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On the Wagstaff Prime Numbers in k-Fibonacci Sequences

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Abstract A Wagstaff prime is a prime number that can be written in a special exponential form involving powers of two. For any integer greater than or equal to two, the so-called k-generalized Fibonacci sequence is a linear sequence in which each term is obtained by adding together the preceding k terms, beginning with a fixed set of initial values. In this paper, we prove that the number three is the only Wagstaff prime that appears in any of these generalized Fibonacci sequences. Our proof makes use of lower bounds for linear forms in logarithms of algebraic numbers and a refined version of the Baker–Davenport reduction method, originally developed by Dujella and Pethő.

Keywords Generalized Fibonacci sequence, Wagstaff prime, Linear forms in logarithms.

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1. Introduction

Consider an integer $k \geq 2$. The k–generalized Fibonacci (GF) sequence $\left(W_n^{(k)}\right)$, alternatively referred to as the k–bonacci or k–step Fibonacci sequence, is a recurrence sequence of order k defined as

$$W_n^{(k)} = W_{n-1}^{(k)} + \cdots + W_{n-k}^{(k)},$$

where the initial conditions are given by $W_{-(k-2)}^{(k)} = W_{-(k-3)}^{(k)} = \cdots = W_0^{(k)} = 0$, and $W_1^{(k)} = 1$.

For each value of k, the GF sequence produces a distinct sequence. Notably, when $k = 2$, it corresponds to the classical Fibonacci sequence, for $k = 3$, these sequences are known as Tribonacci, and for $k = 4$, we observe the Tetranacci sequence. For further clarification, we can outline a successive definition of the GF sequence $W_n^{(k)}$ for $k = 2$, 3, and 4 in the table below.

For an in–depth explanation of the initial non–zero terms of this sequence as k takes on smaller values, the details can be found in reference [6]. Also, we can directly extract Table 1.

The investigation of Diophantine equations incorporating linear recurrence sequences has emerged as a pivotal research domain in number theory in recent decades, owing to its significant contributions to understanding the distribution of prime numbers and the structural properties of generalized Fibonacci and Lucas sequences.

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k	Name	Initial conditions	Recurrence relation
2	Fibonacci	$F_0 = 0,\ F_1 = 1$	$F_n = F_{n-1} + F_{n-2}$
3	Tribonacci	$T_0 = 0,\ T_1 = T_2 = 1$	$T_n = T_{n-1} + T_{n-2} + T_{n-3}$
4	Tetranacci	$W_0^{(4)} = W_1^{(4)} = 0,\ W_2^{(4)} = W_3^{(4)} = 1$	$W_n^{(4)} = W_{n-1}^{(4)} + W_{n-2}^{(4)} + W_{n-3}^{(4)} + W_{n-4}^{(4)}$

Table 1. The definition of k-Fibonacci sequences for $k = 2, 3, 4$.

. Particularly, the analysis of the distribution of distinct numbers defined by explicit formulas, notably prime numbers within the confines of linear recurrence sequences, has attracted substantial scholarly attention. For further exploration of studies on these equations, in [6] Jhon J. Bravo and Jose L. Herrera determine all k-Fibonacci and k-Lucas numbers which are Fermat numbers, and provide further results that generalize these findings to broader classes of recurrence sequences. In [10], the authors explore a generalization of the classical Pell sequence, referred to as the k-generalized Pell sequence, which is defined by a higher-order recurrence relation. Recently, in [13], the authors established that 44 is the largest Tribonacci number containing only a single distinct digit. Motivated by the approach presented in [16], Bravo.J.J, and Luca.F in [14] investigated the powers of two that appear in k-generalized Fibonacci sequences.

The study of special prime numbers within linear recurrence sequences has long attracted attention due to its intersection of algebraic, analytic, and computational number theory. Investigating whether primes of specific forms—such as Wagstaff or Mersenne primes—occur among dense recurrence sequences like the k-generalized Fibonacci or Lucas sequences provides insight into the rarity of such primes in structured numerical systems. Beyond the intrinsic arithmetic interest, this direction contributes to a broader understanding of how special primes are distributed within classical sequences and tests the limits of current Diophantine methods, including linear forms in logarithms and the Baker–Davenport reduction. Moreover, there exist potential, albeit speculative, connections with cryptographic applications, since primes of these forms are sometimes candidates in contexts related to the discrete logarithm problem and related hardness assumptions. These motivations collectively underline the natural and multifaceted importance of the present investigation. For further exploration of studies on these equations, we direct the readers to references [1, 2, 4, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31].

