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b-Local Irregular Chromatic Number of Graphs

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Abstract In this paper, we study a new notation of coloring of graph, namely a b-local irregularity coloring. Suppose $l : V(G) \to \{1, 2, \dots, k\}$ is called vertex irregular k-labeling and $w : V(G) \to N$, where $w(u) = \sum_{v \in N(u)} l(v)$. Every color class has a representative adjacent to at least one vertex in each of the color classes.l is b-local irregularity coloring. The b-local irregular chromatic number denoted by $\chi_{b-lis}(G)$ is the largest of k such that G admits a b-local irregularity coloring. In this paper, we study the b-local irregular chromatic number of graphs namely path, cycle, star, friendship, complete, complete bipartite, and Wheel graph.

Keywords b-coloring, local irregularity, vertex coloring, chromatic number

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1. Introduction

In this paper, we consider a simple and connected graph $G(V, E)$, where V and E are vertex and edge sets respectively, for more detail at [1]. Suppose $l : V(G) \to \{1, 2, \dots, k\}$ is called vertex irregular k-labeling and $w : V(G) \to N$, where $w(u) = \sum_{v \in N(u)} l(v), l$ is called local irregularity vertex coloring. The minimum cardinality of local irregularity vertex coloring is called the chromatic local irregular number, denoted by $\chi_{lis}(G)$. The former focuses on assigning colors (represented by positive integers) to the vertices of a graph such that the sum of the colors assigned to the neighbors of any two adjacent vertices is distinct. This concept, introduced and expanded by Kristiana et al. [2], has been applied to various graph classes including paths, cycles, trees, and more recently, bicyclic graphs [3].

Irving and Manlove [6] introduced the b-chromatic number. The b-chromatic number of a graph G is the largest positive integer k such that G admits a proper k- coloring in which every color class has a representative adjacent to at least one vertex in each of the other color classes. Such a coloring is called a b-coloring. b-coloring has gained prominence for its requirement that each color class must contain at least one vertex that is adjacent to vertices in all other color classes. This ensures that the coloring reflects a form of representative.

There are some previous results of b-coloring in some families graph, namely the regular graph [11], the windmill graph [4], bipartite graph [10] and the graph resulting operation namely the cartesian product [7], the product of the path and cycle [9], the corona and shadow in C_n^k [8], and corona product [5].

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In this study, we initiate to combine the two notion, namely local irregularity vertex coloring and b-coloring. We name for this combination as b-local irregularity coloring. This new concept defines a coloring of a graph where (1) for every edge uv, the sum of the colors of the neighbors of u is not equal to that of v (local irregularity condition), and (2) every color class contains at least one vertex that is adjacent to all other color classes (b-coloring condition). The b-local irregular chromatic number of a graph is the minimum number of colors required to satisfy both conditions simultaneously. Some research results related to local irregularity coloring and b-coloring used in this paper are as follows:

Proposition 1
If $S_n; K_n; P_n; C_n; W_n$ are respectively the star graph, the complete graph, the path graph, the cycle and the wheel graph on n vertices, then $\varphi(S_n) = 1; \varphi(K_n) = n; \varphi(P_n) = \varphi(C_n) = 3; \varphi(W_n) = 4, n \geq 5$

Proposition 2
For every $n, m \in N$, we have $\varphi(K_{(n,m)}) = 2$, where $K_{(n,m)}$ is a complete bipartite graph.

Proposition 3
If $S_n; K_n; P_n; C_n; W_n$ are respectively the star graph, the complete graph, the path graph, the cycle graph and the wheel graph on n vertices, then $\chi_{lis}(S_n) = 2; \chi_{lis}(K_n) = n; \chi_{lis}(P_n) = \chi_{lis}(C_n) = 3; \chi_{lis}(W_n) = 4, n \geq 5$

Proposition 4
For every $n, m \in N$, we have $\chi_{lis}(K_{(n,m)}) = 2$, where $K_{(n,m)}$ is a complete bipartite graph.

2. Results and Discussion

First, we define the new concept of local irregularity vertex coloring and b-coloring as follows.

