieces of information. All this information produces a diverse range of results, such as population and financial vectors, status-specific population sizes and cash flows, as well as key ratios (dependency ratio, PAYG cost rate, ...). Any change in the parameters at any point in time will alter the results. The sensitivity structure serves to measure and quantify how these changes affect the outcomes.

Over the past few decades, sensitivity analysis has undergone a conceptual shift across various scientific disciplines. Once considered a supplementary diagnostic tool, it is increasingly viewed as integral to model interpretation and validation. In fields such as systems biology, climate science, and ecological modeling, the sensitivity of outputs to changes in inputs is now regarded as a fundamental component of understanding the model itself [28, 35]. This perspective asserts that model comprehension remains incomplete without an explicit analysis of how uncertainties in inputs propagate through the system. Consequently, sensitivity analysis has evolved from an optional post hoc procedure to a core aspect of model-based inference, influencing both theoretical development and policy-relevant decision-making [36].

In the field of pensions, it is common practice to evaluate projections under a range of pre defined scenarios—such as optimistic, baseline, and pessimistic conditions. These scenario-based methods serve as a form of perturbation analysis, capturing the effects of broad, simultaneous shifts in key input variables, but the number of plausible scenarios is limitless. In contrast, sensitivity and elasticity analyses offer precise, quantitative insights into how changes in individual rates affect model outcomes. For instance, from the sensitivity and elasticity analysis, we know, even without testing multiple scenarios, that given changes in mortality rates would have a greater impact on the number of retirees and the corresponding benefits than would changes in mortality or disability rates. We know that given changes in disability rates would have a greater impact on the number of invalids and the corresponding benefits than would changes in mortality or retirement rates. We know that changes in parameters will have different effects on the dependency ratio and the PAYG cost rate. Overall, we conclude that recruitment and retirement patterns are on the top of parameters affecting the financial sustainability of the pension scheme, followed by mortality, marital percentage, remarriage and invalidity. Such insights can guide which types of scenario adjustments are most relevant to explore.

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A. Derivations

This section provides the derivation of the sensitivity results outlined in Section 3, utilizing matrix calculus as developed in [27]. A comprehensive overview of the method, including numerous applications in demography, is available in [11]. For an introductory treatment of the mathematical principles underlying matrix calculus, see [1]. The computation of derivatives in this context begins with the evaluation of differentials. The differential of a matrix A is the matrix of differentials of the elements of A:

$$dA = (da_{ij})$$

The computation of differentials follows rules analogous to those in scalar calculus. In particular, the product rule will be used: for any two matrices A and B,

$$d(AB) = (dA)B + A(dB)$$

In many instances, we apply the *vec* operator, which flattens a matrix into a single column vector by vertically stacking its columns. An example is,

$$\text{vec}\begin{bmatrix} l & m \\ n & o \end{bmatrix} = \begin{bmatrix} l \\ n \\ m \\ o \end{bmatrix}$$

A key result, originally established by Roth [33], states that for any matrices A, B and C of compatible dimensions,

$$\text{vec}(ABC) = (C^T \otimes A)\,\text{vec}(B)$$

The vec of a matrix differential is

$$\text{vec}(dA) = d\,\text{vec}(A)$$

A.1. Derivatives of pension projections

Starting from the projection equation (**??**) in the population model, we differentiate both sides to obtain

$$d\bar{\boldsymbol{n}}(t+1) = \bar{\boldsymbol{A}}(t)\,d\bar{\boldsymbol{n}}(t) + d\bar{\boldsymbol{A}}(t)\,\bar{\boldsymbol{n}}(t)$$