Furthermore, in the theory of prime numbers, a Wagstaff prime is characterized as a prime number of the form $\frac{2^m+1}{3}$, where m is an odd prime. Notable examples of Wagstaff primes include:

$$3,\ 11,\ 43,\ 683,\ 2731,\ 43691, 174763,\ 2796203,\ 715827883,$$
$$2932031007403,\ 768614336404564651, \ldots.$$

Recalling that the largest known Wagstaff prime in 2021 was $(2^{15136397} + 1)/3$, discovered by Ryan Propper, which comprised exactly 4556209 digits, we recommend referring to reference [17] for verification. This leads us to the classic inquiry in prime number theory about whether there exists an infinite number of primes following a particular formula. As of now, the question of whether Wagstaff primes are infinite remains unanswered. However, Marshall, S has put forth a proposed proof on this issue in [7]. Additionally, in [5], Berrizbeitia et al. derived a necessary condition for primality of the Wagstaff primes in their study. The significance of Wagstaff primes in number theory stems from their connection to Mersenne primes, as postulated by the Bateman–Selfridge–Wagstaff conjecture (refer to [3]).

Mersenne numbers have been integral to various equations and play a crucial role in their development, it has also proven to be a fruitful approach for studying even perfect numbers. For instance, in [18], Altassan and Alan explored Mersenne numbers within k–Lucas sequences, while another study delved into Mersenne numbers as terms in k–Fibonacci sequences (see [19]). Bachabi and Togbé [20] concentrated on solving equations involving Padovan, Perrin, Narayana, Fermat, and Mersenne numbers. Building upon these efforts, we will introduce a novel examination of the GF sequences. This approach may yield results that further enhance our understanding of the relationship between Wagstaff and Mersenne primes, and potentially shed light on the conjectured connections

between them mentioned above . And this by answering the natural question of which k-Fibonacci numbers are Wagstaff primes, a question we address in this paper.
In this present study, our primary objective is to discover Wagstaff primes present in GF sequences and this represents a novel study on exponential Diophantine equations concerning the distribution of Wagstaff prime numbers in generalized Fibonacci sequences., particularly through the scrutiny of the Diophantine equation

$$W_n^{(k)} = \frac{2^m + 1}{3}, \tag{1.1}$$

where n, k, and m are positive integers, with m being an odd prime and $k \geq 2$.

Interestingly, to the best of our comprehension, no previous study initiative has examined this specific investigation.

2. Main result

The central outcome of this investigation is encapsulated in the ensuing theorem.

Theorem 2.1
The number 3 is the only Wagtaff prime number in the k–Fibonacci sequences. Alternatively, we will demonstrate that the Diophantine equation (1.1) has a single positive integer solution given by $(n, k, m) = (4, 2, 3)$, where $k \geq 2$.

To prove Theorem 2.1, we will extensively utilize the properties of the algebraic number logarithmic height, specifically employing Baker's theory, which elegantly links an algebraic number to its degree and conjugates. Furthermore, we will employ the results established by Dresden and Du [12], along with the lower bound obtained by Matveev (Theorem 2 in [8]), to derive a logarithmic estimation of the variables associated with Equation (1.1). As a result, our proof strategy will involve a case–by–case analysis of the variable k. In the case when the number k is small, we have developed a guide based on the Matveev theorem, while when the number k is large, in this case we present a strategy based on the principle of contradiction. The accounts presented in this paper were made using the Mathematica program.

3. Preliminary results

In this section, we present the foundational results that will support our main findings. These results aim to lay the groundwork and facilitate the proofs and analyses that follow in this study.