Definition 1
Let $l : V(G) \to \{1, 2, \ldots, k\}$ is called vertex irregular k-labeling and $w : V(G) \to N$ where $w(u) = \sum_{v \in N(u)} l(v), l$ is called b-local irregularity coloring if:

1. $opt(l) = min\{max\{l_i\}, l_i$ vertex irregular labeling$\}$
2. for every $uv \in E(G); w(u) \neq w(v)$
3. every color class has a representative adjacent to at least one vertex in each of the color classes.

Definition 2
The b-local irregular chromatic number denoted by $\chi_{b-lis}(G)$ is the largest of k such that G admits a b-local irregularity coloring.

Lemma 1
For any graph G with $\chi_{lis}(G)$ is local irregular chromatic number and $\varphi(G)$ is b-chromatic number, then $max\{\chi_{lis}(G), \varphi(G)\} \leq \chi_{b-lis}(G) \leq \Delta(G) + 1$.

Observation 1
Let deg (u) and deg(v) be a degree of two vertices in $V(G)$. If two adjacent vertices u, v has $|deg(u) - deg(v)| \neq 0$ the $opt(G) = 1$

Observation 2
Let deg (u) and deg(v) be a degree of two vertices in $V(G)$. If two adjacent vertices u, v has $|deg(u) - deg(v)| = 0$ the $opt(G) \geq 2$

We determine the b-local irregular chromatic number of graphs namely path (P_n), cycle (C_n), star (S_n), friendship (Fr_n), complete $(K - n)$, complete bipartite $(K_{n,m})$ and Wheel (W_n).

Theorem 1
Let P_n be a path graph, for $n \geq 4$, the b-local irregular chromatic number is $\chi_{b-lis}(P_n) = 3$

Proof
Let P_n be a path graph with the vertex set $V(P_n) = \{x_i : 1 \leq i \leq n\}$ and the edge set $E(P_n) = \{x_i x_{i+1} : 1 \leq i \leq n-1\}$. The maximum (Δ) degree of P_n is 2 and based on Observation 2, $opt(l)(P_n) = 2$. Based on Proposition 1, 3 and Lemma 1, $max\{\chi_{lis}(P_n), \varphi(P_n)\} = max\{3, 3\} \leq \chi_{b-lis}(P_n) \leq \Delta(P_n) + 1 = 2 + 1 = 3$.
Case 1. When n is even
Furthermore, we define $l : V(P_n) \to \{1, 2\}$ as follows:

$$l(x_i) = \begin{cases} 1, & i \text{ is even, } i \neq 2, n \\ 2, & i \text{ is odd, } i = 2, n \end{cases}.$$

The labeling provides vertex-weights as follows:

$$w(x_i) = \begin{cases} 2, & i \text{ is odd, } i = n\,; i \neq 3, n-1 \\ 3, & i = 3, n-1 \\ 4, & i \text{ is even,} i \neq n \end{cases}.$$

Based on Definition 1 and vertex weight, we have three color classes, namely $C_1 = \{x_i : i \ is\ odd; i = 1, n; i \neq 3, n-1\}, C_2 = \{x_i : i = 3, n-1\}$ and $C_3 = \{x_i : i\ is\ even\}$. By taking $x_1 \in C_1$, then there is $x_1 x_2 \in E(P_n)$ where $x_2 \in C_3$ and $x_3 \in C_2$, then there is $x_2 x_3 \in E(P_n)$ where $x_2 \in C_3$. It shows that every color class has a representative adjacent to at least one vertex in each other color classes. Thus, the definition of b-local irregularity coloring is satisfied.
Case 2. When n is odd
Furthermore, we define $l : V(P_n) \to \{1, 2\}$ as follows:

$$l(x_i) = \begin{cases} 1, & i \text{ is odd, } i \neq 1, n \\ 2, & i \text{ is even, } i = 1, n \end{cases}.$$

The labeling provides vertex-weights as follows:

$$w(x_i) = \begin{cases} 2, & i \text{ is even, } i = 1, n\,; i \neq 2, n-1 \\ 3, & i = 2, n-1 \\ 4, & i \text{ is odd,} i \neq 1, n \end{cases}.$$

