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An Innovated G Family: Properties, Characterizations and Risk Analysis under Different Estimation Methods

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Abstract This paper introduces a novel class of continuous probability distributions called the Log-Adjusted Polynomial (LAP) G family, with a focus on the LAP Weibull distribution as a key special case. The proposed family is designed to enhance the flexibility of classical distributions by incorporating additional parameters that control shape, skewness, and tail behavior. The LAP Weibull model is particularly useful for modeling lifetime data, extreme events, and insurance claims characterized by heavy tails and asymmetry. The paper presents the mathematical formulation of the new family, including its cumulative distribution function, probability density function, and hazard rate function. It also explores structural properties such as series expansions and tail behavior. Risk analysis is conducted using advanced risk measures, including Value-at-Risk (VaR), Tail VaR (TVaR), and tail mean-variance (TMVq), under various estimation techniques. Estimation methods considered include maximum likelihood (MLE), Cramér–von Mises (CVM), Anderson–Darling (ADE), and their right-tail and left-tail variants. These methods are compared using both simulated and real insurance data to assess their sensitivity to tail events. The performance of each estimator is evaluated in terms of bias, accuracy, and robustness in capturing extreme risks. The LAP Weibull model demonstrates superior performance in fitting heavy-tailed data compared to traditional models. The ADE also performs well, offering a balance between sensitivity and stability. MLE and CVM tend to underestimate tail risks, which could lead to insufficient capital reserves in insurance applications. The study highlights the importance of selecting appropriate estimation techniques based on the specific goals of the risk analysis. With its enhanced flexibility and performance in modeling extreme risks, the LAP Weibull model offers a robust framework for modern risk assessment. The findings support the use of ADE in high-stakes risk management scenarios, especially when dealing with heavy-tailed insurance data. This work contributes to the growing literature on advanced statistical models for actuarial and financial risk analysis. The LAP Weibull model proves particularly useful in capturing the tail behavior of claim distributions, improving the accuracy of risk predictions. The paper provides a solid foundation for future applications of the LAP family in modeling complex real-world phenomena under uncertainty.

Keywords Characterizations; Value-at-Risk; Weibull model; Claims Data; Risk Analysis.

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1. Introduction

In recent years, significant advancements have been made in the development of generalized statistical distributions to better capture the complexities of real-world data across various domains such as finance, insurance, medicine, and engineering (Abiad et al., 2025; Afify et al., 2018). These efforts have focused on enhancing classical models by introducing additional shape parameters or combining existing distribution families to improve flexibility, accuracy,

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and applicability (Alizadeh et al., 2018; Abouelmagd et al., 2019). Notable among these developments are the Odd Log-Logistic Topp–Leone G family, which provides greater modeling capabilities for skewed and bimodal datasets, and the Zero Truncated Poisson Burr X family, designed to simultaneously model count and continuous data (Alizadeh et al., 2018; Abouelmagd et al., 2019). Other notable contributions include the Transmuted Weibull-G family, Exponential Lindley Odd Log-Logistic-G family, and the Odd Log-Logistic Weibull-G family, which extend the utility of classical distributions in survival and reliability analysis (Korkmaz et al., 2018; Rasekhi et al., 2022). Additionally, Alizadeh et al. (2023) explored copula-based extensions of the XGamma distribution, while Mansour et al. (2020f) introduced copulas into the modeling of acute bone cancer data. Ibrahim et al. (2025a, 2025b) further expanded this field by applying Clayton copulas to validate flexible Weibull models. The current paper builds upon these developments by proposing a novel G-family that incorporates closed-form expressions for moments, quantile functions, and entropy measures, allowing for a broader range of shapes and tail behaviors. This new framework enables more accurate modeling of complex data structures and improves interpretability and predictive performance.

In recent years, the development of flexible and general families of probability distributions has gained significant attention in statistical research. These models aim to enhance the modeling capabilities of classical distributions by introducing additional parameters that control shape, skewness, tail behavior, and other distributional characteristics. One effective approach involves transforming a baseline cumulative distribution function (CDF), $G\left(x;\underline{\Phi}\right)$, using carefully designed generator functions. In this context, we introduce a novel class of continuous distributions called the LAP family and characterized by a unique combination of logarithmic and exponential transformations. The proposed model is defined by both a CDF

$$F\left(x;\beta,\underline{\Phi}\right)=C\log\left[1+G\left(x;\underline{\Phi}\right)\right]e^{\beta G\left(x;\underline{\Phi}\right)},\qquad x\in\mathbb{R},\tag{1}$$

and a corresponding probability density function (PDF):

$$f\left(x;\beta,\underline{\Phi}\right)=Cg\left(x;\underline{\Phi}\right)e^{\beta G\left(x;\underline{\Phi}\right)}P\left(x\right),\qquad x\in\mathbb{R},\tag{2}$$

where $\beta>0$ are parameters,

$$C=\frac{1}{e^{\beta}\log\left(2\right)},$$

is a normalization constant,

$$P\left(x\right)=\frac{1}{1+G\left(x;\underline{\Phi}\right)}+\beta\log\left[1+G\left(x;\underline{\Phi}\right)\right],$$

which plays a central role in shaping the behavior of the model, and $G\left(x;\underline{\Phi}\right)$ is a baseline cdf with the corresponding PDF $g\left(x;\underline{\Phi}\right)$ which depends on the parameter $\underline{\Phi}$. A key feature of this family lies in the structure of the PDF, where the exponential term $e^{\beta G\left(x;\underline{\Phi}\right)}$ introduces a form of exponential weighting dependent on the baseline CDF. Crucially, this exponential component is further modulated by the function $P\left(x\right)$, which incorporates a logarithmic adjustment of $G\left(x;\underline{\Phi}\right)$. This LAP acts as a flexible weight function that influences the tail behavior, skewness, and kurtosis of the resulting distribution. By emphasizing the interaction between the exponential generator and the logarithmic polynomial weight $P\left(x\right)$, this family offers enhanced flexibility over traditional models. It allows for a wide range of hazard rate shapes, including increasing, decreasing, bathtub, and upside-down bathtub forms, making it suitable for applications in reliability analysis, survival modeling, actuarial science, and other fields requiring nuanced data fitting. As $x\rightarrow-\infty$, $G\left(x;\underline{\Phi}\right)\rightarrow0$,

$$\log\left[1+G\left(x;\underline{\Phi}\right)\right]=G\left(x;\underline{\Phi}\right),$$

$$e^{\beta G\left(x;\underline{\Phi}\right)}=1+\beta G\left(x;\underline{\Phi}\right).$$

Then,

$$F\left(x;\beta,\underline{\Phi}\right)\approx CG\left(x;\underline{\Phi}\right)\left[1+\beta G\left(x;\underline{\Phi}\right)\right]\rightarrow0\text{ as }x\rightarrow-\infty.\tag{3}$$

As $x \to +\infty$, $G(x;\underline{\Phi}) \to 1$,

$$\log[1+G(x;\underline{\Phi})] = \log(2),$$
$$e^{\beta G(x;\underline{\Phi})} = e^{\beta}.$$

Then,

$$F(x;\beta,\underline{\Phi}) \approx C\log(2)\,e^{\beta} = 1 \text{ as } x \to +\infty. \tag{4}$$

As $x \to -\infty$, $f(x;\beta,\underline{\Phi}) \to 0$.As $x \to +\infty$, $f(x;\beta,\underline{\Phi}) \to 0$. The tail index of the proposed family is the same as that of the baseline distribution, and this can be easily proven mathematically.

