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Quantile-Based Extropy Measure for Record Statistics

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Abstract Compared to their distribution function-based extropy measure (Lad et al. 2015), the quantile-based extropy measures have a few special characteristics (Krishnan et al. 2020). The present communication deals with the study of the quantile-based extropy measure for record statistics. In this context, a generalized model for which there is no cdf or pdf is examined, and several examples are provided for illustration purposes. Additionally, we examine the dynamic version of the suggested extropy measure for record statistics and provide characterization results for that. Finally, we investigate the suggested extropy measure in the F^γ family of distributions.

Keywords Extropy measure; F^γ-family; Hazard rate function; Quantile function; Record value.

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1. Introduction

The term 'record' was first introduced by K. N. Chandler [5]. Suppose $\{X_n, n \geq 1\}$ be a sequence of independently and identically distributed (iid) random variables with $F(x)$ cumulative distribution function (cdf) and $f(x)$ probability density function (pdf), then X_m will be called an upper record value if X_m is greater than all X_i for $i < m$. Similarly, by the replacement of $\{X_n\}$ by $\{-X_n\}$ the lower record values can be obtained. The pdf of m^{th} upper record value is given by

$$f_m(x) = \frac{1}{\Gamma(m)}\{-\ln(\bar{F}(x))\}^{m-1} f(x),\ -\infty < x < +\infty \tag{1}$$

and

$$f^L_m(x) = \frac{1}{\Gamma(m)}\{-\ln(F(x))\}^{m-1} f(x); \quad -\infty < x < +\infty\ , \tag{2}$$

is the pdf of m^{th} lower record value, where $\Gamma(m)$ is the gamma function. For more details and applications of these statistics, see [2] and [6].

"The theory of information" is an individual area of study that emerged from Shannon's proposal for entropy [24]. Further, during the study of the "classical record model" [5] considers the observations from a continuous probability distribution for the underlying sample from which records are obtained and studies the stochastic behavior of random record values. For more details and applications of Shannon entropy based on record value, refer to [22],[3], [4], [30], [1], [16, 17], and [11, 12].

One of the two explanations for a probability distribution is a quantile function or a distribution function. In many situations, quantile functions are more sway to distribution functions. Also, many probability distributions can not be expressed by specific density functions but they can be tracked by some quantile density functions refer to Hankel and Lee [7], Van Staden and Loots [28], and [18]. However, in life testing studies using a quantile

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approach, one need not wait for the failure results of a part to provide useful estimations. The quantile function $Q(w)$ corresponding to a nonnegative random variable X and distribution function $F(x)$, is given as

$$Q(w) = F^{-1}(w) = \inf\{x \mid F(x) \geq w\}, \quad 0 \leq w \leq 1. \tag{3}$$

The derivative of the quantile function is $q(w)$ and $f(Q(w))$, which are known as the quantile density function and density quantile function, respectively [19]. From (3), we obtain $F(Q(w)) = w$ and by differentiating with respect to w we obtain a relationship between quantile density function and density quantile function, defined as

$$q(w)f(Q(w)) = 1.$$

The hazard quantile function and reversed hazard quantile function are important quantile measures that are helpful in reliability analysis. These are defined as

$$H(w) = \frac{1}{(1-w)q(w)}, and \tag{4}$$

$$\bar{H}(w) = \frac{1}{(w)q(w)}, \tag{5}$$

respectively.
The complementary dual to Shannon's entropy measure is the extropy measure (Lad et al. 2015), the differential extropy measure is defined as

$$J(X) = -\frac{1}{2}\int_0^\infty f^2(x)dx \tag{6}$$

For the study the concept of extropy and residual extropy for order statistics and record value refer to [21] and [20]. The residual and past extropy of k-record values[9]. Symmetric properties of extropy and some characterization results for record value [27], and [8].
Recently the study of information measures based on quantile function has found much attention among researchers. The quantile version of extropy (6) has been considered [10], which is defined as

$$\begin{aligned} L(X) &= -\int_0^1 \frac{1}{q(w)}dw \\ &= -\int_0^1 \frac{1}{(1-w)H(w)}dw. \end{aligned} \tag{7}$$

$L(X)$ gives a quantile version of the extropy that measures the uncertainty of X and $H(w)$ is the hazard quantile function.
Quantile-based information measure[25]. Quantile-based entropy of order statistics [26] . For a more details study on quantile-based measures, we refer to [13], [29], [26], and [10].
A study on quantile-based shannon entropy for record statistics has been studied [14], and quantile-based cumulative residual extropy of order statistics has been proposed and studied by Sathar and Vijyan [23]. The present study introduces the results of quantile-based extropy measures for record statistics and related measures. Motivated by these, we proposed and studied the quantile-based extropy measures based on record statistics in this research paper.
The results are described throughout the manuscript as follows: In Section 2, a quantile version of the extropy of m^{th} upper record statistics is introduced and expressed in the form of expectation. A comparative study of the proposed measure for a generalized model and some examples concludes the study of proposed measures for the various models that do not have any closed-form expression for pdf or cdf but have simple or tractable quantile functions or quantile density functions presented in Section 3. In Section 4, the Dynamic quantile extropy measure of m^{th} record, characterization results, bounds, and results for some specific lifetime distributions are studied. In Section 5, the proposed measure has been studied for F^γ-family of distributions, and Section 6 concludes this paper.

2. Quantile-based Extropy Measure of Record Statistics

Using (3), we have $F(q(w)) = w$, then the quantile version of the pdfs of m^{th} upper record (1) and lower record (2) values become respectively as

$$\mathfrak{f}_m(w) = \frac{\{-\ln(1-w)\}^{m-1}}{\Gamma(m)q(w)}; \quad 0 \leq w \leq 1, \tag{8}$$

and

$$\mathfrak{f}_m^L(w) = \frac{\{-\ln(w)\}^{m-1}}{\Gamma(m)q(w)}; \quad 0 \leq w \leq 1. \tag{9}$$

The survival function and hazard rate function for m^{th} upper record value and reverse hazard rate function for m^{th} lower record value are

$$\bar{F}_m(w) = \frac{\Gamma(m; -\ln(1-w))}{\Gamma(m)}; \quad 0 \leq w \leq 1, \tag{10}$$

$$H_m(w) = \frac{1}{\Gamma(m; -\ln(1-w))q(w)}\{-\ln(1-w)\}^{m-1}; \quad 0 \leq w \leq 1, \tag{11}$$

and

$$\bar{H}_m^L(w) = \frac{1}{\gamma(m; -\ln(1-w))q(w)}\{-\ln(1-w)\}^{m-1}; \quad 0 \leq w \leq 1, \tag{12}$$

respectively, where $\Gamma(m; -\ln(1-w))$ and $\gamma(m; -\ln(1-w))$ are incomplete upper and lower gamma functions. Analogous to (6), we proposed the quantile-based extropy for record statistics, which is defined as

$$L\mathfrak{f}^m(w) = -\frac{1}{2}\int_0^1 \{\mathfrak{f}_m(w)\}^2 q(w)\, dw; \quad 0 \leq w \leq 1 \tag{13}$$

for $m = 1$, equation (13) reduces to the quantile version of extropy for the parent distribution, a result obtained by [10].