Based on Definition 1 and vertex weight, we have three color classes, namely $C_1 = \{x_i : i\ is\ even; i = 1, n; i \neq 2, n-1\}, C_2 = \{x_i : i = 2, n-1\}$ and $C_3 = \{x_i : i\ is\ odd; i \neq 1, n\}$. By taking $x_1 \in C_1$, then there is $x_1 x_2 \in E(P_n)$ where $x_2 \in C_2$ and $x_3 \in C_3$, then there is $x_2 x_3 \in E(P_n)$ where $x_2 \in C_2$. It shows that every color class has a representative adjacent to at least one vertex in each other color classes. Thus, the definition of b-local irregularity coloring is satisfied. Based on Definition 2 and the two cases above, it is concluded $\chi_{b-lis}(P_n) = 3$. □

Theorem 2
Let C_n be a cycle graph, for $n \geq 4$, the b-local irregular chromatic number is $\chi_{b-lis}(C_n) = 3$

Proof
Let C_n be a cycle graph with vertex set $V(C_n) = \{x_i : 1 \leq i \leq n\}$ and edge set $E(C_n) = \{x_i x_{i+1} : 1 \leq i \leq n-1\} \bigcup \{x_n x_1\}$. The Maximum$(\Delta)$ degree of C_n is 2 and and based on Observation 2, $opt(C_n) = 3$. Based on Proposition 1, 3 and Lemma 1, $max\{\chi_{lis}(C_n), \varphi(C_n)\} = max\{3, 3\} \leq \chi_{b-lis}(C_n) \leq \Delta(C_n) + 1 = 2 + 1 = 3$.
Case 1. When $n \equiv 0 \mod 3$
We define $l : V(C_n) \to \{1, 2, 3\}$ as follows:

$$l(x_i) = \begin{cases} 1, & i \equiv 1 \mod 3 \\ 2, & i \equiv 2 \mod 3 \\ 3, & i \equiv 0 \mod 3 \end{cases}$$

The labeling provides vertex-weights as follows:

$$w(x_i) = \begin{cases} 3, & i \equiv 0 \mod 3 \\ 4, & i \equiv 2 \mod 3 \\ 5, & i \equiv 1 \mod 3 \end{cases}$$

Based on Definition 1 and vertex weight, we have three color classes, namely $C_1 = \{x_i : i \equiv 0 \mod 3\}$, $C_2 = \{x_i : i \equiv 2 \mod 3\}$, and $C_3 = \{x_i : i \equiv 1 \mod 3\}$. By taking $x_2 \in C_2$, then there are $x_3 \in C_3, x_1 \in C_1$ such that $x_2x_3, x_1x_2 \in E(C_n)$, and if we take $x_3 \in C_3$, the there is $x_4 \in C_1$ such that $x_3x_4 \in E(C_n)$. It shows that every color class has a representative adjacent to at least one vertex in each other color classes. Thus, the definition of b-local irregularity coloring is satisfied.

Case 2. When $n \equiv 1 \mod 3$
We define $l : V(C_n) \to \{1, 2, 3\}$ as follows:

$$l(x_i) = \begin{cases} 1, & i \equiv 1 \mod 3 \\ 2, & i \equiv 2 \mod 3 \\ 3, & i \equiv 0 \mod 3 \end{cases}$$

The labeling provides vertex-weights as follows:

$$w(x_i) = \begin{cases} 3, & i \equiv 0 \mod 3,\ i = 1 \\ 4, & i \equiv 2 \mod 3,\ i = n \\ 5, & i \equiv 1 \mod 3,\ i \neq 1, n \end{cases}$$

Based on Definition 1 and vertex weight, we have three color classes, namely $C_1 = \{x_i : i \equiv 0 \mod 3,\ i = 1\}$, $C_2 = \{x_i : i \equiv 2 \mod 3,\ i = n\}$, and $C_3 = \{x_i : i \equiv 1 \mod 3,\ i \neq 1, n\}$. By taking $x_4 \in C_3$, then there are $x_3 \in C_1, x_5 \in C_2$ such that $x_4x_3, x_4x_5 \in E(C_n)$ and if we take $x_3 \in C_1$, the there is $x_2 \in C_3$ such that $x_3x_2 \in E(C_n)$. It shows that every color class has a representative adjacent to at least one vertex in each other color classes. Thus, the definition of b-local irregularity coloring is satisfied.