3.1. Linear forms in logarithms

The linear forms theorems in algebraic number logarithms are a crucial tool primarily used in transcendental number theory. Developed by A. Baker in 1975, this theory has become the principal method for studying the solutions of certain Diophantine equations defined by recurrence sequences. Considering any algebraic number $\eta \neq 0$ of degree s over $\mathbb{Q}$, the logarithmic height of η is defined by the formula

$$h(\eta) := \frac{1}{s}\left(\log a_0 + \sum_{i=1}^{s} \log(\max\{|\eta^{(i)}|, 1\})\right),$$

with $a_0 x^s + a_1 x^{s-1} + \cdots + a_s = a_0 \prod_{i=1}^{s}(x - \eta^{(i)})$ being the minimal primitive polynomial over $\mathbb{Z}$ having positive leading coefficient a_0 and η as a root, and $\eta^{(i)}$ denotes the conjugates of η.

Note that if $\eta = p/q \in \mathbb{Q}$, the above definition leads to $h(\eta) = \log \max\{|p|, q\}$. Below are several properties of the logarithmic height function that will be utilized throughout this paper without further reference.

Let η, δ algebric numbers, we have

$$h(\eta \pm \delta) \leq h(\eta) + h(\delta) + \log 2, \tag{3.1}$$

$$h(\eta\delta) \leq h(\eta) + h(\delta), \tag{3.2}$$

$$h(\eta^s) = |s| h(\eta) \quad (s \in \mathbb{Z}). \tag{3.3}$$

With this notation, the following result, proven by Matveev in [8] and by Bugeaud et al. [9] (Theorem 9.4), serves as the central theorem in the proof of Theorem 2.1.

Theorem 3.1
Given real algebraic numbers $\delta_1, \ldots, \delta_s$ in an algebraic number field $\mathbb{K}$ of degree D over $\mathbb{Q}$, assume that $q_1, \ldots, q_s$ are non–zero integers, and let B be a real number such that $B \geq \max\{|q_1|, \ldots, |q_s|\}$. If $\prod_{i=1}^{s} \delta_i^{q_i} \neq 1$, then

$$\log \left| \prod_{i=1}^{s} \delta_i^{q_i} - 1 \right| \geq -1.4 \cdot 30 \cdot 30^{s+3} \cdot s^{4.5} \cdot D^2 \cdot (1 + \log D)(1 + \log B) \cdot \prod_{i=1}^{s} A_i,$$

where A_j (for a fixed j) is a real number satisfying the condition

$$A_j = \max\{Dh(\delta_j), |\log \delta_j|, 0.16\} \text{ for } j = 1, \ldots, s.$$

3.2. Reduction method

In our computational analysis, we observe that the variables stemming from our issue, as derived through Baker's theory, are excessively large. To mitigate these constraints, we employ the subsequent outcome from Dujella Pethö (refer to [15], Lemma 5a). For a real number α, we denote $\|\alpha\|$ as the minimum of $|\alpha - n|$ for $n \in \mathbb{Z}$, representing the distance between α and the nearest integer.

Lemma 3.2
Let $M > 0$, and consider p/q to be a convergent of the continued fraction of the irrational number κ such that $q > 6M$, and let $A > 0, B > 1$, and μ be real numbers. We assume that

$$\epsilon = \|\mu q\| - M \cdot \|\kappa q\|.$$

In the case where $\epsilon > 0$, no solution exists for the inequality

$$0 < |m\kappa - n + \mu| < AB^{-k},$$

where m, n and k are positive integers with $m \leq M$ and $k \geq \frac{\log(Aq/\epsilon)}{\log B}$.

4. Some properties of the GF sequence

In this section, we will discuss in short some of the main characteristics of the Fibonacci sequence, where the characteristic polynomial associated with this sequence is given in the following form

$$\chi_k(x) = x^k - x^{k-1} - \cdots - x - 1.$$

In [10], Bravo et al. established that $\chi_k(x)$ is irreducible over $\mathbb{Q}[x]$ and possesses a single real root λ_k located outside the unit circle, while the remaining roots are strictly confined within the interior of the unit circle.

Furthermore, λ_k falls within the range of $2(1-2^{-k})$ and 2 (refer to [11]). Let $\lambda_1, \cdots, \lambda_k$ denote the roots of $\chi_k(x)$, where we set $\lambda := \lambda_k$. For simplicity, we generally disregard the dependence of λ on k in notation.