In the next Table 1, we evaluate the quantile-based hazard rate function for record statistics, for some specific lifetime distributions. In the next theorem, we will express the proposed measure (13) in the form of expectation

Distribution function	Quantile density function $q(w)$	Hazard quantile function $H_m(w)$
Exponential	$\frac{1}{\lambda(w)},\ \lambda > 0$	$\frac{\lambda\{-\ln(1-w)\}^{m-1}(1-w)}{\Gamma(m;-\ln(1-w))}$
Uniform	$\frac{1}{\beta-\alpha},\ \alpha,\ \beta > 0$	$\frac{\{-\ln(1-w)\}^{m-1}}{\Gamma(m;-\ln(1-w))(\beta-\alpha)}$
Rescaled beta	$R[1-(1-w)^{\frac{1}{c}}],\ C, R > 0$	$\frac{C\{-\ln(1-w)\}^{m-1}}{R\Gamma(m;-\ln(1-w))(1-w)^{\frac{1}{C}-1}}$
Generalized Pareto	$\frac{b}{a}\left[(1-w)^{-\frac{a}{a+1}}-1\right],\ a > -1\ b > 0$	$\frac{\{-\ln(1-w)\}^{m-1}(1-w)}{\Gamma(m;-\ln(1-w))\frac{b}{a}\left[(1-w)^{-\frac{a}{a+1}}-1\right]}$
Folder Crammer	$\frac{w}{\theta(1-w)},\ \theta > 0$	$\frac{\theta\{-\ln(1-w)\}^{m-1}(1-w)^2}{\Gamma(m;-\ln(1-w))w}$

Table 1. $H_m(w)$ for some specific lifetime distributions.

and gamma distribution.

Theorem 2.1
Quantile-based extropy measure (13) can be expressed as

$$L\mathfrak{f}^m(w) = -\frac{\Gamma(2m-1)}{2\{\Gamma(m)\}^2} E\left[\frac{1}{q(1-e^{T^*})}\right],$$

where $T^* \sim \Gamma(2m-1)$ and E is the expectation.

Proof
Substituting equations (8) and (10), in equation (13) we obtained

$$L\mathfrak{f}^{m}(w) = -\frac{1}{2\{\Gamma(m)\}^2}\int_0^1 \frac{1}{\{-\ln(1-w)\}^{-2(m-1)}q(w)}dw,$$

putting $-\ln(1-w) = z$, $dw = e^{-z}dz$, we obtain

$$L\mathfrak{f}^{m}(w) = -\frac{1}{2\{\Gamma(m)\}^2}\int_0^\infty \frac{z^{2(m-1)}e^{-z}}{q(1-e^{-z})}dz, \tag{14}$$

equation (14) can be written as

$$L\mathfrak{f}^{m}(w) = -\frac{\Gamma(2m-1)}{2\{\Gamma(m)\}^2}E\left[\frac{1}{q(1-e^{T^*})}\right].$$

So, the result follows. □

Theorem 2.2
If $Y = rX + p$, with $r,\ p\ > 0$, then $L\mathfrak{f}^{m}_{Y}(w) = \frac{1}{r}\,L\mathfrak{f}^{m}_{X}(w)$.

Proof
We have, $Y = rX + p$, then
$F_Y(y) = P[Y \le (y)] = P[rX + p \le (y)] = F_X\left(\frac{y-p}{r}\right)$.
Taking $F_X\left(\frac{y-p}{r}\right) = w$, we get $Q_Y(w) = rQ_X(w) + p$, we have

$$q_Y(w) = \frac{1}{r\,q_X(w)}. \tag{15}$$

Extropy measure (13) corresponding to random variable Y is given as

$$L\mathfrak{f}^{m}_{Y}(w) = -\frac{1}{2\{\Gamma(m)\}^2}\int_0^1 \frac{1}{\{-\ln(1-w)\}^{-2(m-1)}q_Y(w)}dw,$$

using equation (15) we obtained

$$\begin{aligned} L\mathfrak{f}^{m}_{Y}(w) &= -\frac{1}{2\{\Gamma(m)\}^2}\int_0^1 \frac{\{-\ln(1-w)\}^{2(m-1)}}{q_X(w)}dw \\ &= -\frac{1}{2r\{\Gamma(m)\}^2}\int_0^1 \frac{\{-\ln(1-w)\}^{2(m-1)}}{q_X(w)}dw\,, \\ L\mathfrak{f}^{m}_{Y}(w) &= \frac{1}{r}\,L\mathfrak{f}^{m}_{X}(w). \end{aligned}$$

This proves the result. □

3. $L\mathfrak{f}^{m}(w)$ of a Generalized Model

Taking a quantile density function (qdf) of a generalized model into consideration, which is defined as

$$q(w) = A(-\ln(1-w))^{B}(1-w)^{C}w^{D}(\ln w)^{E}, \tag{16}$$

where the model's parameters are $A, B, C, D,\ and\, E$. Proposed $L\mathfrak{f}^{m}(w)$ for generalized model (16) is defined as

$$L\mathfrak{f}^{m}(w) = -\frac{1}{2A\{\Gamma(m)\}^2}\int_0^1 \frac{(1-w)^{-C}w^{-D}+3}{\{-\ln(1-w)\}^{2-2m+B}(\ln w)^{E}}dw. \tag{17}$$