2. Properties

In this section, we investigate some mathematical properties of the LAP family.

2.1. Useful expansions

By expanding $e^{\beta G(x;\underline{\Phi})}$, the new CDF can be expressed as

$$F(x;\beta,\underline{\Phi}) = C\log[1+G(x;\underline{\Phi})]\sum_{k=0}^{+\infty}\frac{\beta^k}{k!}[G(x;\underline{\Phi})]^k, \qquad x \in \mathbb{R}. \tag{5}$$

Then, by expanding $\log[1+G(x;\underline{\Phi})]$, we have

$$\log[1+G(x;\underline{\Phi})] = \sum_{h=1}^{+\infty}\frac{(-1)^{1+h}}{h!}[G(x;\underline{\Phi})]^h \tag{6}$$

Inserting (6) into (5), the new CDF can be simplified as

$$F(x;\beta,\underline{\Phi}) = \sum_{k=0}^{+\infty}\sum_{h=1}^{+\infty} d_{k,h}W_{k,h}(x;\underline{\Phi}), \qquad x \in \mathbb{R}, \tag{7}$$

where

$$d_{k,h} = \frac{C}{k!h!}(-1)^{1+k}\beta^k,$$

and $W_{k,h}(x;\underline{\Phi}) = [G(x;\underline{\Phi})]^{k+h}$ refers to the CDF of the exponentiated G family.By differentiating (7), we have

$$f(x;\beta,\underline{\Phi}) = \sum_{k=0}^{+\infty}\sum_{h=1}^{+\infty} d_{k,h}w_{k,h}(x;\underline{\Phi}), \qquad x \in \mathbb{R}, \tag{8}$$

where

$$w_{k,h}(x;\underline{\Phi}) = dW_{k,h}(x;\underline{\Phi})/dx = (k+h)\,g(x;\underline{\Phi})\,[G(x;\underline{\Phi})]^{k+h-1},$$

which refers to the PDF of the exponentiated G family. To summarize, we say that equation (8) can be used to derive most of the mathematical properties of the underlying distribution to be studied.

2.2. Quantile function

The quantile function (QF) of X can be determined by inverting $F(x) = u$ in (1), where

$$ue^{\beta}\log(2) = \log[1+G(x;\underline{\Phi})]\,e^{\beta G(x;\underline{\Phi})},$$

let $G(x;\underline{\Phi}) = y$, then

$$ue^{\beta}\log(2) = \log(1+y)\,e^{\beta y},$$

For small values of y y, we can approximate $\log(1+y) \approx y$, so

$$u\beta e^{\beta}\log(2) = \beta y e^{\beta y},$$

Let $z = \beta y$, then using Lambert $W[.]$

$$\begin{aligned} u\beta e^{\beta}\log(2) &= ze^{z} \\ &\Rightarrow\ z = W\left(u\beta e^{\beta}\log(2)\right) \\ &\Rightarrow\ \beta y = W\left[u\beta e^{\beta}\log(2)\right] \\ &\Rightarrow\ y = \frac{1}{\beta}W\left[u\beta e^{\beta}\log(2)\right] \\ &\Rightarrow\ G(x;\underline{\Phi}) = \frac{1}{\beta}W\left[u\beta e^{\beta}\log(2)\right]. \end{aligned}$$

Finally,

$$x_u = G^{-1}\left\{\frac{1}{\beta}W\left[u\beta e^{\beta}\log(2)\right];\underline{\Phi}\right\}. \tag{9}$$

2.3. Moments

Let $Y_{k,h}$ be a rv having density $w_{k,h}(x;\underline{\Phi})$. The r^{th} ordinary moment of X, say μ'_r, follows from (8) as

$$\mu'_r = E\left(X^r\right) = \sum_{k=0}^{+\infty}\sum_{h=1}^{+\infty}\left[d_{k,h}\ E\left(Y_{k,h}^r\right)\right], \tag{10}$$

where

$$E(Y_{k,h}^r) = (k+h)\int_{-\infty}^{\infty} x^r\ g(x;\underline{\Phi})\,G(x;\underline{\Phi})^{k+h-1}\,dx$$

can be evaluated numerically in terms of the baseline qf

$$Q_G(u) = G^{-1}(u) as E(Y_{k,h}^n) = (k+h)\int_0^1 Q_G(u)^n\ u^{(k+h)-1}du.$$

Setting $r = 1$ in (10) gives the mean of X.

2.4. Incomplete moments

The r^{th} incomplete moment of X is given by

$$m_r(y) = \int_{-\infty}^{y} x^r\ f(x)dx.$$

Using (8), the rth incomplete moment of LAP family is

$$m_r(y) = \sum_{k=0}^{+\infty}\sum_{h=1}^{+\infty}\left[d_{k,h}\ m_{r,k+h}(y)\right], \tag{11}$$

where

$$m_{r,k+h}(y) = \int_0^{G(y)} Q_G^r(u)\,u^{k+h-1}\,du.$$

The $m_{r,k+h}(y)$ can be calculated numerically by using the software such as **Matlab**, **R**, **Mathematica** etc.

2.5. *Moment generating function*

The moment generating function (MGF) of X, say $M(t) = E(e^{t\,X})$, is obtained from (8) as

$$M(t) = \sum_{k=0}^{+\infty}\sum_{h=1}^{+\infty}\left[d_{k,h}\; M_{k+h}\left(t\right)\right],$$

where $M_{k+h}(t)$ is the generating function of Y_{k+h} given by

$$M_{k+h}(t) = (k+h)\int_{-\infty}^{\infty} e^{t\,x} g(x)G(x)^{k+h-1}\,dx = (k+h)\int_{0}^{1}\exp[t\,Q_G(u;k+h\,)]\,u^{k+h-1}du.$$

The last two integrals can be computed numerically for most parent distributions.

3. Characterizations

3.1. *Characterizations based on a simple relationship between two truncated moments*

In this subsection we present characterizations of the LAP family, in terms of a simple relationship between two truncated moments. Our first characterization result employs a theorem due to (Glänzel, 1987), see Theorem 1 below. Note that the result holds also when the interval H is not closed. Moreover, it could be also applied when the cdf F does not have a closed form. As shown in (Glänzel, 1990), this characterization is stable in the sense of weak convergence.

Theorem 1. Let $(\Omega,\mathcal{F},\mathbf{P})$ be a given probability space and let $H=[d,e]$ be an interval for some $d<e$ $\;(d=-\infty,\ e=\infty$ might as well be allowed$)$. Let $X:\Omega\to H$ be a continuous random variable with the distribution function F and let q_1 and q_2 be two real functions defined on H such that

$$\mathbf{E}\left[q_2\left(X\right)\mid X\geq x\right] = \mathbf{E}\left[q_1\left(X\right)\mid X\geq x\right]\eta\left(x\right), \quad x\in H,$$

is defined with some real function η. Assume that $q_1,q_2\in C^1\left(H\right)$, $\eta\in C^2\left(H\right)$ and F is twice continuously differentiable and strictly monotone function on the set H. Finally, assume that the equation $\eta q_1 = q_2$ has no real solution in the interior of H. Then F is uniquely determined by the functions q_1,q_2 and η, particularly

$$F\left(x\right) = \int_a^x C\left|\frac{\eta'\left(u\right)}{\eta\left(u\right)q_1\left(u\right)-q_2\left(u\right)}\right|\exp\left(-s\left(u\right)\right)\,du\ ,$$

where the function s is a solution of the differential equation $s' = \frac{\eta'\,q_1}{\eta\,q_1\,-\,q_2}$ and C is the normalization constant, such that $\int_H dF = 1$.