Case 3. When $n \equiv 2 \mod 3$
We define $l : V(C_n) \to \{1, 2, 3\}$ as follows:

$$l(x_i) = \begin{cases} 1, & i \equiv 1 \mod 3,\ i \neq 1 \\ 2, & i \equiv 2 \mod 3,\ i = 1 \\ 3, & i \equiv 0 \mod 3 \end{cases}$$

The labeling provides vertex-weights as follows:

$$w(x_i) = \begin{cases} 3, & i \equiv 0 \mod 3,\ i = n \\ 4, & i \equiv 2 \mod 3,\ i = 1,\ i \neq 2, n \\ 5, & i \equiv 1 \mod 3,\ i = 2,\ i \neq 1 \end{cases}$$

Based on Definition 1 and vertex weight, we have three color classes, namely $C_1 = \{x_i : i \equiv 0 \mod 3,\ i = n\}$, $C_2 = \{x_i : i \equiv 2 \mod 3,\ i = 1,\ i \neq 2, n\}$, and $C_3 = \{x_i : i \equiv 1 \mod 3,\ i = 2,\ i \neq 1\}$. By taking $x_2 \in C_3$, then there are $x_3 \in C_1, x_1 \in C_2$ such that $x_1x_2, x_2x_3 \in E(C_n)$ and if we take $x_5 \in C_2$, the there is $x_6 \in C_1$ such that $x_5x_6 \in E(C_n)$. It shows that every color class has a representative adjacent to at least one vertex in each other color classes. Thus, the definition of b-local irregularity coloring is satisfied. Based on Definition 2 and the three cases above, it is concluded $\chi_{b-lis}(C_n) = 3$. □

Illustration of the b-local irregularity coloring of cycle graph, see Figure 1.

Theorem 3
Let S_n be a star graph, for $n > 2$, the local b-irregular chromatic number is $\chi_{b-lis}(S_n) = 2$.

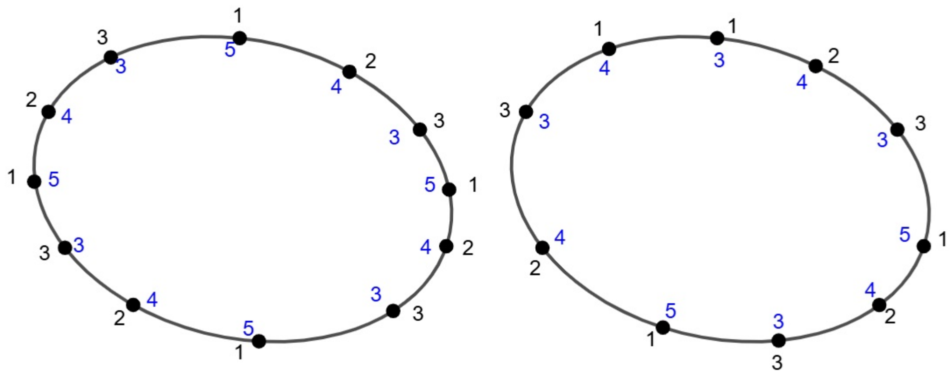

Figure 1. Cycle graph C_{12} and C_{10}.

Proof
Let S_n be a star graph with vertex set $V(S_n) = \{x_i : 1 \le i \le n\}$ and edge set $E(S_n) = \{x_1x_i : 2 \le i \le n\}$. The Maximum($\Delta$) degree of S_n is $n-1$ and based on Observation 1,The $opt(S_n) = 1$. Based on Proposition 1, 3 and Lemma 1, $max\{\chi_{lis}(S_n), \varphi(S_n)\} = max\{2, 2\} = 2 \le \chi_{b-lis}(S_n) \le \Delta(S_n) + 1 = 1 + 1 = 2$.
Furthermore, we define $l : V(S_n) \to \{1\}$ as $l(x_i) = 1$. The labeling provides vertex-weights as follows:

$$w(x_i) = \begin{cases} 1, & 2 \le i \le n \\ n-1, & i = 1 \end{cases}$$

Based on Definition 1 and vertex weight we have two color classes namely $C_1 = \{x_i : 2 \le i \le n\}$ and $C_2 = \{x_1\}$. By taking $x_1 \in C_2$, then there is $x_1x_2 \in E(S_n)$ where $x_2 \in C_1$. It shows that every color class has a representative adjacent to at least one vertex in each other color classes. Thus, the definition of b-local irregularity coloring is satisfied. Based on Definition 2, it concluded $\chi_{b-lis}(S_n) = 2$ □