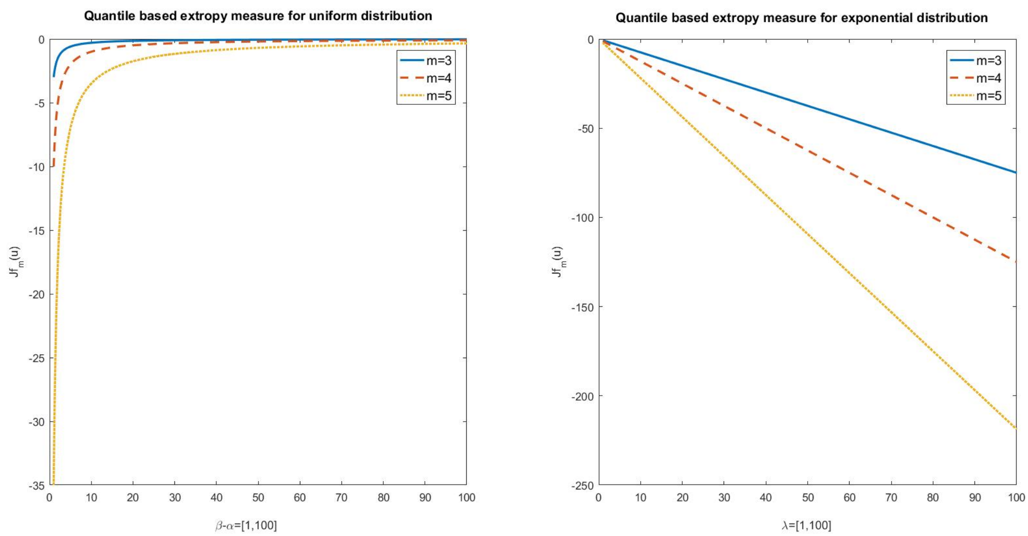

Figure 1. Quantile-based extropy measure (2.6) for m={3, 4, 5}.

Equation (17) provides quantile versions of several lifetime distributions and their corresponding values for different parameter values. Quantile-based extropy of m^{th} upper record value for various lifetime distributions are given in Table 1. For $m = 1$, $L\mathfrak{J}^m(w)$ for the exponential distribution, uniform distribution, Pareto-II distribution, rescaled beta distribution, and generalized pareto distribution will reduce to results obtained by [10]

Considering the importance of some specific probability distributions in real-life events, we study the proposed $L\mathfrak{J}^m(w)$ measure (2.6) for various distributions in the next subsection.

3.1. Comparison of Quantile-Based Extropy Measures $L\mathfrak{J}^m(w)$ for Various Distributions

If,

$L\mathfrak{J}^m(U_w) = -\frac{\Gamma(2m-1)}{2(\alpha-\beta)\{\Gamma(m)\}^2}$, and $L\mathfrak{J}^m(E_w) = -\frac{\theta\Gamma(2m-1)}{2^m\{\Gamma(m)\}^2}$ are the quantile-based extropy of m^{th} upper record values for Uniform and Exponential distributions, respectively. Then the ratio used for comparison is

$$R(m) = \frac{L\mathfrak{J}^m(E_w)}{L\mathfrak{J}^m(U_w)} = \theta(\alpha - \beta)2^{1-m}.$$

Interpretation:

$$|L\mathfrak{J}^m(E_w)| > |L\mathfrak{J}^m(U_w)| \quad \Leftrightarrow \quad R(m) > 1,$$
$$|L\mathfrak{J}^m(U_w)| > |L\mathfrak{J}^m(E_w)| \quad \Leftrightarrow \quad R(m) < 1.$$

The threshold occurs when

$$m = 1 + \log_2\big(\theta(\alpha - \beta)\big).$$

As $m \to \infty$, $R(m) \to 0$, hence $L\mathfrak{J}^m(U_w)$ dominates asymptotically.

Figure 1: Illustrates the behavior of the quantile-based extropy measure $L\mathfrak{J}^m(w)$ for the Uniform and Exponential distributions across the parameter ranges $\beta - \alpha \in [1, 100]$ and $\lambda \in [1, 100]$, respectively, for record orders $m = 3, 4, 5$. For the Uniform distribution (left panel), the extropy measure increases smoothly from large negative values near zero parameter values and gradually approaches zero as the spread $\beta - \alpha$ widens. This indicates that uncertainty decreases slowly as the distribution becomes more dispersed. Among the record orders, the results show that $m = 3$ yields the highest extropy values, followed by $m = 4$ and $m = 5$, implying that higher-order records exhibit lower levels of uncertainty. The curves converge closely for large values of $\beta - \alpha$, demonstrating reduced sensitivity to the parameter at wider ranges.

Parameters	Distribution	Quantile Function $q(w)$	$L\mathfrak{z}^m(w)$
$A=\beta-\alpha;\beta>\alpha,$ $B=C=D=E=0$	Uniform	$\alpha+(\beta-\alpha)u$	$L\mathfrak{z}^m(U_w)=\frac{\Gamma(2m-1)}{2(\alpha-\beta)\{\Gamma(m)\}^2}$
$A=\frac{1}{\theta};\theta>0,C=-1,$ $B=D=E=0$	Exponential	$-\theta\ln(1-w)$	$L\mathfrak{z}^m(U_w)=-\frac{\theta\Gamma(2m-1)}{2^m\{\Gamma(m)\}^2}$
$A=\frac{1}{\theta};\theta>0,C=-(A+1),$ $B=D=E=0$	Classical Pareto	$(1-w)^{-\frac{1}{\theta}}$	$L\mathfrak{z}^m(C_w)=-\frac{(\theta)^{2m}\Gamma(2m-1)}{2^m(2\theta+1)^{2m-1}\{\Gamma(m)\}^2}$
$A=\frac{\gamma}{c},C=-\left(\frac{1}{c}+1\right);\alpha,c>0,$ $B=D=E=0$	Pareto-II	$\alpha((1-w)^{-\frac{1}{c}}-1)$	$L\mathfrak{z}^m(P_w)=-\frac{c^{2m}\Gamma(2m-1)}{2\alpha(2c+1)^{2m-1}\{\Gamma(m)\}^2}$
$A=\frac{b}{a+1},C=-\left(\frac{2a+1}{a+1}\right);a>-1,$ $b>0,B=D=E=0$	Generalized Pareto	$\frac{b}{a}\left[(1-w)^{-\frac{a}{a+1}}-1\right]$	$L\mathfrak{z}^m(G_w)=-\frac{(a+1)^{2m}\Gamma(2m-1)}{2b(3a+2)^{2m-1}\{\Gamma(m)\}^2}$
$A=\frac{1}{\beta\lambda^{\frac{1}{\beta}}},B=\frac{1}{\beta}-1;\lambda,\beta>0,$ $C=-1,D=E=0$	Weibull	$\left(\frac{-\ln(1-w)}{\lambda}\right)^{\frac{1}{\beta}}$	$L\mathfrak{z}^m(W_w)=-\frac{\beta\lambda^{\frac{1}{\beta}}\Gamma(2m-1)}{2^{(2m-1)\beta+1}\{\Gamma(m)\}^2}$
$A=\frac{R}{c},C=\frac{1}{c}-1;C,R>0,$ $B=D=E=0$	Rescaled Beta	$R[1-(1-w)^{\frac{1}{c}}]$	$L\mathfrak{z}^m(R_w)=-\frac{c^{2m}\Gamma(2m-1)}{2R(2c-1)^{2m-1}\{\Gamma(m)\}^2}$
$A=\frac{1}{\theta};\theta>0,C=-2,$ $B=D=E=0$	Folder Crammer	$\frac{w}{\theta(1-w)}$	$L\mathfrak{z}^m(C_w)=-\frac{\theta\Gamma(2m-1)}{2.3^{2m+1}\{\Gamma(m)\}^2}$
$A=\sigma\beta(\beta+1),C=1,$ $D=\beta-1,B=E=0$	Govindarajulu's	$\theta+\sigma\{(\beta+1)w^\beta-\beta w^{\beta+1}\}$	$L\mathfrak{z}^m(G_w)=-\frac{1}{2\Gamma(m)\}^2\sigma\beta(\beta+1)}\int_0^1\frac{(-\ln(1-w))^{2m-2}}{(w^{\beta-1}-w^\beta)}dw$