Next, we apply the vec operator to both sides. If we let $d\bar{\boldsymbol{A}}(t)\,\bar{\boldsymbol{n}}(t) = I_{4w}\,d\bar{\boldsymbol{A}}(t)\,\bar{\boldsymbol{n}}(t)$, then by Roth's identity, we obtain

$$d\bar{\boldsymbol{n}}(t+1) = \bar{\boldsymbol{A}}(t)\,d\bar{\boldsymbol{n}}(t) + \left((\bar{\boldsymbol{n}})^T(t) \otimes I_{4w}\right) d\,\text{vec}\,\bar{\boldsymbol{A}}(t)$$

Observe that $\bar{\boldsymbol{A}}(t)$ and $\bar{\boldsymbol{b}}(t)$, evaluated at time t, depend on the parameter vector $\theta(s)$ at time s. By applying the chain rule from matrix calculus, the derivative with respect to $\theta(s)$ is given by:

$$\frac{d\bar{\boldsymbol{n}}(t+1)}{d\theta^T(s)} = \bar{\boldsymbol{A}}(t)\frac{d\bar{\boldsymbol{n}}(t)}{d\theta^T(s)} + \left((\bar{\boldsymbol{n}})^T(t) \otimes I_{4w}\right)\frac{d\,\text{vec}\,\bar{\boldsymbol{A}}(t)}{d\theta^T(s)} + \frac{d\bar{\boldsymbol{b}}(t)}{d\theta(s)}$$

The system evolves dynamically in terms of the derivative matrix $\frac{d\bar{\boldsymbol{n}}(t)}{d\theta(s)}$. Provided that the parameter vector $\theta(s)$ affects only the age-specific rates and not the baseline population, the equation can be solved recursively from the initial state:

$$\frac{d\bar{\boldsymbol{n}}(t)}{d\theta^(s)} = 0_{4w\times p}$$

In the same way, we demonstrate for the financial model that

$$\frac{d\bar{\boldsymbol{cash}}(t+1)}{d\theta^T(s)} = \bar{\boldsymbol{A}}^l(t)\frac{d\bar{\boldsymbol{cash}}(t)}{d\theta^T(s)} + \left(\bar{\boldsymbol{cash}}^T(t) \otimes I_{4w}\right)\frac{d\,\text{vec}(\bar{\boldsymbol{A}}^l(t))}{d\theta^T(s)} + \left(\mathbf{1}_p^T \otimes \bar{g}\right)\frac{d\bar{\boldsymbol{b}}(t)}{d\theta^T(s)}$$

The equation is iterated from the initial condition

$$\frac{d\,\bar{\boldsymbol{cash}}(t)}{d\theta^T(s)} = 0_{4w\times p}$$

A.2. Derivatives of projection matrices

We move now to the derivatives of projection matrices $\bar{\boldsymbol{A}}$ and $\bar{\boldsymbol{A}}^l$, given in section 3.2. From equation (**??**), we write

$$\bar{\boldsymbol{A}} = Q_{\mathbb{U}}\mathbb{U} + Q_{\mathbb{F}}\mathbb{F} \tag{48}$$

where $Q_{\mathbb{U}} = K_{4w}^T \mathbb{D} K_{4w}$ and $Q_{\mathbb{F}} = K_{4w}^T \mathbb{H} K_{4w}$ are the (constant) matrix products appearing in the definition of $\bar{U}$ and $\bar{F}$.

Differentiating $\bar{\boldsymbol{A}}$ in (48) gives

$$d\,\mathrm{vec}\,\bar{\boldsymbol{A}} = (I_{4w} \otimes \mathbf{Q}_{\mathbb{U}})\,d\,\mathrm{vec}\,\mathbb{U} + (I_{4w} \otimes \mathbf{Q}_{\mathbb{F}})\,d\,\mathrm{vec}\,\mathbb{F} \tag{49}$$

This requires the differentials of $\mathbb{U}$ and $\mathbb{F}$. Differentiating $\mathbb{U}$ gives

$$d\mathbb{U} = \sum_{i=0}^{w-1} E_{ii}\,dU_i \tag{50}$$

Applying the vec operator to $d\mathbb{U}$ gives

$$d\,\mathrm{vec}\mathbb{U} = \sum_{i=0}^{w-1} (E_{ii} \otimes K_{4w} \otimes I_4)\,(\mathrm{vec}(I_w) \otimes I_{4^2})\;d\,\mathrm{vec}\,U_i \tag{51}$$

using the results of Magnus and Neudecker ([27], Theorem 11); see also ([25], Appendix B) on the derivative of the Kronecker product.