Theorem 4
Let W_n be a wheel graph, for $n > 4$, the b-local irregular chromatic number is $\chi_{b-lis}(W_n) = 4$

Proof
Let W_n be a wheel graph with vertex set $V(W_n) = \{x_i : 1 \le i \le n\} \cup \{x\}$ and edge set $E(W_n) = \{x_ix_{i+1} : 1 \le i \le n-1\} \bigcup \{x_nx_1\} \bigcup \{x_ix : 1 \le i \le n\}$. The maximum($\Delta$) $= n - 1$ and based on Observation 2, $opt(W_n) = 3$. Based on Proposition 1, 3 and Lemma 1, $max\{\chi_{lis}(W_n), \varphi(W_n)\} = max\{4, 4\} \le \chi_{b-lis}(W_n) \le \Delta(W_n) + 1 = n + 1 = 3$.
Case 1. When $n \equiv 0 \mod 3$
We define $l : V(W_n) \to \{1, 2, 3\}$ as follows:

$$l(x) = 1$$
$$l(x_i) = \begin{cases} 1, & i \equiv 1 \mod 3 \\ 2, & i \equiv 2 \mod 3 \\ 3, & i \equiv 0 \mod 3 \end{cases}$$

The labeling provides vertex-weights as follows:

$$w(x) = 2n \; w(x_i) = \begin{cases} 4, & i \equiv 0 \mod 3 \\ 5, & i \equiv 2 \mod 3 \\ 6, & i \equiv 1 \mod 3 \end{cases}$$

Based on Definition 1 and vertex weight, we have three color classes, namely $C_1 = \{x_i : i \equiv 0 \mod 3\}$, $C_2 = \{x_i : i \equiv 2 \mod 3\}$, $C_3 = \{x_i : i \equiv 1 \mod 3\}$ and $C_4 = \{x\}$. By taking $x_2 \in C_2$, then there are $x_3 \in C_3, x_1 \in C_1, x \in C_4$ such that $x_2x_3, x_1x_2, xx_2 \in E(W_n)$. By taking $x_3 \in C_3$, the there is $x_4 \in C_1$ such that $x_3x_4 \in E(W_n)$ and if we take $x_1 \in C_1$, then there is $x \in C_4$ such that $xx_1 \in E(W_n)$. It shows that every color class has a representative adjacent to at least one vertex in each other color classes. Thus, the definition of b-local irregularity coloring is satisfied.
Case 2. When $n \equiv 1 \mod 3$
We define $l : V(W_n) \to \{1, 2, 3\}$ as follows:

$$l(x) = 1$$
$$l(x_i) = \begin{cases} 1, & i \equiv 2 \mod 3,\ i = 1 \\ 2, & i \equiv 0 \mod 3 \\ 3, & i \equiv 1 \mod 3,\ i \neq 1 \end{cases}$$

The labeling provides vertex-weights as follows:

$$w(x) = 2n + 1\ w(x_i) = \begin{cases} 4, & i \equiv 1 \mod 3,\ i \neq 1,\ i = 2 \\ 5, & i \equiv 0 \mod 3,\ i = 1 \\ 6, & i \equiv 2 \mod 3,\ i \neq 2 \end{cases}$$

Based on Definition 1 and vertex weight, we have three color classes, namely $C_1 = \{x_i : i \equiv 1 \mod 3,\ i \neq 1,\ i = 2\}$, $C_2 = \{x_i : i \equiv 0 \mod 3,\ i = 1\}$, $C_3 = \{x_i : i \equiv 2 \mod 3,\ i \neq 2\}$ and $C_4 = \{x\}$. By taking $x_4 \in C_1$, then there are $x_3 \in C_3, x_5 \in C_2, x \in C_4$ such that $x_3x_4, x_4x_5, xx_4 \in E(W_n)$. By taking $x_6 \in C_3$, the there is $x_5 \in C_2$ such that $x_5x_6 \in E(W_n)$ and if we take $x_5 \in C_2$, then there is $x \in C_4$ such that $xx_5 \in E(W_n)$. It shows that every color class has a representative adjacent to at least one vertex in each other color classes. Thus, the definition of b-local irregularity coloring is satisfied.
Case 3. When $n \equiv 2 \mod 3$
We define $l : V(W_n) \to \{1, 2, 3\}$ as follows:

$$l(x) = 1$$
$$l(x_i) = \begin{cases} 1, & i \equiv 2 \mod 3,\ i \neq 2 \\ 2, & i \equiv 0 \mod 3,\ i = 1, 2, \\ 3, & i \equiv 1 \mod 3,\ i \neq 1 \end{cases}$$

The labeling provides vertex-weights as follows:

$$w(x) = 2n\ w(x_i) = \begin{cases} 4, & i \equiv 1 \mod 3 \\ 5, & i \equiv 0 \mod 3,\ i \neq 3,\ i = 2 \\ 6, & i \equiv 2 \mod 3,\ i \neq 2,\ i = 3 \end{cases}$$

Based on Definition 1 and vertex weight, we have three color classes, namely $C_1 = \{x_i : i \equiv 1 \mod 3\}$, $C_2 = \{x_i : i \equiv 0 \mod 3,\ i \neq 3,\ i = 2\}$, $C_3 = \{x_i : i \equiv 2 \mod 3,\ i \neq 2,\ i = 3\}$ and $C_4 = \{x\}$. By taking $x_2 \in C_2$, then there are $x_3 \in C_3, x_1 \in C_1, x \in C_4$ such that $x_2x_3, x_1x_2, xx_2 \in E(W_n)$. By taking $x_3 \in C_3$, the there is $x_4 \in C_1$ such that $x_3x_4 \in E(W_n)$ and if we take $x_1 \in C_1$, then there is $x \in C_4$ such that $xx_1 \in E(W_n)$. It shows that every color class has a representative adjacent to at least one vertex in each other color classes. Thus, the definition of b-local irregularity coloring is satisfied. Based on Definition 2 and the three cases above, it is concluded $\chi_{b-lis}(W_n) = 4$. □

Theorem 5
Let Fr_n be a friendship graph, for $n > 4$, the b-local irregular chromatic number is $\chi_{b-lis}(Fr_n) = 3$

Proof
Let Fr_n be a friendship graph with vertex set $V(Fr_n) = \{x\} \cup \{x_i : 1 \leq i \leq n\}$ and edge set $E(Fr_n) = \{x_i x_{i+1} : 1 \leq i \leq n-1\} \cup \{x_n x_1\} \cup \{x x_i : 1 \leq i \leq n\}$. The maximum$(\Delta) = n$, and based on Observation 2, $opt(Fr_n) = 3$. Based on Proposition 1, 3 and Lemma 1, $max\{\chi_{lis}(Fr_n), \varphi(Fr_n)\} = max\{3,3\} = 3 \leq \chi_{b-lis}(W_n) \leq \Delta(Fr_n) + 1 = 2n+1$.
We define $l : V(Fr_n) \rightarrow \{1,2,3\}$ as follows:

$$l(x) = 1$$
$$l(x_i) = \begin{cases} 1, & i \ is \ even \\ 2, & i \ is \ odd \end{cases}$$

The labeling provides vertex-weights as follows:

$$w(x) = 3n$$
$$w(x_i) = \begin{cases} 2, & i \ is \ odd \\ 3, & i \ is \ even \end{cases}$$

Based on Definition 1 and vertex weight, we have two color classes, namely $C_1 = \{x_i : i \ is \ odd\}$, $C_2 = \{x_i : i \ is \ even\}$, and $C_3 = \{x\}$. By taking $x_1 \in C_1$, there is $x_2 \in C_2, x \in C_3$, such that $x_1x_2; x_1x \in E(Fr_n)$ and if we take $x_2 \in C_2$ there is $x \in C_3$ such that $x_2x \in E(Fr_n)$. It shows that every color class has a representative adjacent to at least one vertex in each other color classes. Thus, the definition of b-local irregularity coloring is satisfied. Based on Definition 2 , it is concluded $\chi_{b-lis}(Fr_n) = 3$. □

Illustration of the b-local irregularity coloring of friendship graph, see Figure 2.