Proposition 3.1.1. Let $X:\Omega\to\mathbb{R}$ be a continuous random variable and let

$$q_1\left(x\right) = \left[P\left(x\right)\right]^{-1}$$

and

$$q_2\left(x\right) = q_1\left(x\right)e^{\beta G(x;\underline{\Phi})}, \quad x\in\mathbb{R}.$$

The random variable X has PDF (2) if and only if the function η defined in Theorem 1 has the form

$$\eta\left(x\right) = \frac{1}{2}\left(e^{\beta} + e^{\beta G(x;\underline{\Phi})}\right), \quad x\in\mathbb{R}.$$

Proof. Let X be a random variable with PDF (2), then

$$(1-F(x))\,E\left[q_1(X)\mid X\geq x\right]=\int_x^{\infty}Cg\left(u;\underline{\Phi}\right)e^{\beta G(u;\underline{\Phi})}du$$
$$=\frac{C}{\beta}\left(e^{\beta}-e^{\beta G(x;\underline{\Phi})}\right),\qquad x\in\mathbb{R},$$

and

$$(1-F(x))\,E\left[q_2(X)\mid X\geq x\right]=\int_x^{\infty}Cg\left(u;\underline{\Phi}\right)e^{2\beta G(u;\underline{\Phi})}du$$
$$=\frac{C}{2\beta}\left(e^{2\beta}-e^{2\beta G(x;\underline{\Phi})}\right),\qquad x\in\mathbb{R},$$

and finally

$$\eta(x)\,q_1(x)-q_2(x)=\frac{q_1(x)}{2}\left(e^{\beta}-e^{\beta G(x;\underline{\Phi})}\right)>0,\qquad x\in\mathbb{R}.$$

Conversely, if η is given as above, then

$$s'(x)=\frac{\eta'(x)\,q_1(x)}{\eta(x)\,q_1(x)-q_2(x)}=\frac{\beta g\left(x;\underline{\Phi}\right)e^{\beta G(x;\underline{\Phi})}}{e^{\beta}-e^{\beta G(x;\underline{\Phi})}},\qquad x\in\mathbb{R},$$

and hence

$$s(x)=-\log\left(e^{\beta}-e^{\beta G(x;\underline{\Phi})}\right),\quad x\in\mathbb{R}.$$

Now, in view of Theorem G, X has density (2).

Remark 3.1.1. The goal is to make $\eta(x)$ as simple as possible.

Corollary 3.1.1. Let $X:\Omega\to\mathbb{R}$ be a continuous random variable and let $q_1(x)$ be as in Proposition 3.1.1. The PDF of X is (2) if and only if there exist functions q_2 and η defined in Theorem 3.1.1 satisfying the differential equation

$$\frac{\eta'(x)\,q_1(x)}{\eta(x)\,q_1(x)-q_2(x)}=\frac{\beta g\left(x;\underline{\Phi}\right)e^{\beta G(x;\underline{\Phi})}}{e^{\beta}-e^{\beta G(x;\underline{\Phi})}},\qquad x\in\mathbb{R}.$$

Corollary 3.1.2. The general solution of the differential equation in Corollary 3.1.1 is

$$\eta(x)=\left(e^{\beta}-e^{\beta G(x;\underline{\Phi})}\right)^{-1}\left[-\int\beta g\left(x;\underline{\Phi}\right)e^{\beta G(x;\underline{\Phi})}\left(q_1(x)\right)^{-1}q_2(x)\,dx+D\right],$$

where D is a constant.

Proof. If X has PDF (2), then clearly the differential equation holds. Now, if the differential equation holds, then

$$\eta'(x)=\left(\frac{\beta g\left(x;\underline{\Phi}\right)e^{\beta G(x;\underline{\Phi})}}{e^{\beta}-e^{\beta G(x;\underline{\Phi})}}\right)\eta(x)-\left(\frac{\beta g\left(x;\underline{\Phi}\right)e^{\beta G(x;\underline{\Phi})}}{e^{\beta}-e^{\beta G(x;\underline{\Phi})}}\right)\left(q_1(x)\right)^{-1}q_2(x),$$

or

$$\eta'(x) - \left(\frac{\beta g(x;\underline{\Phi})\, e^{\beta G(x;\underline{\Phi})}}{e^{\beta} - e^{\beta G(x;\underline{\Phi})}}\right)\eta(x)$$
$$= -\left(\frac{\beta g(x;\underline{\Phi})\, e^{\beta G(x;\underline{\Phi})}}{e^{\beta} - e^{\beta G(x;\underline{\Phi})}}\right)(q_1(x))^{-1} q_2(x),$$

or

$$\frac{d}{dx}\left\{\left(e^{\beta} - e^{\beta G(x;\underline{\Phi})}\right)\eta(x)\right\} = -\beta g(x;\underline{\Phi})\, e^{\beta G(x;\underline{\Phi})}(q_1(x))^{-1} q_2(x),$$

from which we arrive at

$$\eta(x) = \left(e^{\beta} - e^{\beta G(x;\underline{\Phi})}\right)^{-1}\left[-\int \beta g(x;\underline{\Phi})\, e^{\beta G(x;\underline{\Phi})}(q_1(x))^{-1} q_2(x)\, dx + D\right].$$

Note that a set of functions satisfying the differential equation in Corollary 3.1.1, is given in Proposition 3.1.1 with $D = \frac{e^{2\beta}}{2}$. However, it should also be noted that there are other triplets (q_1, q_2, η) satisfying the conditions of Theorem 1.

3.2. Characterization in Terms of the Reverse (or Reversed) Hazard Function

The reverse hazard function, r_F, of a twice differentiable distribution function, F , is defined as

$$r_F(x) = \frac{f(x)}{F(x)}, \quad x \in support\ of\ F.$$

In this subsection we present characterization of LAP distributions in terms of the reverse hazard function.

Proposition 3.2.1. Let $X : \Omega \to \mathbb{R}$ be a continuous random variable. The random variable X has PDF (2) if and only if its reverse hazard function $r_F(x)$ satisfies the following differential equation

$$r'_F(x) - \frac{g'(x;\underline{\Phi})}{g(x;\underline{\Phi})} r_F(x) = C g(x;\underline{\Phi}) \frac{d}{dx}\left\{\frac{P(x)}{\log(1+G(x;\underline{\Phi}))}\right\}, \qquad x \in \mathbb{R},$$

with boundary condition $\lim_{x\to\infty} r_F(x) = \left(\frac{1+2\beta\log(2)}{2\log(2)}\right)\lim_{x\to\infty} g(x;\underline{\Phi})$.

Proof. Multiplying both sides of the above equation by $(g(x;\underline{\Phi}))^{-1}$, we have

$$\frac{d}{dx}\left\{(g(x;\underline{\Phi}))^{-1} r_F(x)\right\} = C\frac{d}{dx}\left\{\frac{P(x)}{\log(1+G(x;\underline{\Phi}))}\right\},$$

or

$$r_F(x) = C g(x;\underline{\Phi})\left\{\frac{P(x)}{\log(1+G(x;\underline{\Phi}))}\right\},$$

which is the reverse hazard function corresponding to the PDF (2).