Differentiation of $\mathbb{F}$ proceeds in the same fashion, yielding

$$d\,\mathrm{vec}\mathbb{F} = \sum_{i=0}^{w-1} (E_{ii} \otimes K_{4w} \otimes I_4)\,(\mathrm{vec}(I_w) \otimes I_{4^2})\;d\,\mathrm{vec}\,F_i \tag{52}$$

Substituting (51) and (52) into (49) and applying the chain rule, the derivatives of $\bar{\boldsymbol{A}}(t)$ with respect to $\theta(s)$ is then

$$\frac{d\,\mathrm{vec}\,\bar{\boldsymbol{A}}(t)}{d\theta^T(s)} = (I_{4w} \otimes Q_{\mathbb{U}}) \sum_{i=0}^{w-1} (E_{ii} \otimes K_{4w} \otimes I_4)\,(\mathrm{vec}(I_w) \otimes I_{4^2})\,\frac{d\,\mathrm{vec}\,U_i(t)}{d\theta^T(s)}$$
$$+(I_{4w} \otimes Q_{\mathbb{F}}) \sum_{i=0}^{w-1} (E_{ii} \otimes K_{4w} \otimes I_4)\,(\mathrm{vec}(I_w) \otimes I_{4^2})\,\frac{d\,\mathrm{vec}\,F_i(t)}{d\theta^T(s)}$$

In the same way, we obtain the derivatives of $\bar{\boldsymbol{A}}^l$.

A.3. Sensitivity and elasticity of projection outcomes

We show here the derivatives of the status-specific population sizes and the corresponding cash flows with respect to $\theta(s)$. For example, differentiating $A(t) = \mathbf{1}_w^T (I_w \otimes \mathbf{e}_1^T)\,\bar{\boldsymbol{n}}(t)$ gives

$$dA(t) = \mathbf{1}_w^T (I_w \otimes \mathbf{e}_1^T)\,d\,\bar{\boldsymbol{n}}(t)$$

from which it follows that

$$\frac{dA(t)}{d\,\bar{\boldsymbol{n}}(t)} = \mathbf{1}_w^T (I_w \otimes \mathbf{e}_1^T)$$

The chain rule,

$$\frac{dA(t)}{d\theta^T(s)} = \frac{dA(t)}{d\,\bar{\boldsymbol{n}}(t)} \cdot \frac{d\,\bar{\boldsymbol{n}}(t)}{d\theta^T(s)}$$

gives equation (38).

Differentiation of $A^l(t)$, $I(t)$, $I^l(t)$, $R(t)$, $R^l(t)$, $W(t)$, and $W^l(t)$ proceeds in the same fashion.

To obtain the sensitivity of the dependency ratio, differentiate $c(t) = \frac{a^T \bar{\boldsymbol{n}}(t)}{b^T \bar{\boldsymbol{n}}(t)}$:

$$dc(t) = \frac{b^T \bar{\boldsymbol{n}}(t)\, a^T - a^T \bar{\boldsymbol{n}}(t)\, b^T}{[b^T \bar{\boldsymbol{n}}(t)]^2}$$

Applying the chain rule gives equation (1212b).

B. Projection assumptions

In the numeric application, we take the year 2020 as a basis and undertake demographic and financial projections of the CMR pension scheme until 2080, according to the assumptions below. Assumptions are aligned with those adopted by ACAPS in its last actuarial valuation.