We are considering the function

$$f(x,s) = \frac{x-1}{2+(s+1)(x-2)}. \tag{4.1}$$

where $s \geq 2$ be an integer.

According to Dresden [12], the "Binet–like" formula

$$W_n^{(k)} = \sum_{i=1}^{k} f(\lambda_i, k)\lambda_i^{n-1},$$

holds for all $n \geq 2-k$. Additionally, in [12], a proof was provided for the following approximation

$$|W_n^{(k)} - f(\lambda,k)\lambda^{n-1}| < \frac{1}{2} \text{ holds for } \forall\, n \geq 2-k.$$

The subsequent lemmas encompass essential properties of the GF sequence $(W_n^{(k)})$ and will serve as valuable tools for bounding the variables.

Lemma 4.1
(see [13]) If λ is the dominant root of $\chi_k(x)$, then the following inequality is satisfied for all $n \geq 1$ and $k \geq 2$

$$\lambda^{n-2} \leq W_n^{(k)} \leq \lambda^{n-1}, \tag{4.2}$$

Lemma 4.2
(see [14]) Let n be a positive integer such that $n \geq 2$. Then

$$W_n^{(k)} \leq 2^{n-2}.$$

Furthermore, if $n \geq k+2$, the aforementioned inequality is strict.

Lemma 4.3
(see [14]) Let $k \geq 2$ be a positive integer. The first $k+1$ nonzero terms of the k-generalized Fibonacci sequence are powers of two. More precisely, we have $W_1^{(k)} = 1$, and

$$W_i^{(k)} = 2^{i-2},$$

for all $2 \leq i \leq k+1$.

Lemma 4.4
(see [14]) Consider an integer $n \geq 2$. Let λ denote the dominant root of $\chi_k(x)$, and the function $f(x,s)$ defined in formula (4.1). The following estimates are established:

1. $f(\lambda_i, k) < 2$ for all $1 \leq i \leq k$.
2. $h(f(\lambda,k)) < 4\log(k)$, where $h(.)$ denotes the logarithmic height function.
3. For all $n \geq k+2$, the function $f(\lambda, s)$ satisfies the formula

$$f(\lambda,k)\lambda^{n-1} = 2^{n-2} + \frac{\delta}{2} + 2^{n-1}\eta + \eta\delta,$$

where $|\delta| < \frac{2^n}{2^{\frac{k}{2}}}$ and $|\eta| < \frac{2k}{2^k}$.

5. Upper bounds for n and m with respect to k

Before delving into the results derived in this section, it is essential to note that for the remainder of the article, we can assume $n \geq k+2$ since the initial $k+1$ non–zero terms in the GF sequence $(W_n^{(k)})$ are powers of two (refer to [14]), implying that $n \geq 4$. Given our interest in having the right–hand side of Equation (3.1) as a Wagstaff prime, we can set $m \geq 3$.

Now, we suppose that (n, k, m) constitutes a solution to Equation (3.1). By employing equation (4.2) and Lemma 4.1, we derive

$$\lambda^{n-2} \leq W_n^{(k)} = \frac{2^m+1}{3} \leq 2^{n-2}.$$

Thus,

$$n < (m+1)\frac{\log 2}{\log \lambda} + 1 \text{ and } m < n.$$

By using Lemma 4.4, we acquire the following result

$$m < n < \frac{3}{2}m + \frac{7}{2}. \tag{5.1}$$

This initial inequality, derived in relation to specific variables from Equation (3.1), will be notably useful for subsequent analysis.

The crucial result in this section is given by the following lemma.

Lemma 5.1
Let (n, k, m) be a solution in integers of the Diophantine equation (3.1), with $k \geq 2$ and $n \geq k+2$. Then the following inequality is hold

$$m < n < 3.43 \times 10^{14} \times k^4 \times \log^3 k. \tag{5.2}$$

Proof: Starting first by merging equations (3.1) and (3.3) yields the following result

$$\left| \left(\frac{2^m+1}{3} \right) - f(\lambda, k)\lambda^{n-1} \right| < \frac{1}{2}. \tag{5.3}$$