Table 1 Quantile-based extropy $L\mathfrak{z}^m(w)$ for various lifetime distributions.

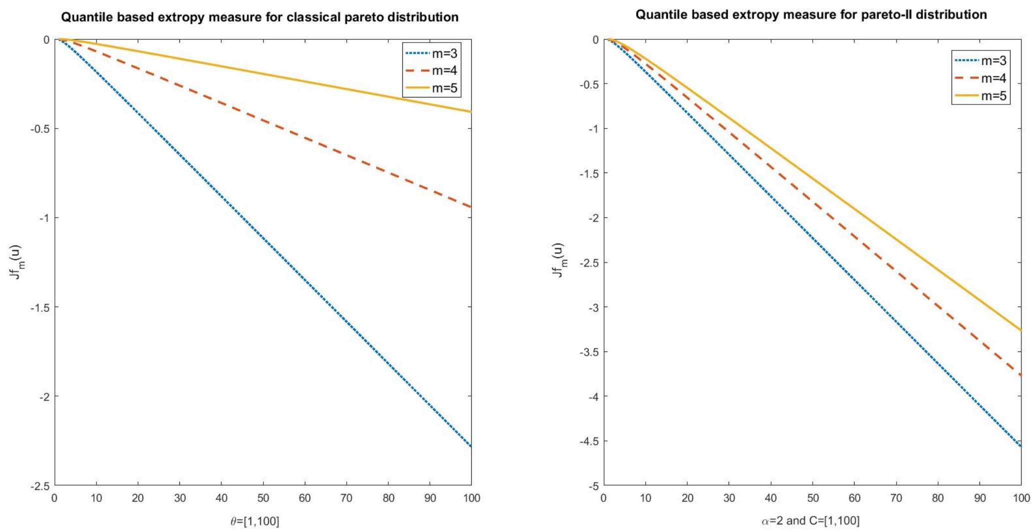

Figure 2. Quantile-based extropy measure (2.6) for different values of$m = 3, 4, \, and \, 5$.

In contrast, the Exponential distribution plot (right panel) reveals a distinctly different pattern. Here, the extropy decreases linearly and rapidly as the rate parameter λ increases, reflecting strong sensitivity of uncertainty to shifts in distribution concentration. The magnitude of extropy reduction is much more pronounced than in the Uniform case, with separation among the curves for $m = 3, 4$, and 5 becoming significantly wider for increasing λ, indicating that higher-order records are more strongly affected by parameter variation.

Corollary 3.1

If $L\mathfrak{f}^{m}(C_w) = -\frac{\theta^{2m}\,\Gamma(2m-1)}{2^m\,(2\theta+1)^{2m-1}\,[\Gamma(m)]^2}$, and $L\mathfrak{f}^{m}(P_w) = -\frac{c^{2m}\,\Gamma(2m-1)}{2\alpha\,(2c+1)^{2m-1}\,[\Gamma(m)]^2}$ are the quantile-based extropy of m^{th} upper record values for classical Pareto and Pareto-II distributions, respectively. Then the ratio used for comparison is

$$R = \frac{L\mathfrak{f}^{m}(C_w)}{L\mathfrak{f}^{m}(P_w)} = \frac{\frac{\theta^{2m}}{2^m(2\theta+1)^{2m-1}}}{\frac{c^{2m}}{2\alpha(2c+1)^{2m-1}}} = \frac{(\theta/c)^{2m}\cdot 2\alpha\cdot(2c+1)^{2m-1}}{2^m(2\theta+1)^{2m-1}}.$$

Or equivalently:

$$R = \frac{\alpha}{2^{m-1}}\left(\frac{\theta}{c}\right)^{2m}\left(\frac{2c+1}{2\theta+1}\right)^{2m-1}$$

Observations from the ratio R:

- $R > 1 \implies f^m(C_w) > f^m(P_w)$ in magnitude.
- $R < 1 \implies f^m(P_w) > f^m(C_w)$ in magnitude.
- As m increases, the powers $2m$ and $2m-1$ dominate, so even small differences in θ/c or $(2c+1)/(2\theta+1)$ can significantly affect which function is larger.

Figure 2 illustrates the behavior of the quantile-based extropy measure $L\mathfrak{f}^{m}(w)$ for the Classical Pareto and Pareto-II distributions for different record orders $m = 3, 4, 5$. For the Classical Pareto distribution, evaluated over the scale parameter $\theta \in [1, 100]$, the extropy measure exhibits a decreasing linear trend, indicating a progressive reduction in uncertainty as θ increases. The curve corresponding to $m = 3$ demonstrates the steepest decline, followed by $m = 4$ and $m = 5$, implying that higher-order record values yield relatively smaller reductions in extropy. This suggests that the effect of increasing scale is more dominant for lower record orders.