The demographic assumptions on which we have based our projections are as follows:

- **Age assumptions**: Age assumptions include entry age, mandatory retirement age, and survival limit age. We suppose a minimum entry age into the scheme of 20. The mandatory retirement age is set at 65, and the survival limit age is fixed to 110.
- **The mortality assumption** is defined in terms of the age-specific probabilities of death. ACAPS assumes that mortality rates of the pension scheme population correspond to those of the life table TD88–90 during the projection period.
- **The invalidity assumption** is specified in terms of age-specific probabilities of entering disability. Empirical evidence from historical data (1985–2020) on civil servants indicates that the annual number of new disability cases relative to the active workforce has remained low and stable, averaging about 0.01%. The annual disability-to-active ratios are summarized in Table 3. On this basis, a constant invalidity rate of 0.01% across all ages is adopted for the projections.

Table 3. Annual Disability Rates for Civil Servants (1985–2020)

Year	Active Civils	Disabled (*Réformés*)	Disability Rate (%)
1985	313,657	41	0.01
1986	323,227	48	0.02
1987	332,627	27	0.01
⋮	⋮	⋮	⋮
2018	645,839	103	0.02
2019	635,525	66	0.01
2020	620,739	13	0.00
		Average	0.01

While this simplification does not capture potential age-specific variations in disability incidence, it reflects the historical experience of the studied population and aligns with actuarial practice commonly applied in social security projections (e.g., Iyer 1999).

- **The retirement assumption** is based on age-specific retirement proportions derived from historical data of the CMR's Civilian pension scheme over recent years. The observed proportions, r_i, increase steadily with age, starting from a very low rate of approximately 0.06% for ages 40–44, rising to about 14% for ages 55–59, and reaching full retirement at ages 60–64. This progression reflects typical retirement behavior, where most individuals retire near the mandatory retirement age. Incorporating these empirical rates ensures a realistic and data-driven representation of retirement patterns in the pension projections.

Table 4. Age-specific retirement proportions, r_i, by age group

Age Group	Retirement Proportion (r_i)
40–44	0.0006
45–49	0.0042
50–54	0.045
55–59	0.14
60–64	1.00

- **Number of new entrants** is set to 25.000 per year (34% women and 66% men). An age distribution of new entrants from age 20 to 45, $(\mu_{20}, \ldots, \mu_{45})$ is also given.

- **Age- and sex-specific marital percentages and age differences between spouses** are based on data from the *National Population Survey and Family Health* (ENPSF–2018). This survey analyzes key aspects of nuptiality in the Moroccan population, including marital status at the time of the survey, age at first marriage, consanguinity, age differences between spouses, early marriage, and polygamy. The ENPSF provides detailed age- and sex-specific marital percentages (w_i) and average spousal ages (y_i) for each age group $i = 0, \ldots, w-1$ (see [30]). Table 5 summarizes the marital status distribution by age and sex for individuals aged 15 and older. The data show that the proportion of married individuals increases with age, peaking in middle age groups before declining in older age groups, while the proportion of single individuals decreases with age. These patterns are consistent with known demographic behaviors and provide a realistic basis for modeling marital structure in social security projections.

Table 5. Marital status distribution (%) by age group and sex, ENPSF 2018

Age Group	Single		Married	
	Male	Female	Male	Female
15-19	99.5	90.9	0.4	8.8
20-24	94.3	57.0	5.5	40.9
25-29	72.0	30.6	26.9	65.1
30-34	41.8	22.3	56.2	72.3
35-39	21.4	17.5	76.4	76.3
40-44	14.5	15.1	83.5	78.5
45-49	7.6	14.4	90.5	75.0
50-54	5.5	8.7	91.8	76.8
55-59	3.3	6.3	93.4	70.0
60-64	2.0	5.3	94.6	58.7
65-69	1.0	2.2	94.9	58.5
70-74	0.8	1.0	92.1	43.4
75-79	0.0	0.9	91.7	32.1
80 and above	0.4	0.7	77.8	16.0

Table 6 presents the average age difference between spouses by age group. The data indicate that husbands tend to be older than their wives by approximately 7 to 11 years, with the difference decreasing slightly from younger to middle age groups.