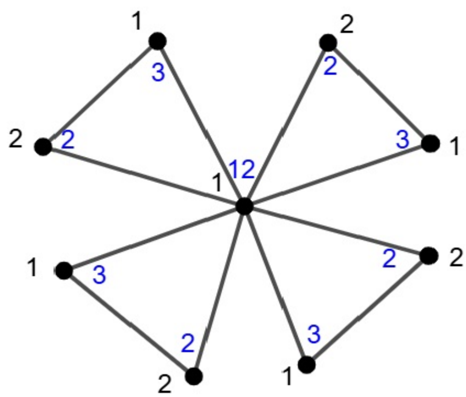

Figure 2. friendship graph Fr_4.

Theorem 6
Let K_n be a complete graph, for $n \geq 4$ the b-local irregular chromatic number is $\chi_{b-lis}(K_n) = n$.

Proof
Let $V(K_n) = \{v_i : 1 \leq i \leq n\}$ be the vertex set of complete graph. Every vertex is adjacent to all other vertices in the graph K_n (i.e., $\delta(K_n) = \Delta(K_n) = n-1$). Since, the labeling of every vertex in K_n is different, we have $opt(K_n) = n$. Furthermore, we define $l : V(K_n) \rightarrow \{1,2,\ldots,n\}$ as $l(x_i) = i, 1 \leq i \leq n$. The vertex-weight is $w(x_i) = \frac{n(n+1)}{2} - i$. The weight of all the vertices are distinct and each weight is adjacent to all other weights. It shows that every color class has a representative adjacent to at least one vertex in each other color classes. Thus, the definition of b-local irregularity coloring is satisfied. Based on Definition 2 , it is concluded $\chi_{b-lis}(K_n) = n$. □

Theorem 7
Let $K_{n,m}$ be a complete bipartite graph, for $n \geq 2$ and$m \geq 2$, the b-local irregular chromatic number is $\chi_{b-lis}(K_{n,m}) = 2$.

Proof
Let $K_{n,m}$ be a complete bipartite graph with vertex set $V(K_{n,m}) = \{x_i : 1 \leq i \leq n\} \bigcup \{y_j : 1 \leq j \leq m\}$ and edge set $E(K_{n,m}) = \{x_i x_j : 1 \leq i \leq n, 1 \leq j \leq m\}$. The maximum($\Delta$) degree of $K_{n,m}$ is $\begin{cases} n, & n \geq m \\ m, & m \geq n \end{cases}$. Based on Observation 2, $opt(K_{n,m}) = 2$. Based on Proposition 2, 3 and Lemma 1, $max\{\chi_{lis}(K_{n,m}), \varphi(K_{n,m})\} = max\{2, 2\} = 2 \leq \chi_{b-lis}(K_{n,m}) \leq \Delta(K_{n,m}) + 1 = n + 1$.
Furthermore, we define $l : V(K_{n,m}) \to \{1, 2\}$ as follows:

$$l(x_i) = 1, 1 \leq i \leq n$$
$$l(y_j) = 2, 1 \leq j \leq m$$

The labeling provides vertex-weights as follows:

$$w(x_i) = 2m, 1 \leq i \leq n$$
$$w(y_j) = n, 1 \leq j \leq m$$

Based on Definition 1 and vertex weight we have two color classes namely $C_1 = \{x_i : 1 \leq i \leq n\}$ and $C_2 = \{y_j : 1 \leq j \leq m\}$. By taking $x_1 \in C_1$, then there is $y_1 \in C_2$ such that $y_1 x_1 \in E(K_{n,m})$. It shows that every color class has a representative adjacent to at least one vertex in each other color classes. Thus, the definition of b-local irregularity coloring is satisfied. Based on Definition 2, it concluded $\chi_{b-lis}(K_{n,m}) = 2$ □

3. Concluding Remarks

In this paper, we have studied the b-local irregularity coloring and obtained the b-local irregular chromatic number of some graphs like path, cycle, star, wheel, friendship, complete and complete bipartite graph.

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