In contrast, for the Pareto-II distribution, plotted against the scale parameter $c \in [1, 100]$ with a fixed shape parameter $\alpha = 2$, the extropy measure also decreases linearly but at a noticeably faster rate than in the Classical

Pareto case. The separation between curves becomes more pronounced as c increases, reflecting stronger sensitivity of uncertainty to parameter variation in the Pareto-II distribution.
If, $L\mathfrak{f}^m(G_w) = -\frac{c^{2m}\,\Gamma(2m-1)}{2R\,(2c-1)^{2m-1}\,[\Gamma(m)]^2}$, and $f^m(F_w) = -\frac{\theta\,\Gamma(2m-1)}{2\cdot 3^{2m+1}\,[\Gamma(m)]^2}$ are the quantile-based extropy of m^{th} upper record values for classical Pareto and Pareto-II distributions, respectively. Then the ratio used for comparison is

$$R_T = \frac{f^m(R_w)}{f^m(F_w)} = \frac{\frac{c^{2m}}{2R(2c-1)^{2m-1}}}{\frac{\theta}{2\cdot 3^{2m+1}}} = \frac{c^{2m}\cdot 3^{2m+1}}{R\,\theta\,(2c-1)^{2m-1}} = \frac{3\,(3c)^{2m}}{R\,\theta\,(2c-1)^{2m-1}}.$$

Observations:

- Both functions are negative, so they have the same sign.
- The magnitude comparison depends strongly on c, R, θ, and m.
- As m increases, the powers $2m$ and $2m-1$ dominate, so even small differences in c can significantly affect which function is larger.
- If c is large relative to 3 and R is small, then $f^m(R_w)$ quickly dominates $f^m(F_w)$.
- If c is small or R is large, then $f^m(F_w)$ may have a larger magnitude than $f^m(R_w)$.

Example 3.1
If X be a power distributed random variable with quantile density function $q(w) = \frac{\gamma}{\beta} w^{\frac{1}{\beta}-1}$ $\forall \gamma, \beta > 0$. The quantile version of extropy for m^{th} lower record value is defined as

$$L\mathfrak{f}^L_{m}(w) = -\frac{(\beta k)^{2m}\Gamma(2m-1)}{2\gamma(2\beta-1)^{2m-1}\{\Gamma(m)\}^2}.$$

Example 3.2
If X is an Inverted exponential distribution following random variable with quantile density function $q(w) = \frac{\lambda}{w(-\ln w)^2}$ $\forall \lambda > 0$. Then the quantile version of extropy of m^{th} lower record value is defined as

$$L\mathfrak{f}^L_{m}(w) = -\frac{\Gamma(2m-1)}{2\lambda\{\Gamma(m)\}^2}.$$

In many situations, quantile functions are more sway to distribution functions. In the following examples, some distributions have been studied for $L\mathfrak{f}^m(w)$ for which q(.) exists.

Example 3.3
[7] introduced a lambda family of distributions that has several applications in reliability, known as the Davis distribution, whose quantile density function is defined as

$$q(w) = Gw^{\lambda_1-1}(1-w)^{-\lambda_2-1}\{\lambda_1(1-w)+\lambda_2 w\}; G, \lambda_1, \lambda_2 \geq 0. \tag{18}$$

The quantile version of the extropy of m^{th} upper record for Davis distribution is given as

$$L\mathfrak{f}^m(w) = -\frac{1}{2G\{\Gamma(m)\}^2}\int_0^1 \frac{\{-\ln(1-w)\}^{2m-2}}{w^{\lambda_1-1}(1-w)^{-\lambda_2-1}\{\lambda_1(1-w)+\lambda_2 w\}}dw, \tag{19}$$

which can be easily computed numerically. As $\lambda_2 \longrightarrow 0$, (19) corresponds to the Power distribution. Also as $\lambda_1 \longrightarrow 0$, (19) reduces to the Pareto I distribution. When $\lambda_1 = \lambda_2 > 0$, then (19) corresponds to Log Logistic distribution.

Example 3.4
Let us consider Van Staden-Loots distribution with quantile function

$$q(w) = \lambda_1 + \lambda_2\left[\left(\frac{(1-\lambda_3)}{\lambda_4}\right)(w^{\lambda_4}-1) - \frac{\lambda_3}{\lambda_4}\left\{(1-w)^{\lambda_4}-1\right\}\right], \text{ where } \lambda_i > 0 \text{ for } i = 1,2,3,4$$

and the quantile density function of Van Staden-Loots distribution is given as

$$q(w) = \lambda_2 \left[(1-\lambda_3)w^{\lambda_4 - 1} + \lambda_3(1-w)^{\lambda_4-1}\right]. \tag{20}$$

Thus, the quantile-based extropy measure of m^{th} upper record for Van Staden-Loots distributions is given as

$$L\mathfrak{f}^m(w) = -\frac{1}{2\{\Gamma(m)\}^2}\int_0^1 \frac{(-\ln(1-w))^{2m-2}}{\lambda_2(1-\lambda_3)w^{\lambda_4-1} + \lambda_3(1-w)^{\lambda_4-1}}dw, \tag{21}$$

for $\lambda_1 \longrightarrow 0$, (21) reduces to the Exponential distribution Also as $\lambda_4 \longrightarrow 1$, (21) reduces to the Uniform distribution and as $\lambda_2 = 2, \lambda_3 = 1/2, \lambda_4 = 0$, (21) reduces to the Logistic distribution with parameter equals to 1.

Example 3.5
A five-parameter Lambda family of distributions introduced by Gilchrist (2000), which quantile density function is defined as

$$q(w) = \lambda_2\left(\frac{1-\lambda_3}{2}\right)\left[(1-w)^{\lambda_5-1} + w^{\lambda_4-1}\right]. \tag{22}$$

Utilizing this family, Tarsitano (2005) provided some close approximations to a number of symmetric and asymmetric distributions and suggested, using this model in situations when the actual situation under examination does not suggest a particular distributional form. For this family, the quantile-based extropy of the m^{th} upper record is defined as

$$L\mathfrak{f}^m(w) = -\frac{1}{2\{\Gamma(m)\}^2}\int_0^1 \frac{\{-\ln(1-w)\}^{2m-2}}{\lambda_2\left(\frac{1-\lambda_3}{2}\right)\left[(1-w)^{\lambda_5-1} + w^{\lambda_4-1}\right]}dw, \tag{23}$$

for $\lambda_4 \longrightarrow 0$, (23) reduces to the exponential distribution. This family also includes the Generalized Tuckey Lambda family of distributions, when $\lambda_4 \longrightarrow \infty, \lambda_5 \longrightarrow 0$, power distribution when $\lambda_5 \longrightarrow \infty$ and $|\lambda_4| < \infty$, and the generalized Pareto distribution when $\lambda_4 \longrightarrow \infty$ and $|\lambda_5| < \infty$.