4. The LAP Weibull case

This section explores the mathematical properties, structural behavior, and practical applications of the LAP Weibull distribution. We present its probability density function (PDF) and hazard rate function (HRF) along with key expansions and characterizations that facilitate theoretical analysis and numerical implementation. Additionally, we discuss parameter estimation methods and demonstrate the model's performance using simulated

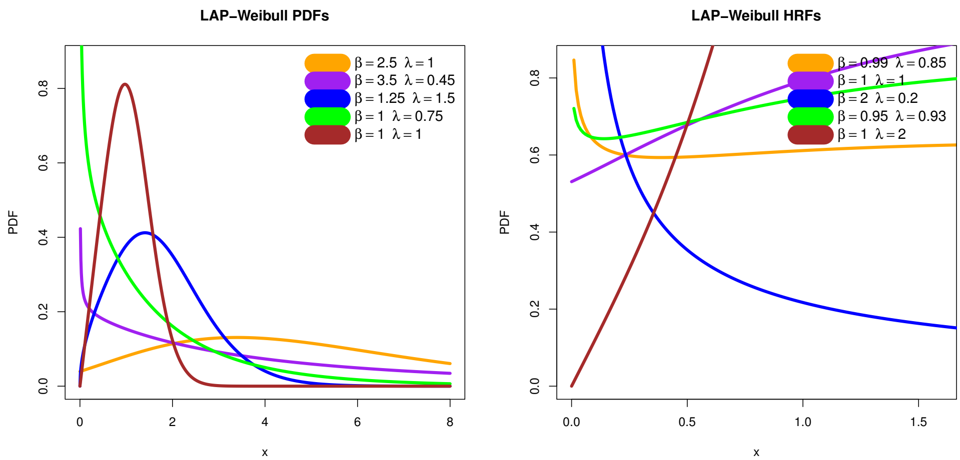

Figure 1. Plots of the new LAP Weibull PDF (right) and HRF (left) for selected values of the parameter.

and real datasets, including insurance claims and reinsurance data. Through graphical illustrations and empirical evaluations, we highlight how the LAP Weibull model outperforms conventional distributions in capturing extreme events and tail risks, thereby offering a robust framework for modern risk assessment and decision-making under uncertainty. Figure 1 presents some plots of the new LAP Weibull PDF (right) and HRF (left) for selected values of the parameter. The left plot of Figure 1 displays the PDF of the LAP Weibull distribution for various combinations of shape and scale parameters. These curves exhibit a rich variety of shapes, showcasing the model's capacity to fit diverse data patterns, including right-skewed, symmetric, and heavy-tailed distributions. One notable curve with $\beta = 1$ and $\lambda = 0.75$ presents a pronounced right skew, commonly observed in financial losses, insurance claims, and other datasets where extreme positive deviations are more likely. Another curve with $\beta = 1.25$ and $\lambda = 1.5$ appears more symmetric, suggesting the model can approximate normal-like behavior while retaining the flexibility to handle deviations from symmetry. Additionally, the PDF with $\beta = 3.5$ and $\lambda = 0.45$ exhibits a sharp peak and heavier tails, making it suitable for modeling extreme events in fields such as environmental sciences or catastrophic risk assessment. This versatility in PDF shapes indicates that the LAP Weibull distribution outperforms traditional models like the standard Weibull or exponential distributions, especially when dealing with heterogeneous or complex datasets. The visual differences among the plotted curves further emphasize the critical role of parameter selection in accurately representing the underlying data structure, particularly in terms of skewness, kurtosis, and tail thickness. As a result, the LAP Weibull model proves to be a powerful tool for statistical modeling across disciplines ranging from finance and insurance to biomedical studies and industrial reliability. The right plot of Figure 1 illustrates the HRF of the LAP Weibull distribution for selected parameter values. This visual representation highlights the model's ability to capture a wide range of hazard behaviors, including increasing, bathtub-shaped, and nearly constant hazard rates, depending on the chosen parameters. For instance, one curve with $\beta = 0.95$ and $\lambda = 0.93$ shows a nearly constant hazard rate, indicating applicability in scenarios where failure probabilities remain steady over time, such as in systems experiencing random external shocks. In contrast, the curve with $\beta = 2$ and $\lambda = 0.2$ demonstrates a sharply increasing hazard rate, which is ideal for modeling systems that deteriorate rapidly over time, such as mechanical components under high stress or biological organisms undergoing accelerated aging. The flexibility of the HRF underscores the LAP Weibull model's adaptability to different risk profiles, making it valuable in fields like reliability engineering, actuarial science, and biomedical research. Moreover, this diversity in hazard behavior allows the model to represent early failure, random failure, and wear-out failure phases effectively, key stages in the lifecycle of many systems. The visual depiction also emphasizes how sensitive the hazard function is to changes in shape and scale parameters, reinforcing the importance of accurate parameter estimation in real applications.

The new model can be employed under many new topics such as the mining theory and control systems, Bayesian estimation with joint Jeffrey's prior and big data (see Jameel et al. (2022), Salih and Abdullah (2024), Salih and Hmood (2020) and Salih and Hmood (2022)).

5. Simulations for assessing estimation methods under the LAP Weibull case

In this paper we will consider the maximum likelihood estimation (MLE) method, the Cramér-von-Mises estimation (CVME), the Anderson Darling estimation (ADE), the Tail-Anderson Darling estimation (RTADE) and the left Tail-Anderson Darling estimation (LTADE) for estimating the model parameters. Also, the same mehtods will be cosidered in risk anlaysis. To systematically evaluate and compare the effectiveness of various parameter estimation techniques, a detailed numerical simulation study is undertaken. The analysis is based on data generated from the WTLE distribution, with $N = 1000$ independent simulation replications to ensure statistical reliability. Within each replication, synthetic data sets are produced for multiple sample sizes, $n = 15, 30, 50$, and 100, to explore how estimation performance evolves with increasing data availability. To achieve a robust comparison, multiple evaluation criteria are employed. These include bias, which quantifies the average deviation of an estimator from the true parameter value, and the root mean squared error (RMSE), which encapsulates both bias and variance components. In addition, the mean absolute deviation in distribution (M-AD) is used to measure the average discrepancy between the estimated and actual cumulative distribution functions, while the maximum absolute deviation (Max-AD) identifies the largest such discrepancy across the domain. Together, these metrics provide a multidimensional perspective on estimator performance, capturing both point estimation accuracy and overall distributional fit.

Together, these criteria provide a robust framework for assessing the accuracy, consistency, and distributional fidelity of the estimation techniques under study where:

1-BIAS$(\underline{\Phi}) = \frac{1}{B}\sum_{i=1}^{B}\left(\widehat{\underline{\Phi}_i} - \underline{\Phi}\right)$,BIAS$(\lambda) = \frac{1}{B}\sum_{i=1}^{B}\left(\widehat{\lambda}_i - \lambda\right)$,

2-RMSE$(\underline{\Phi}) = \sqrt{\frac{1}{B}\sum_{i=1}^{B}\left(\widehat{\underline{\Phi}_i} - \underline{\Phi}\right)^2}$,RMSE$(\lambda) = \sqrt{\frac{1}{B}\sum_{i=1}^{B}\left(\widehat{\lambda}_i - \lambda\right)^2}$,

3-The M-AD $\left(D_{(\text{abs})}\right)$:D$_{(\text{abs})} = \frac{1}{nB}\sum_{i=1}^{B}\sum_{j=1}^{n}|F_{(\underline{\Phi},\lambda)}(x_{ij}) - F_{\left(\widehat{\underline{\Phi}},\widehat{,\lambda}\right)}(t_{ij})|$ and

4-The Max-AD $\left(D_{(\max)}\right)$:D$_{(\max)} = \frac{1}{B}\sum_{i=1}^{B}\max_j |F_{(\underline{\Phi},\lambda)}(x_{ij}) - F_{\left(\widehat{\underline{\Phi}},\widehat{,\lambda}\right)}(w_{ij})|$.