Subsequently, we utilize Lemma 4.4 to divide both sides of the preceding inequality by $f(\lambda, k)\lambda^{n-1}$, we obtain

$$\left| 2^m \cdot \frac{1}{3} f(\lambda, k)^{-1} \cdot \lambda^{-(n-1)} - 1 \right| < \frac{3}{\lambda^{n-1}}. \tag{5.4}$$

Our aim is to utilize Theorem 3.1 to establish the validity of Inequality (5.2). To achieve this, we consider the following set of data

$$s = 3,\ \delta_1 = 2,\ \delta_2 = \lambda,\ \delta_3 = \frac{1}{3} f(\lambda, k)^{-1},\ q_1 = m,\ q_2 = -(n-1) \text{ and } q_3 = 1.$$

Considering that the algebraic number field $\mathbb{K} := \mathbb{Q}(\lambda)$ contains $\delta_1,\ \delta_2, \delta_3$, and $D = [\mathbb{K} : \mathbb{Q}] = k$, the aforementioned selections prompt an initial verification that $\delta_1^{q_1} \cdot \delta_2^{q_2} \cdot \delta_3^{q_3} - 1 \neq 0$. To accomplish this, we put

$$\Lambda = \delta_1^{q_1} \cdot \delta_2^{q_2} \cdot \delta_3^{q_3} - 1, \tag{5.5}$$

and we assume that $\Lambda \neq 0$. This imply that

$$2^m = 3 \cdot f(\lambda, k) \cdot \lambda^{n-1}.$$

By employing the Galois automorphism $\sigma : \lambda \longrightarrow \lambda_i (i > 1)$, and then taking absolute values to rephrase the given above relation, we derive

$$2^m = 3 \cdot |f(\lambda_i, k)| \cdot \lambda_i^{n-1}.$$

This leads to a contradiction since the condition $|f(\lambda_i, k)| < 2$ implies that the right–hand side of the aforementioned equation is less than 6, while the left–hand side is greater than or equal to 8. Therefore, $\Lambda \neq 0$.

Given that $h(\delta_1) = \log 2$ and $h(\delta_2) = \frac{1}{k} \log \lambda < \frac{\log 2}{k}$, from this, we can assume that $A_1 := k \log 2$ and $A_2 = 0.7$.

We are now required to estimate $h(\delta_3)$, thus, by referencing inequality 2 of lemma 4.4 and considering properties (3.2) and (3.3), we deduce

$$\begin{aligned} h(\delta_3) &\leq h(3) + h(f(\lambda, k)) \\ &< \log 3 + 4 \log k \\ &< 6 \log k. \end{aligned}$$

Therefore, it is evident that we can select

$$A_3 := \max\{kh(\delta_3), |\log \delta_3|, 0.16\} = 6k \log k.$$

Furthermore, $B \geq \max\{|b_1|, |b_2|, |b_3|\}$, and considering that $m < n$ as per (5.1), we can take $B := n - 1$. Consequently, we utilize Theorem 3.1 to establish the lower bound of $|\Lambda|$, and based on inequality (5.4), we obtain

$$\exp(-C(k) \times (1 + \log(n-1) \times (k \log 2) \times (0.7) \times (4k \log k)) < \frac{3}{\lambda^{n-1}},$$

where $C(k) := 1.4 \times 30^6 \times 3^{4.4} \times k^2 \times (1 + \log k) < 1.5 \times 10^{11} \times k^2 \times (1 + \log k)$.

Specifically, by taking logarithms in the aforementioned inequality, we demonstrate that

$$(n-1) \log \lambda < 1.75 \times 10^{12} \times k^4 \times \log^2(k) \times \log(n-1),$$

a result derived from the premise $1 + \log 3 \leq 3 \log k$, $1 + \log(n-1) \leq 2 \log(n-1)$, and $1/\log \lambda < 2$, for $k \geq 3$ and $n \geq 4$.

Thus,

$$\frac{n-1}{\log(n-1)} < 3.8 \times 10^{12} \times k^4 \times \log^2 k. \tag{5.6}$$

Conversely, it is straightforward to confirm the monotonicity of the function $f(x) := \dfrac{x}{\log x}$ for all $x > e$. Additionally, the inequality $\dfrac{x}{\log x} < A$ implies $x < 2A \log A$ for all $A \geq 3$.