4. Dynamic Quantile Extropy for Record Statistics

$L\mathfrak{f}^m(w)$ is not useful for a system that has survived for some units of time. In such cases, dynamic forms of these measures become more important and useful for measuring uncertainty.
Residual extropy of m^{th} upper record value is given as

$$J(f_m;t) = -\frac{1}{2}\int_t^\infty \left(\frac{\mathfrak{f}_m(x)}{\bar{F}_m(t)}\right)^2 dx. \tag{24}$$

Differentiating equation (24) on both sides with respect to t and using $h_m(X_t) = \frac{\mathfrak{f}_m(t)}{\bar{F}_m(t)}$, where $h_m(X_t)$ is hazard function for m^{th} upper record, we obtained

$$J^{'}(f_m;t) = \frac{1}{2}\left[h_m^2(X_t) + 4h_m(X_t)J(f_m;t)\right],$$

the above equation can be written as

$$h_m^2(X_t) + 4h_m(X_t)J(f_m;t) - 2J^{'}(f_m;t) = 0, \tag{25}$$

if $J^{'}(f_m;t) > 0 \; or \; = 0$ equation (25) has only one positive root and if $J^{'}(f_m;t) < 0$ then (25) has only two positive roots, which are given by

$$h_m(X_t) = -2J^{'}(f_m;t) \pm \left(4J^2(f_m;t) + 2J^{'}(f_m;t)\right)^{\frac{1}{2}}$$

Now we propose quantile-based residual extropy of m^{th} upper record statistics, which is defined as

$$L(f_m; w) = -\frac{1}{2}\int_w^1 \left(\frac{\mathfrak{f}_m(p)}{\bar{F}_m(p)}\right)^2 q(p)\, dp. \tag{26}$$

It can be written as

$$L(f_m; w) = -\frac{1}{2\{\Gamma(m; -\ln(1-w))\}^2}\int_w^1 \frac{(-\ln(1-p))^{2(m-1)}}{q(p)}\, dp, \tag{27}$$

where $\Gamma(m; -\ln(1-w))$ is incomplete upper gamma function.
$L(f_m; w)$ measures the expected uncertainty contained in the conditional density about the predictability of an outcome of X until $100(1-w)$ % point of distribution. Differentiate the eq. (27), we get

$$q(w) = \frac{(-\ln(1-w))^{2(m-1)}}{2(\Gamma(m; -\ln(1-w)))L'(f_m; w) + 4\Gamma'(m; -\ln(1-w)]\Gamma(m; -\ln(1-w))L(f_m; w)}. \tag{28}$$

However equation (28) provide a directed relationship between the quantile density function $q(w)$ and $L(f_m; w)$ which shows that it uniquely determine the underlying distribution. The equation (28) can be expressed as

$$H_m(w) = -4L(f_m; w) - 2\frac{\Gamma(m; -\ln(1-w))}{\Gamma'(m; -\ln(1-w)}L'(f_m; w). \tag{29}$$

The main difference between (25) and (30) is (30) provides a linear equation that presents a unique hazard function corresponding to the extropy measure of m^{th} upper record statistics but from (25), $J(f_m; t)$ may determine the distribution uniquely.

Theorem 4.1
X be a random variable with a quantile density function

$$q(w) = \frac{1}{\lambda(1-w)}, \quad \lambda > 0$$

if and only if

$$L(f_m; w) = -\frac{\lambda(\Gamma(2m-1); -2\ln(1-w))}{2^{2m}[\Gamma(m; -\ln(1-w))]^2}.$$

Proof
We only prove the converse part; the direct part is obvious. Using (27) we obtained

$$\begin{aligned}
L(f_m; w) &= -\frac{\lambda(\Gamma(2m-1); -2\ln(1-w))}{2^{2m}[\Gamma(m; -\ln(1-w))]^2}\\
-\frac{1}{2[\Gamma(m; -\ln(1-w))]^2}\int_w^1 \frac{(-\ln(1-p))^{2(m-1)}}{q(p)}\, dp &= -\frac{\lambda(\Gamma(2m-1); -2\ln(1-w))}{2^{2m}[\Gamma(m; -\ln(1-w))]^2}\\
\int_w^1 \frac{(-\ln(1-p))^{2(m-1)}}{q(p)}\, dp &= \frac{\lambda}{2^{2m}}(2\Gamma(2m-1); -2\ln(1-w))\\
\int_w^1 \frac{(-\ln(1-p))^{2(m-1)}}{q(p)}\, dp &= \lambda\int_{-\ln(1-w)}^{\infty} t^{2m}e^{-2t}dt.
\end{aligned}$$

Substituting $t = -\ln(1-p)$, we obtained

$$\int_w^1 \frac{(-\ln(1-p))^{2(m-1)}}{q(p)}\, dp = \lambda\int_w^1 -\ln(1-p)^{2m}(1-p)dp,$$

differentiating on both sides with respect to w, we obtained

$$\frac{(-\ln(1-p))^{2(m-1)}}{q(p)} = \lambda(-\ln(1-w)^{2m}(1-w)$$
$$q(w) = \frac{1}{\lambda(1-w)}$$

This proves the desired result. □

Theorem 4.2
X be a random variable with a quantile density function

$$q(w) = \frac{R}{C}(1-w)^{\left(\frac{1-C}{C}\right)}, \;\; C,\, R > 0$$

if and only if

$$L(f_m; w) = -\frac{C^{2m}(\Gamma(2m-1); -\left(\frac{2C-1}{C}\right)\ln(1-w))}{2R(2C-1)^{2m-1}[\Gamma(m; -\ln(1-w))]^2}.$$

Proof
We only prove the converse part; the direct part is obvious. Using (27) we obtained

$$L(f_m; w) = -\frac{C^{2m}(\Gamma(2m-1); -\left(\frac{2C-1}{C}\right)\ln(1-w))}{2R(2C-1)^{2m-1}[\Gamma(m; -\ln(1-w))]^2}$$
$$-\frac{1}{2[\Gamma(m; -\ln(1-w))]^2}\int_w^1 \frac{(-\ln(1-p))^{2(m-1)}}{q(p)}\,dp = -\frac{C^{2m}(\Gamma(2m-1); -\left(\frac{2C-1}{C}\right)\ln(1-w))}{2R(2C-1)^{2m-1}[\Gamma(m; -\ln(1-w))]^2}$$
$$\frac{C}{R}\left(\frac{C}{2C-1}\right)^{2m-1}\left[\Gamma(2m-1); -\left(\frac{2C-1}{C}\right)\ln(1-w)\right] = \int_w^1 \frac{(-\ln(1-p))^{2(m-1)}}{q(p)}\,dp,$$

the above expression can be written as

$$\frac{C}{R}\int_{-\ln(1-w)}^{\infty} t^{2m-2}e^{--\left(\frac{2C-1}{C}\right)t}dt = \int_w^1 \frac{(-\ln(1-p))^{2(m-1)}}{q(p)}\,dp.$$