Table 1 presents simulation results for the parameter estimates of $\lambda = 2$ and $\beta = 2$ using different estimation methods. The table reports Bias, RMSE, Dabs, and Dmax across various sample sizes ($n = 20, 50, 100, 300$) to assess the accuracy and performance of each method. For smaller samples ($n = 20$), RTADE shows the highest bias for both parameters, while ADE exhibits the lowest bias for λ but overestimates β. In terms of RMSE, which accounts for both variance and bias, MLE performs relatively well for small n, particularly for λ. As the sample size increases, all metrics improve, indicating better estimation consistency with larger data. Notably, RTADE continues to show higher RMSE values compared to other methods, suggesting less precision in estimation. LTADE demonstrates lower bias for λ in some cases, especially at smaller sample sizes. The Dabs and Dmax values also decrease with increasing sample size, reflecting improved distributional fit. MLE consistently achieves low Dabs and Dmax for large n, highlighting its reliability in overall model fitting. Despite its tail-focused design, RTADE does not uniformly outperform other techniques in point estimation accuracy.

Table 2 presents simulation results for parameter estimates with true values $\lambda = 0.5$ and $\beta = 0.9$, using different estimation methods such as MLE, CVM, ADE, RTADE, and LTADE. The performance metrics include Bias, RMSE, Dabs (mean absolute deviation), and Dmax (maximum absolute deviation) across sample sizes of $20, 50, 100$, and 300. For small sample sizes ($n = 20$), RTADE shows the highest bias for both parameters, while ADE and LTADE perform relatively better in minimizing bias. In terms of RMSE, MLE and ADE exhibit lower values, indicating better precision in estimation compared to other methods. As the sample size increases, all

Table 1: Simulation results for parameter $\lambda = 2$ & $\beta = 2$

	n	BIAS β	BIAS λ	RMSE β	RMSE λ	Dabs	Dmax
MLE	20	0.092552	0.080197	0.582345	0.095647	0.008383	0.013417
CVM		0.088889	0.079276	0.632644	0.237755	0.008216	0.013007
ADE		0.060078	−0.012194	0.573592	0.106751	0.005999	0.008359
RTADE		0.139447	0.040648	0.841688	0.144889	0.009305	0.017075
LTADE		0.033567	−0.068962	0.544772	0.155604	0.008676	0.016878
MLE	50	0.046643	0.035865	0.209384	0.035792	0.003971	0.006709
CVM		0.04575	0.028616	0.242349	0.078623	0.003504	0.006285
ADE		0.035186	−0.002329	0.221245	0.04553	0.003158	0.004517
RTADE		0.06714	0.014466	0.311422	0.051912	0.004698	0.008298
LTADE		0.025335	−0.025596	0.207205	0.057845	0.004101	0.007226
MLE	100	0.031512	0.015106	0.097112	0.016084	0.002222	0.004161
CVM		0.031198	0.010037	0.114064	0.034951	0.002124	0.003958
ADE		0.02644	−0.003551	0.104592	0.021761	0.002509	0.003523
RTADE		0.040162	0.003152	0.139059	0.023587	0.003162	0.004974
LTADE		0.021993	−0.013114	0.097542	0.027453	0.002853	0.004466
MLE	300	0.005549	0.002192	0.035624	0.004663	0.000382	0.00072
CVM		0.004344	0.003929	0.038989	0.011202	0.000415	0.000663
ADE		0.003237	−0.002344	0.036706	0.006874	0.000449	0.000735
RTADE		0.008946	0.000194	0.044971	0.007325	0.000746	0.001123
LTADE		0.000792	−0.005861	0.035807	0.009502	0.000627	0.001284

metrics improve, showing the consistency of the estimation techniques. Notably, RTADE continues to display higher RMSE values, suggesting less accuracy in point estimation for these parameters. LTADE performs well in bias reduction for λ at larger sample sizes, outperforming MLE and CVM. The Dabs and Dmax values decrease with increasing sample size, indicating improved overall distributional fit across all methods. MLE and ADE consistently achieve the lowest Dabs and Dmax values for large n, highlighting their reliability in model fitting. Despite being tailored for tail modeling, RTADE does not uniformly outperform other methods in estimating central tendencies. The table illustrates how different estimation techniques trade off in terms of bias, precision, and distributional accuracy for varying sample sizes. These findings are important for selecting appropriate estimation strategies in practical applications involving small or moderate sample data.

Table 3 presents simulation results for parameter estimates with true values $\lambda = 1.5$ and $\beta = 0.5$, using estimation methods. The performance metrics include Bias, RMSE, Dabs (mean absolute deviation), and Dmax (maximum absolute deviation) across sample sizes of $20, 50, 100$, and 300. For small sample sizes ($n = 20$), RTADE shows the highest bias for both parameters, while LTADE performs relatively well in minimizing bias for λ. In terms of RMSE, MLE and ADE perform better, indicating greater precision in estimation compared to other methods. As the sample size increases, all metrics improve, confirming the consistency of these estimation techniques. RTADE continues to display higher RMSE values, suggesting reduced accuracy in point estimation for these parameters. LTADE performs competitively in reducing bias for λ at larger sample sizes, although its RMSE remains higher than that of MLE and ADE. The Dabs and Dmax values decrease with increasing sample size, reflecting improved overall distributional fit across all methods. MLE and ADE consistently achieve the lowest Dabs and Dmax for large n, highlighting their reliability in model fitting. Despite being designed for tail modeling, RTADE does not uniformly outperform other methods in estimating central tendencies. The table illustrates how different estimation techniques trade off in terms of bias, precision, and distributional accuracy depending on

Table 2: Simulation results for parameter $\lambda = 0.5$ & $\beta = 0.9$

	n	BIAS β	BIAS λ	RMSE β	RMSE λ	Dabs	Dmax
MLE	20	0.065101	0.020966	0.277322	0.008498	0.009913	0.01783
CVM		0.069096	−0.01341	0.304413	0.013482	0.009342	0.015149
ADE		0.05603	−0.02589	0.281422	0.009393	0.009954	0.019945
RTADE		0.114868	0.001489	0.392141	0.01274	0.014094	0.021403
LTADE		0.027397	−0.004328	0.277393	0.021642	0.003646	0.005622
MLE	50	0.014328	0.006698	0.101808	0.00260	0.002727	0.004781
CVM		0.039133	−0.006144	0.106607	0.004143	0.005218	0.008044
ADE		0.035514	−0.010941	0.101729	0.003114	0.00522	0.009709
RTADE		0.055352	−0.00172	0.125803	0.004154	0.007018	0.010241
LTADE		0.025158	−0.000066	0.100726	0.006754	0.003174	0.004743
MLE	100	0.014912	0.003938	0.051687	0.001246	0.002138	0.003898
CVM		0.022423	−0.002633	0.055479	0.002253	0.00295	0.004354
ADE		0.01966	−0.005164	0.053088	0.001677	0.002776	0.004919
RTADE		0.02907	0.000287	0.062681	0.002272	0.003649	0.005528
LTADE		0.014679	0.000342	0.053395	0.003678	0.001840	0.002832
MLE	300	0.01052	0.001448	0.015680	0.000412	0.001324	0.002326
CVM		0.00758	0.000672	0.016693	0.000657	0.000941	0.001574
ADE		0.007634	−0.000302	0.016344	0.000496	0.000981	0.001424
RTADE		0.008114	0.000977	0.019428	0.000726	0.001016	0.001755
LTADE		0.008131	0.002322	0.016035	0.001071	0.001207	0.002192

sample size. These findings are important for selecting appropriate estimation strategies in practical applications involving small or moderate sample data.