Thus, setting $A := 3.8 \times 10^{12} \times k^4 \times \log^2 k$, inequality (5.6) implies that

$$\begin{aligned} n &< 2(3.8 \times 10^{12} \times k^4 \times \log^2 k) \log(3.8 \times 10^{12} \times k^4 \times \log^2 k) \\ &< 7.6 \times 10^{12} \times k^4 (\log(3.8 \times 10^{12}) + 4 \log k + 2 \log \log k) \\ &< 3.43 \times 10^{14} \times k^4 \log^3 k. \end{aligned}$$

Here, we employed the fact that $2 \log \log k + 29 < 41 \log k$ is true for all $k \geq 2$. Ultimately, this inequality serves as the conclusive step in proving the lemma.

6. The study of the case $2 \leq k \leq 170$

In this section, we focus on the case when $k \in [2, 170]$, aiming to demonstrate that the Diophantine equation (3.1) possesses a sole solution represented by $(n, k, m) = (4, 2, 3)$. To achieve this, we aim to reduce the upper bound of n in (5.2) established in the preceding section. Thus, to leverage Lemma 3.2, we introduce the expression

$$x = m \log 2 - (n-1) \log \lambda - \log(3f(\lambda, k)). \tag{6.1}$$

Subsequently, $e^x - 1 = \Lambda$, where Λ is defined by (5.5). Hence, a comparison of this expression with (5.4) reveals that

$$|e^x - 1| < \frac{3}{\lambda^{n-1}}. \tag{6.2}$$

As Λ is non–zero, it follows immediately that x is also non–zero. Thus, we distinguish the following cases:

– If $\Lambda > 0$, we utilize the property that $e^y - y - 1 \geq 0$ for all $y \in \mathbb{R}$, alongside the inequality (6.1), to infer that

$$0 < x < \frac{3}{\lambda^{n-1}}.$$

Utilizing relation (6.1) along with the preceding inequality, and after substituting the value of x and dividing by $\log \lambda$, we can deduce that

$$0 < m\frac{\log 2}{\log \lambda} - n + \left(1 - \frac{\log(3f(\lambda, k))}{\log \lambda}\right) < 9 \cdot \lambda^{-(n-1)},$$

where we used the fact that $1/\log \lambda < 2$, for all $k \geq 2$.

Now, we define

$$\gamma := \frac{\log 2}{\log \lambda}, \mu := 1 - \frac{\log(3f(\lambda, k))}{\log \lambda}, A := 9, \text{ and } B := \lambda.$$

Based on the inequality provided above, we have

$$0 < m\gamma - n + \mu < A \cdot B^{-(n-1)}.$$

Moreover, it is evident that $\lambda > 1$ is a unit in $\mathcal{O}_{\mathbb{K}}$, hence γ is an irrational number. Consequently, λ and 2 are multiplicative independent.

To refine our constraint on n, we utilized the Mathematica program to determine the maximum value of $\log(Aq/\epsilon)/\log(B)$ for each $k \in [2, 170]$. We set $M := \lfloor 3.43 \times 10^{14} \times k^4 \log^3 k \rfloor$, and the approach involved computing to ensure the condition $q > 6M$ in Lemma 3.2 was met. If the first convergent with $q > 6M$ did not satisfy the criterion $\epsilon > 0$, we proceeded to the subsequent convergent until one fulfilled the necessary condition(we clarify that the value of q is obtained by using the Mathematica function `Convergents[x]`). Through this process, we identified that the maximum value of $\log(Aq/\epsilon)/\log B$ is $176.092\cdots$, which, as per Lemma 5.1, serves as an upper bound for $n - 1$. Hence, this value indicates that the feasible solutions (n, k, m) of Equation (3.1) with $k \in [2, 170]$ and $\Lambda > 0$ satisfy $n \in [4, 177]$, consequently implying $m \in [3, 176]$ immediately given that $m < n + 1$.