Substituting $t = -\ln(1-p)$, we obtained

$$\frac{C}{R}\int_w^1 \{-\ln(1-p)\}^{2m-2}e^{\left(\frac{2C-1}{C}\right)\ln(1-w)}\left(\frac{1}{1-p}\right)dp = \int_w^1 \frac{(-\ln(1-p))^{2(m-1)}}{q(p)}\,dp$$

Differentiating the above expression with respect to w

$$q(w) = \frac{R}{C}(1-w)^{\left(\frac{1-C}{C}\right)}.$$

This provides the desired result. □

Theorem 4.3
If $L'(f_m; w)$ is increasing in $w\ \forall\ 0 < w < 1$, i.e, $(L'(f_m; w) \geq 0)$ then

$$L(f_m; w) \leq \frac{(-\ln(1-w)^{m-1})}{4\Gamma'(m; -\ln(1-w))}H(X_w).$$

Proof
Differentiating (27) w.r.t. w then we have,

$$L'(f_m;w) = \frac{d}{dw}\left(\frac{-1}{2(\Gamma(m;-\ln(1-w)))^2}\int_w^1 \frac{(-\ln(1-p))^{2(m-1)}}{q(p)}\right)dp$$
$$= \frac{\Gamma'(m;-\ln(1-w))}{(\Gamma(m;-\ln(1-w)))^3}\int_w^1 \frac{(-\ln(1-p))^{2(m-1)}}{q(p)}dp + \frac{1}{(\Gamma(m;-\ln(1-w)))^2}.$$

If $L'(f_m;w)$ is increasing in w, i.e, $(L'(f_m;w)\geq 0)$. Then

$$\frac{\Gamma'(m;-\ln(1-w))}{(\Gamma(m;-\ln(1-w)))^3}\int_w^1 \frac{(-\ln(1-p))^{2(m-1)}}{q(p)}dp + \frac{1}{(\Gamma(m;-\ln(1-w)))^2} \geq 0;$$
$$\frac{-2\Gamma'(m;-\ln(1-w))}{(\Gamma(m;-\ln(1-w)))}J(f_m;u) + \frac{(-\ln(1-w))^{2m-1}}{2(\Gamma(m;-\ln(1-w)))^2 q(w)} \geq 0;$$
$$\frac{(-\ln(1-w))^{m-1}}{4(\Gamma'(m;-\ln(1-w)))}H(X_w) - L(f_m;w) \geq 0,$$

we get,

$$L(f_m;w) \leq \frac{(-\ln(1-w))^{m-1}}{4(\Gamma'(m;-\ln(1-w)))}H(X_w) \tag{30}$$

□

By considering the generalized model (16) and different values of parameters, quantile extropy for record statistics of m^{th} upper record for residual lifetime for various distributions have been summarized in Table 2.

Theorem 4.4
Let us consider X and Y to be two r.v. such that $X \leq_{H_m(w)} \geq Y$ then $X \leq_{L(f_m;w)} Y$.

Proof
Let $X \leq_{H_m(w)} \leq Y$. So that $q_X(w) \leq q_Y(w)$ implies that

$$\frac{\{-\ln(1-w)\}^{m-1}}{\Gamma(m;-\ln(1-w))q_X(w)} \leq \frac{\{-\ln(1-w)\}^{m-1}}{\Gamma(m;-\ln(1-w))q_Y(w)}$$
$$\frac{1}{(\Gamma(m;-\ln(1-w)))^2}\frac{\{-\ln(1-w)\}^{2(m-1)}}{q_X(w)} \leq \frac{1}{(\Gamma(m;-\ln(1-w)))^2}\frac{\{-\ln(1-w)\}^{2(m-1)}}{q_Y(w)}$$
$$-\frac{1}{2(\Gamma(m;-\ln(1-w)))^2}\int_w^1 \frac{\{-\ln(1-w)\}^{2(m-1)}}{q_X(w)} \geq -\frac{1}{2(\Gamma(m;-\ln(1-w)))^2}\int_w^1 \frac{\{-\ln(1-w)\}^{2(m-1)}}{q_Y(w)}$$
$$X \leq_{L(f_m;w)} Y.$$

This proves the result. □

In some practical situations, uncertainty may have more to do with a past existence than what's to come. Similar to the quantile-based residual extropy of m^{th} record, we study the quantile-based extropy of m^{th} record for the inactivity/ past time. Quantile extropy of m^{th} record value for past lifetime is given by

$$\bar{L}(f_m;u) = \frac{-1}{2}\int_0^w \left(\frac{\mathrm{f}_m(p)}{F_m(p)}\right)^2 q(p)\,dp. \tag{31}$$

It can also be written as

$$\bar{L}(f_m;u) = \frac{-1}{2\{\gamma(m;-\ln(1-w)\}^2}\int_0^w \frac{(-\ln(1-p))^{2(m-1)}}{q(p)}dp,$$
$$= \frac{-1}{2}\frac{1}{\gamma(m;-\ln(1-w))}\int_0^w (-\ln(1-p))^{m-1}H(X_p)dp, \tag{32}$$