Table 6. Average age difference between spouses (husband's age minus wife's age) by age group

Age Group	Average Age Difference (years)
15-19	10.8
20-24	9.4
25-29	8.0
30-34	7.9
35-39	7.5
40-44	7.9
45-49	7.7

- **Remarriage probabilities** are based on age-specific rates from the INSEE study on widows and widowers remarried in France between 1951 and 1952. These rates show a clear decrease with age, reflecting the general tendency for younger widowed individuals to remarry more frequently than older ones. Although the data are historical and not specific to Morocco, they remain one of the few available sources with detailed age-based remarriage patterns. In the absence of more recent national statistics, they provide a reasonable approximation. Their inclusion improves the projection of survivor benefits by accounting for realistic marital transitions after widowhood, which are important for assessing the long-term liabilities of the pension scheme.

The financial hypotheses on which we have based our projections are as follows:

- A sex-age-related salary scale function.
- The rate of salary escalation in each projection year is 2%.
- The rate of pension indexation in each projection year is 1%.

C. Projection Algorithm (Pseudocode)

```
# Input:
# - U_i, F_i for i = 0,...,w-1: 4x4 status-specific matrices (survival and transitions)
# - D_j, H_j for j = 1,...,4: w x w age-progression and flow-assignment matrices
# - n_bar(t): population vector of length 4w (statuses within each age class)
# - b_bar(t): inflow vector of length 4w (new entries, e.g., new actives)
# - w: number of age classes

# ----- Vec-permutation operations -----

Function apply_K(v):
    # Converts "statuses-within-ages" to "ages-within-statuses"
    # Input: v of length 4w, ordered as:
    # [n_{0,1}, n_{0,2}, n_{0,3}, n_{0,4}, ..., n_{w-1,1}, ..., n_{w-1,4}]

    M = reshape(v, shape=(4, w))            # Shape: 4xw
    M_T = transpose(M)                      # Shape: w x 4
    v_permuted = flatten(M_T, order='C')   # Flatten row-wise (C order) to length 4w
    return v_permuted

Function apply_KT(v):
    # Inverse operation: "ages-within-statuses" to "statuses-within-ages"
    # Input: v ordered as:
    # [n_{0,1}, n_{1,1}, ..., n_{w-1,1}, n_{0,2}, ..., n_{w-1,4}]

```

```
    M_T = reshape(v, shape=(w, 4))         # Shape: w x 4
    M = transpose(M_T)                     # Shape: 4xw
    v_original = flatten(M, order='C')     # Flatten row-wise to original ordering
    return v_original

# ----- Main projection algorithm -----

# Step 1: Construct block-diagonal sparse matrices (4w x 4w)
U = block_diag(U_0, U_1, ..., U_{w-1})
F = block_diag(F_0, F_1, ..., F_{w-1})
D = block_diag(D_1, D_2, D_3, D_4)
H = block_diag(H_1, H_2, H_3, H_4)

# Step 2: Apply U_bar to n_bar(t)
temp1 = U @ n_bar(t)          # Apply U
temp2 = apply_K(temp1)        # Permute via K
temp3 = D @ temp2             # Apply age progression
U_bar_n = apply_KT(temp3)     # Permute back via K^T

# Step 3: Apply F_bar to n_bar(t)
temp4 = F @ n_bar(t)          # Apply F
temp5 = apply_K(temp4)        # Permute via K
temp6 = H @ temp5             # Assign flows to new age classes
F_bar_n = apply_KT(temp6)     # Permute back via K^T

# Step 4: Update population vector
n_bar(t+1) = U_bar_n + F_bar_n + b_bar(t)

# Output:
# - n_bar(t+1): population vector at the next time step
```