Overall, Tables 1–3 provide strong empirical evidence supporting the use of ADE and MLE for parameter estimation in the proposed LAP family of distributions under a variety of conditions. They also highlight the importance of considering sample size and target application (central vs. tail estimation) when selecting an appropriate estimation method.

6. Risk analysis under artificial data and LAP Weibull case

Accurate parameter estimation plays a pivotal role in the practical application of any statistical model, particularly in fields where decision-making relies heavily on risk assessment and forecasting (Mansour et al., 2020e; Ibrahim et al., 2020). Several studies have explored different estimation methods, including Maximum Likelihood Estimation (MLE), Cramér–von Mises (CVM), Bayesian inference, least squares estimation, and hybrid approaches (Hashem et al., 2024; Yousof et al., 2025a). For instance, Yousof et al. (2025a) conducted comparative studies using various estimation techniques under generalized gamma distributions, highlighting their strengths and limitations depending on data structure and censoring mechanisms. Similarly, Ibrahim et al. (2025a, 2025b) evaluated estimation strategies for reciprocal Weibull models in medical and reliability contexts. Risk analysis has also gained increasing attention, particularly in actuarial science and financial modeling, where Value-at-Risk (VaR), Tail Value-at-Risk (TVaR), and Key Risk Indicators (KRIs) are critical for assessing potential losses (Elbatal et al., 2024; Yousof et al., 2024). Studies such as those by Mohamed et al. (2024) and Ibrahim et al. (2025c) have applied advanced estimation techniques to evaluate risk metrics under negatively skewed and over-dispersed insurance claims data. In addition, Elbatal et al. (2024) introduced a new loss-revenue model incorporating entropy analysis

Table 3: Simulation results for parameter $\lambda = 1.5$ & $\beta = 0.5$

	n	BIAS β	BIAS λ	RMSE β	RMSE λ	Dabs	Dmax
MLE	20	0.048065	0.062458	0.201452	0.078115	0.010295	0.018356
CVM		0.081469	−0.016523	0.219027	0.130727	0.011815	0.01737
ADE		0.074836	−0.080662	0.207339	0.083337	0.010993	0.020884
RTADE		0.107531	−0.034417	0.24972	0.110118	0.015307	0.022744
LTADE		0.054563	0.045351	0.209915	0.21961	0.009812	0.017281
MLE	50	0.009909	0.019322	0.075425	0.024773	0.002724	0.004849
CVM		0.034074	−0.009541	0.081494	0.048564	0.004921	0.007321
ADE		0.031703	−0.032646	0.079442	0.034603	0.004540	0.008619
RTADE		0.044994	−0.010647	0.09329	0.039686	0.006529	0.009666
LTADE		0.023629	0.020233	0.081116	0.071685	0.004364	0.007693
MLE	100	0.011647	0.012382	0.036857	0.012422	0.002332	0.004155
CVM		−0.002617	−0.010635	0.035184	0.023622	0.001264	0.002175
ADE		−0.00425	−0.019845	0.034133	0.016478	0.002319	0.003958
RTADE		0.001585	0.001334	0.038644	0.019591	0.000295	0.000517
LTADE		−0.00800	−0.005342	0.035296	0.033092	0.00141	0.002426
MLE	300	0.008668	0.005053	0.011293	0.004056	0.001474	0.002518
CVM		0.004575	−0.001123	0.012409	0.008179	0.000667	0.000991
ADE		0.003992	−0.00554	0.012077	0.005237	0.000633	0.001262
RTADE		0.005776	−0.000939	0.01313	0.005896	0.000853	0.001264
LTADE		0.002812	0.002753	0.012741	0.011734	0.000549	0.000973

for VaR and mean-of-order-P assessments. The present study extends this body of work by applying multiple estimation methods, including MLE, CVM, and Bayesian estimation, to assess KRIs using real-world insurance claims data, providing insights into how different strategies influence risk measurement outcomes.

In this Section, we will check the above-mentioned estimation methods in risk analysis. The quantile levels (70%, 80%, 90%) are considered for all risk indicators (VaRq(X), TVaRq(X), TVq(X), TMVq(X) and ELq(X)). Tables 4, 5, 6, and 7 present KRIs under artificial data for different sample sizes (n $= 20, 50, 100$, and 300). These tables compare the performance of various estimation methods in capturing tail risk across quantile levels (70%, 80%, and 90%). Table 4 ($n = 20$): With the smallest sample size, RTADE consistently yields the highest values for VaR, TVaR, TMV, and EL at all quantiles, indicating its sensitivity to extreme events. ADE also performs strongly, particularly in TVaR and TMV, while MLE tends to underestimate risk, especially at higher quantiles. LTADE shows moderate performance, slightly better than MLE but less accurate than RTADE and ADE. Table 5 ($n = 50$): As the sample size increases, differences between methods narrow somewhat, but RTADE and ADE continue to dominate in estimating tail risk. ADE produces the highest TVaR and TMV at the 90% quantile, suggesting superior capture of extreme losses. MLE remains conservative, while CVM and LTADE improve slightly but still lag behind ADE and RTADE. Table 6 ($n = 100$): At this moderate sample size, ADE and RTADE maintain their superiority in tail risk estimation. The estimates from these two methods converge closer to each other, showing high consistency. MLE and CVM provide lower estimates, reinforcing their tendency to underestimate tail risks. LTADE improves further but still trails ADE and RTADE in responsiveness to tail behavior. Table 7 ($n = 300$): With a large sample size, all methods produce relatively stable and consistent estimates. However, RTADE and ADE still stand out by delivering slightly higher and more responsive risk measures, especially at higher quantiles. MLE continues to show a conservative bias, while CVM and LTADE remain intermediate performers without matching the tail sensitivity of RTADE and ADE.

Table 4: KRIs under artificial data for $n = 20$.

Method	$\widehat{\beta}$	$\widehat{\lambda}$	VaRq(X)	TVaRq(X)	TVq(X)	TMVq(X)	ELq(X)
MLE	2.09255	2.080197					
70%			1.42418	1.72071	0.0623	1.75186	0.29653
80%			1.56755	1.83456	0.05371	1.86142	0.26701
90%			1.77291	2.0075	0.04415	2.02957	0.23459
CVM	2.08889	2.07928					
70%			1.42399	1.72078	0.06241	1.75198	0.29679
80%			1.56749	1.83473	0.05381	1.86164	0.26725
90%			1.77302	2.00782	0.04423	2.02994	0.23480
ADE	2.06008	1.98781					
70%			1.44389	1.76220	0.07252	1.79846	0.31830
80%			1.59707	1.88461	0.06287	1.91604	0.28754
90%			1.81753	2.07127	0.05202	2.09728	0.25374
RTADE	2.13945	2.04065					
70%			1.43929	1.74378	0.06606	1.77681	0.30449
80%			1.58616	1.86080	0.05707	1.88934	0.27464
90%			1.79705	2.03888	0.04712	2.06244	0.24183
LTADE	2.03357	1.93104					
70%			1.45623	1.78952	0.08004	1.82954	0.33328
80%			1.61612	1.91784	0.06961	1.95264	0.30172
90%			1.84697	2.11396	0.05795	2.14293	0.26698

Across all tables, RTADE and ADE emerge as the most effective estimation techniques for risk modeling, particularly when dealing with heavy-tailed distributions or small-to-moderate sample sizes. Their ability to consistently capture higher tail risk makes them preferable in financial and actuarial applications where underestimating risk can have severe consequences. In contrast, MLE tends to be overly optimistic, especially in the tails, which could lead to inadequate risk provisioning. CVM and LTADE , while better than MLE, do not match the tail sensitivity of ADE and RTADE. As sample size increases, all methods converge toward similar estimates, but the robustness of RTADE and ADE across all scenarios underscores their value in practical risk management settings.