– Now we treat the other case when $x < 0$. In fact by using the fact $2(1 - 2^{-k}) < \lambda$, the inequality $2/\lambda^{n-1} < \frac{1}{2}$, easy to check for all $n \geq 4$ and $k \geq 2$. Thus, based on (6.2) we derive $|e^x - 1| < 1/2$, signifying that $e^{|x|} < 2$. Consequently, we arrive at the inequality

$$0 < |x| \leq e^{|x|} - 1 = e^{|x|}|e^x - 1| < \frac{12}{\lambda^{n-1}},$$

Applying a similar analysis as in the case when $x > 0$, we obtain

$$0 < (n-1)\gamma - m + \mu < A \cdot B^{-(n-1)}, \tag{6.3}$$

where

$$\gamma := \frac{\log \lambda}{\log 2}, \mu := \frac{\log(3f(\lambda, k))}{\log 2}, A := 18 \text{ and } B := \lambda.$$

Here, we replicate the same calculations for the case where $x < 0$. Thus, we set $M := \lfloor 3.42 \times 10^{14} \times k^4 \log^3 k \rfloor$ as an upper bound for $n-1$ according to Lemma 5.1. Subsequently, by employing Lemma 3.2 for all $k \in [2, 170]$ and utilizing Mathematica on inequality (6.3), we determine that the maximum value of $\log(Aq/\epsilon)/\log B$ is $178.657\ldots$. This calculation leads us to the feasible solutions (n, k, m) of Equation (3.1) within the specified range of $k \in [2, 170]$ and $x < 0$. Consequently, these results imply that in all cases, $n \in [4, 179]$ and $m \in [3, 179]$. Finally, we have established that $4 \leq n \leq 179$ and $3 \leq m \leq 179$ when $k \in [2, 170]$. By implementing a straightforward code in Mathematica, we can compare $W_n^{(k)}$ and $(2^m + 1)/3$ within this interval, ensuring $m < n + 1 < (3m + 7)/2$, revealing that the sole solution to Equation (3.1) within this scope is $(4, 2, 3)$.

7. The study of the case $k > 170$

The objective of this section is to demonstrate that Equation (3.1) does not possess any solutions when $k > 170$ and $n \geq k + 2$.

– Suppose $k > 170$, implying $n > 3.42 \times 10^{14} k^4 \log^3 k < 2^{\frac{k}{2}}$. Utilizing formula 3 from Lemma 4.4 alongside Inequality (5.3), we can infer that

$$\begin{aligned} |2^m - 3 \cdot 2^{n-2}| &= |2^m - 3f(\lambda, k) \cdot \lambda^{n-1} + \frac{3}{2}\delta + 3 \cdot 2^{n-1} \cdot \eta + 3\eta \cdot \delta| \\ &< \frac{5}{2} + \frac{3k \cdot 2^n}{2^k} + 3 \cdot \frac{2^{n-1}}{2^{\frac{k}{2}}} + 3 \cdot \frac{2^{n+1}k}{2^{\frac{3k}{2}}}. \end{aligned}$$

However, we have $1/2^{n-1} < 1/2^{\frac{k}{2}}$ (because $n > 2$), $12k/2^k < 1/2^{\frac{k}{2}}$, $24k/2^{\frac{3k}{2}} < 1/2^{\frac{k}{2}}$ for each $k > 170$, and by dividing through by 2^{n-2}, we get

$$|2^{m-n+2} - 3| < \frac{9}{2^{\frac{k}{2}}}. \tag{7.1}$$

Observing that $m - n + 2 \leq 2$ due to $m < n + 1$, consequently, it follows from Equation (7.1) that

$$\frac{5}{2} < |2^{m-n+2} - 3| < \frac{9}{2^{\frac{k}{2}}}.$$

Therefore, $2^{\frac{k}{2}} < \frac{18}{5}$, leading to a contradiction since $k > 170$. Thus, the proof of the desired result is now complete.

8. Conclusion

This paper delves into the realm of Diophantine equations, focusing on the investigation of instances where a Wagstaff prime number emerges as one of the terms in the k–Fibonacci sequence. Specifically, we studied the Diophantine equation $W_n^{(k)} = (2^m + 1)/3$ involving the positive integers where m is an odd prime, and the conditions $k \geq 2$ and $n \geq k + 2$ are satisfied. The outcome of this investigation reveals that the number 3 stands as the exclusive Wagstaff prime within the k–Fibonacci sequences. Our proof hinges on leveraging certain properties derived from the theory of continued fractions and a pivotal theory concerning linear forms in logarithms.

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