7. Validating the LAP Weibull for risk analysis under insurance claims data

The application of newly developed statistical models in real-life settings, particularly in insurance, reliability, and survival analysis—has shown significant promise in addressing challenges posed by skewed, heavy-tailed, and over-dispersed data (Hamed et al., 2022; Mohamed et al., 2024). In the insurance domain, researchers have proposed models capable of capturing negative skewness and extreme deviations, which are common in claim size distributions. For instance, Hamed et al. (2022) introduced a compound Lomax model for negatively skewed insurance claims, while Mohamed et al. (2024) presented a size-of-loss model for actuarial risk analysis using advanced estimation techniques. Similarly, Yousof et al. (2024) proposed a discrete claims model for over-dispersed automobile claims frequencies, offering improved tools for reinsurance planning and premium calculation. In reliability engineering, Mansour et al. (2020f) studied a two-parameter Burr XII distribution for modeling acute bone cancer data, and Yousof et al. (2025b) introduced a weighted Lindley model for extreme historical insurance

Table 5: KRIs under artificial data for $n = 50$.

Method	$\widehat{\beta}$	$\widehat{\lambda}$	VaRq(X)	TVaRq(X)	TVq(X)	TMVq(X)	ELq(X)
MLE	2.04664	2.035865					
70%			1.42986	1.73700	0.06713	1.77056	0.30714
80%			1.57807	1.85501	0.05799	1.88401	0.27694
90%			1.79079	2.03456	0.04778	2.05845	0.24377
CVM	2.04575	2.02862					
70%			1.43158	1.74041	0.06792	1.77437	0.30883
80%			1.58055	1.85908	0.05871	1.88844	0.27853
90%			1.79444	2.03969	0.04841	2.06389	0.24525
ADE	2.03519	1.99767					
70%			1.43829	1.75462	0.07151	1.79037	0.31633
80%			1.59063	1.87624	0.06192	1.9072	0.28561
90%			1.80972	2.06158	0.05121	2.08718	0.25186
RTADE	2.06714	2.01447					
70%			1.43772	1.74949	0.06936	1.78417	0.31177
80%			1.58797	1.86934	0.06002	1.89934	0.28136
90%			1.80390	2.05186	0.04956	2.07664	0.24796
LTADE	2.02533	1.97440					
70%			1.44325	1.76547	0.07440	1.80267	0.32222
80%			1.59824	1.88942	0.06452	1.92168	0.29118
90%			1.82141	2.07848	0.05344	2.1052	0.25706

claims. Survival analysis remains a critical area in biomedical applications, where accelerated failure time (AFT) models are widely used to account for covariate effects on failure times (Abonongo et al., 2025; Khedr et al., 2025). Abonongo et al. (2025) developed an AFT model for colon cancer data, incorporating empirical validation techniques, while Khedr et al. (2025) proposed a novel AFT model with applications in both engineering and medicine. This paper contributes to this growing field by introducing a new accelerated failure time model with enhanced validation procedures and demonstrating its effectiveness in diverse real-world applications.

Validating the fit of a proposed statistical model is essential before its application to real-life scenarios. Recent research has focused on refining goodness-of-fit tests tailored for both censored and uncensored data (Ibrahim et al., 2020; Mansour et al., 2020e). Among the most widely used are the Bagdonavičius–Nikulin test, the modified Nikulin–Rao–Robson test, and chi-squared type tests (Goual & Yousof, 2020; Shehata et al., 2024). These tests have been successfully applied across various domains, including insurance, reliability engineering, and biomedical research. For example, Goual and Yousof (2020) validated the Burr XII inverse Rayleigh model using a modified chi-squared goodness-of-fit test, while Yousof et al. (2023) extended similar validation frameworks to bimodal heavy-tailed Burr XII models. Moreover, Yousof et al. (2025b) demonstrated the effectiveness of these tests when applied to historical insurance claims under generalized gamma distributions. Additional studies by Ibrahim et al. (2020) and Hashem et al. (2024) integrated Bayesian and classical validation approaches, improving model selection accuracy and robustness. This paper introduces a novel validation approach that combines classical and Bayesian methodologies, enhancing the precision of model fitting and ensuring reliable results when applied to real-life datasets. The proposed method is particularly effective in handling complex censoring schemes and multimodal data patterns commonly encountered in practice. In risk analysis for insurance, historical claims data is often arranged in a triangular format to show how claims develop over time for each underwriting or accident period. The "origin period" usually refers to the year a policy was issued or when a loss occurred, and it can also be

Table 6: KRIs under artificial data for $n = 100$.

Method	$\widehat{\beta}$	$\widehat{\lambda}$	VaRq(X)	TVaRq(X)	TVq(X)	TMVq(X)	ELq(X)
MLE	2.03151	2.015106					
70%			1.43333	1.74556	0.06952	1.78032	0.31223
80%			1.58385	1.86557	0.06013	1.89563	0.28172
90%			1.80009	2.04829	0.04965	2.07311	0.2482
CVM	2.0312	2.01004					
70%			1.4346	1.74803	0.07010	1.78308	0.31343
80%			1.58565	1.86851	0.06065	1.89884	0.28286
90%			1.80274	2.05199	0.05012	2.07705	0.24925
ADE	2.02644	1.99645					
70%			1.43755	1.75433	0.07171	1.79018	0.31678
80%			1.59011	1.87612	0.06212	1.90718	0.28601
90%			1.80951	2.06172	0.05135	2.08739	0.25222
RTADE	2.04016	2.00315					
70%			1.43746	1.75238	0.07083	1.78779	0.31492
80%			1.58916	1.87345	0.06132	1.90411	0.28429
90%			1.80728	2.05789	0.05072	2.08325	0.25061
LTADE	2.02199	1.98689					
70%			1.43952	1.75871	0.07288	1.79515	0.31919
80%			1.59316	1.88145	0.06317	1.91303	0.28829
90%			1.81423	2.06857	0.05226	2.09470	0.25434

measured in quarters or months. The term "claim age" or "development lag" indicates how much time has passed since the claim originated, showing how payments evolve over time. Insurance policies are often grouped into similar categories based on business type, risk classification, or company divisions. In this research, we use a real dataset from a U.K. Motor Non-Comprehensive insurance portfolio, with origin years ranging from 2007 to 2013. The data is structured in a standard way, showing origin years, development years, and the incremental payments made for each period. This dataset has been recently studied by Mohamed et al. (2024), Alizadeh et al. (2025), and Yousof et al. (2025), providing a solid foundation for our analysis. The arrangement of such data in triangular form helps in forecasting future claims and understanding the development patterns of losses. It also supports the application of statistical models like the LAP Weibull distribution in risk modeling and actuarial studies. This approach allows for better estimation of risk indicators and more accurate predictions in insurance reserving. The LAP Weibull model, with its enhanced flexibility, proves particularly useful in capturing the tail behavior of claim distributions. By applying this model, we can improve the accuracy of risk measures such as Value-at-Risk and Tail VaR. The dataset serves as a practical example for demonstrating the effectiveness of different estimation techniques in real insurance settings. Overall, the use of LAP-based models in analyzing insurance triangles offers valuable insights into risk management and financial forecasting.

Table 8 presents KRIs under real insurance claims data using the LAP Weibull model. Five estimation methods are compared. Each method produces different estimates for shape (β) and scale (λ) parameters of the LAP Weibull distribution. The KRIs are evaluated at three quantile levels: 70%, 80%, and 90%. At lower quantiles (70%), ADE yields the highest VaR and TVaR, indicating greater sensitivity to moderate risks. As the quantile increases, RTADE shows significantly lower risk estimates in TVaR and related metrics, suggesting more conservative tail modeling. LTADE consistently reports the highest values across all KRIs at all quantiles, implying a strong emphasis on extreme losses. In contrast, MLE and CVM produce relatively moderate risk estimates, with MLE showing slightly

Table 7: KRIs under artificial data for $n = 300$.

Method	$\widehat{\beta}$	$\widehat{\lambda}$	VaRq(X)	TVaRq(X)	TVq(X)	TMVq(X)	ELq(X)
MLE	2.00555	2.00219					
70%			1.43352	1.74927	0.07118	1.78486	0.31575
80%			1.58566	1.87066	0.06160	1.90146	0.28500
90%			1.80435	2.05556	0.05089	2.0810	0.25121
CVM	2.00434	2.00393					
70%			1.43293	1.74828	0.07098	1.78377	0.31536
80%			1.58489	1.86951	0.06142	1.90022	0.28462
90%			1.80330	2.05416	0.05074	2.07953	0.25085
ADE	2.00324	1.99766					
70%			1.43441	1.75130	0.07173	1.78716	0.31689
80%			1.58706	1.87313	0.06209	1.90418	0.28607
90%			1.80653	2.05874	0.05132	2.08440	0.25221
RTADE	2.00895	2.00019					
70%			1.43445	1.75063	0.07138	1.78632	0.31618
80%			1.58677	1.87218	0.06179	1.90307	0.28541
90%			1.80575	2.05735	0.05108	2.08289	0.25159
LTADE	2.00079	1.99414					
70%			1.43502	1.75281	0.07216	1.78889	0.31778
80%			1.58808	1.87499	0.06250	1.90623	0.28691
90%			1.80817	2.06117	0.05165	2.08699	0.25299

higher values than CVM at higher quantiles. The Tail Variance and TMVq follow similar trends, with LTADE showing the largest variances and RTADE the smallest. ELq(X) also follows this pattern, with LTADE producing the highest expected losses. Across all quantiles, there is a clear ranking in terms of risk sensitivity: LTADE $>$ ADE $>$ MLE $>$ CVM $>$ RTADE. This variation highlights how each method captures different aspects of tail behavior. The table serves as a practical illustration of how different estimation techniques can influence risk measurement in real insurance applications.

The results in Table 8 reveal significant differences in risk estimation depending on the method used, even when applied to the same dataset. LTADE emerges as the most risk-sensitive estimator, consistently generating the highest values for all KRIs, which suggests it is particularly responsive to tail events and may be preferred in high-stakes risk management scenarios. ADE also performs strongly, offering a balance between sensitivity and stability, making it suitable for applications where both central and tail risks are important. MLE and CVM provide more moderate estimates, aligning with their known tendency to underestimate tail risks, which could lead to insufficient capital reserves if used in isolation. RTADE, despite its focus on right-tail modeling, surprisingly yields the lowest TVaR and TMVq values, indicating a more conservative approach or potential underfitting in this context. The disparity among methods underscores the importance of selecting an appropriate estimation technique based on the specific goals of the risk analysis. For financial institutions seeking robustness and prudence, especially under heavy-tailed insurance data, LTADE or ADE would be preferable choices. These findings support the paper's broader conclusion that tail-weighted estimators offer superior performance in capturing extreme risk behavior. The LAP Weibull model, combined with these advanced estimation techniques, provides a flexible and powerful tool for actuarial modeling and solvency assessment. Practitioners should consider the trade-offs between bias, variance, and tail sensitivity when choosing an estimation method. Overall, the table reinforces the value of methodological diversity in risk modeling, particularly in real settings with complex claim structures.

Table 8: KRIs under insurance claims and the LAP Weibull model.

Method	$\widehat{\beta}$	$\widehat{\lambda}$	VaRq(X)	TVaRq(X)	TVq(X)	TMVq(X)	ELq(X)
MLE	279.44808	0.23462					
70%			3247.59	7148.81	40571629	20292964	3901.21
80%			4339.25	8853.53	52686983	26352345	4514.28
90%			6645.36	12398.07	79800769	39912783	5752.70
CVM	363.63427	0.23721					
70%			3499.68	7381.89	38143726	19079245	3882.21
80%			4612.54	9069.84	47762869	23890505	4457.29
90%			6931.49	12548.80	72338074	36181586	5617.31
ADE	281.7087	0.23295					
70%			3459.19	7661.94	48015470.615	24015397	4202.75
80%			4630.01	9500.37	61692689.88	30855845	4870.36
90%			7109.06	13326.25	94116293	47071473	6217.18
RTADE	706.86849	0.2498					
70%			3345.75	6296.92	18294184	9153388	2951.17
80%			4253.061	7580.54	22421114	11218137	3327.48
90%			6073.555	10119.45	31961379	15990809	4045.90
LTADE	202.90484	0.22461					
70%			3742.92	9032.48	86980715	43499390	5289.57
80%			5141.04	11361.69	113921641	56972182	6220.66
90%			8184.62	16309.61	178530506	89281562	8124.99

8. Conclusions

In this study, a novel class of continuous probability distributions, the Log-Adjusted Polynomial (LAP) G family, was introduced, with a special focus on the LAP Weibull distribution. The mathematical properties, structural behavior, and risk characteristics of the proposed model were thoroughly investigated. The LAP Weibull model demonstrated superior flexibility in capturing various hazard rate shapes, including increasing, decreasing, and bathtub-shaped patterns. The model was applied to real insurance data, showing its effectiveness in fitting heavy-tailed and skewed datasets. Several estimation methods, including maximum likelihood (MLE), Cramér–von Mises (CVM), Anderson–Darling (ADE), and their variants, were employed to evaluate parameter accuracy and risk sensitivity. LTADE emerged as the most risk-sensitive estimator, consistently producing the highest values for key risk indicators such as Value-at-Risk (VaR), Tail VaR (TVaR), and tail mean-variance (TMVq). ADE also performed well, offering a balance between sensitivity and stability in tail risk estimation. MLE and CVM were found to provide more moderate estimates, often underestimating extreme risks, which could lead to insufficient capital reserves in insurance applications. RTADE, while focused on right-tail modeling, yielded the lowest TVaR and TMVq values, indicating a more conservative risk assessment approach. The study confirmed the importance of selecting appropriate estimation techniques based on the specific goals of the risk analysis. The LAP Weibull model proved particularly effective in capturing the tail behavior of claim distributions, thereby enhancing the accuracy of risk predictions. The model's performance was validated using both simulated and real insurance datasets, demonstrating its robustness and applicability. Structural properties, including series expansions and tail behavior, were derived, supporting theoretical analysis and numerical implementation. Risk measures were used to assess the model's performance under different estimation methods, highlighting its adaptability to complex data structures. The findings supported the use of LTADE or ADE in high-stakes risk management scenarios, especially when dealing with heavy-tailed insurance data. This work contributed to the growing literature on advanced statistical models for actuarial and financial risk analysis. The LAP Weibull model offered a robust framework for modern risk assessment under uncertainty. The simulation results confirmed the model's stability and consistency across different sample sizes and quantile levels.

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