Parameters	Distribution	Quantile Function $q(w)$	$L(f_m;w)$
$A=\beta-\alpha;\beta>\alpha,$ $B=C=D=E=0$	Uniform	$\alpha+(\beta-\alpha)w$	$\frac{-(\Gamma(2m-1);-\ln(1-w))}{2(\beta-\alpha)\Gamma[(m;-\ln(1-w))]^2}$
$A=\frac{1}{\theta};\theta>0,C=-1,$ $B=D=E=0$	Exponential	$-\theta\ln(1-w)$	$\frac{-\theta(\Gamma(2m-1);-\ln(1-w)}{\Gamma[(m;-\ln(1-w))]^2)2^{2m}}$
$A=\frac{1}{\theta};\theta>0,C=-(A+1),$ $B=D=E=0$	Classical Pareto	$(1-w)^{-\frac{1}{\theta}}$	$\frac{-\theta^2\ln(1-w)\Gamma((2m-1);-\frac{1-2\theta}{\theta})}{2(1-2\theta)^{2m-1}[\Gamma(m;-\ln(1-w))]^2}$
$A=\frac{\gamma}{c},C=-\left(\frac{1}{c}+1\right);\alpha,c>0,$ $B=D=E=0$	Pareto-II	$\alpha\left((1-w)^{-\frac{1}{c}}-1\right)$	$\frac{-c^{2m}\ln(1-w)]\Gamma((2m-1);-\frac{1+2c}{c})}{2\alpha(2c+1)^{2m-1}[\Gamma(m;-\ln(1-w))]^2}$
$A=\frac{b}{a+1},C=-\left(\frac{2a+1}{a+1}\right);a>-1,$ $b>0,B=D=E=0$	Generalized Pareto	$\frac{b}{a}\left[(1-w)^{-\frac{a}{a+1}}-1\right]$	$\frac{-1(a+1)^{2m}\ln(1-w)\Gamma((2m-1;-\frac{2+3a}{a+1})}{2b(3a+2)^{2m-1}[\Gamma(m;-\ln(1-w))]^2}$
$A=\frac{1}{\beta\lambda^{\frac{1}{\beta}}},B=\frac{1}{\beta}-1;\lambda,\beta>0,$ $C=-1,D=E=0$	Weibull	$\left(\frac{-\ln(1-w)}{\lambda}\right)^{\frac{1}{\beta}}$	$\frac{-1\beta\lambda^{\frac{1}{\beta}}\Gamma(2m-1)}{2^{(2m-1)\beta+1}\Gamma(m)^2}$
$A=\frac{R}{c},C=\frac{1}{c}-1;c,R>0,$ $B=D=E=0$	Rescaled Beta	$R\left[1-(1-w)^{\frac{1}{c}}\right]$	$\frac{-c^{2m}\Gamma(2m-1;-\frac{2c-1}{c}\ln(1-w))}{2R(2c-1)^{2m-1}[\Gamma(m;-\ln(1-w))]^2}$
$A=\frac{1}{\theta};\theta>0,C=-2,$ $B=D=E=0$	Folder Crammer	$\frac{w}{\theta(1-w)}$	$\frac{-\theta\Gamma(2m-1;-3\ln(1-w))}{2.3^{2m+1}[\Gamma(m;-\ln(1-w))]^2}$
$A=\sigma\beta(\beta+1),C=1,$ $D=\beta-1,B=E=0$	Govindarajulu's	$\theta+\sigma\left((\beta+1)w^{\beta}-\beta w^{\beta+1}\right)$	$\frac{-1}{2\sigma\beta(\beta+1)[\Gamma(m;-\ln(1-w))]^2}\int_w^1\frac{(-\ln(1-p))^{2m-2}}{p^{\beta}(\frac{1}{p}-1)}dp$

Table 2 Quantile based residual extropy $L(f_m;w)$ for various lifetime distributions.

$L(f_m; w)$ provides quantile-based past extropy for record statistics that measure the uncertainty of X, using either the hazard quantile function or the quantile density function.Where $\gamma(m; -\ln(1-w))$ is incomplete lower gamma function.

Remark 4.1
When $m = 1$, (25) and (30) redwce to the quantile residual extropy and quantile past extropy of parent distribution respectively and when $w \to 0$ in (25) and $w \to 1$ in (30), both reduced to (13).

Remark 4.2
$L\mathfrak{f}^m(w)$ (13) has a mathematical relation with $L(f_m; w)$ (25) and $\bar{L}(f_m; w)$ (30), which is given as

$$L\mathfrak{f}^m(w) = \left[\frac{\Gamma(m; -\ln(1-w))}{\Gamma(m)}\right]^2 L(f_m; w) + \left[\frac{\gamma(m; -\ln(1-w))}{\Gamma(m)}\right]^2 \bar{L}(f_m; u)$$

Differentiating equation (30) w.r.t. w. Then we get a direct relationship between the quantile density function q(w) and $\bar{L}(f_m; u)$ which shows that it uniquely determines the underlying distribution as

$$q(w) = \frac{(-\ln(1-w))^{2(m-1)}}{2\gamma(m; -\ln(1-w))[\bar{L}'(f_m, u)\gamma(m; -\ln(1-w)) + \bar{L}(f_m; u)\gamma'(m; -\ln(1-w))]} \tag{33}$$

We will now provide bounds for the quantile-based past extropy measure in terms of reverse hazard rate function for m^{th} record statistics.

Theorem 4.5
If $\bar{L}(f_m; w)$ is increasing (decreasing) in w, i.e, $L'(f_m; w) \geq (\leq) 0$ then

$$\bar{L}(f_m; w) \leq (\geq) \frac{-1}{4} \frac{(-\ln(1-w))^{m-1}}{(\Gamma'(m; -\ln(1-w)))} \bar{H}(X_u).$$

Proof
Differentiating (30) w.r.t. w then we have,

$$\begin{aligned}
\bar{L}'(f_m; u) =& \frac{d}{dw}\left(\frac{-1}{2(\gamma(m; -\ln(1-w)))^2}\int_0^w \frac{(-\ln(1-p))^{2(m-1)}}{q(p)}\,dp\right)\\
=& \frac{\gamma'(m; -\ln(1-w))}{(\gamma(m; -\ln(1-w)))^3}\int_0^w \frac{(-\ln(1-p))^{2(m-1)}}{q(p)}dp - \frac{(-\ln(1-w))^{2(m-1)}}{2[\gamma(m; -\ln(1-w))]^2 q(w)}
\end{aligned}$$

If $\bar{L}'(f_m; u)$ is increasing in w, i.e, $(\bar{L}'(f_m; u) \geq 0)$. Then

$$\frac{\gamma'(m; -\ln(1-w))}{(\gamma(m; -\ln(1-w)))^3}\int_0^w \frac{(-\ln(1-p))^{2(m-1)}}{q(p)}dp - \frac{(-\ln(1-w))^{2(m-1)}}{2[\gamma(m; -\ln(1-w))]^2 q(w)} \geq 0;$$
$$\frac{-2\gamma'(m; -\ln(1-w))}{(\gamma(m; -\ln(1-w)))}\bar{L}(f_m; w) - \frac{(-\ln(1-w))^{m-1}}{2[\gamma(m; -\ln(1-w))]}\bar{H}(X_w) \geq 0, ,$$

$$\bar{L}(f_m; w) \leq (\geq) \frac{-1}{4} \frac{(-\ln(1-w))^{m-1}}{(\Gamma'(m; -\ln(1-w)))} \bar{H}(X_u). \tag{34}$$

we get the required result, □

5. F^γ Family of Distributions

If X is a non-negative continuous random variable and the corresponding cdf is $F(X)$. If we add one parameter $\gamma > 0$, it becomes $[F(x)]^\gamma$ that is richer and more versatile cdf compared to $F(X)$. It allows for both non-monotone and monotone hazard rates. For instance, exponential distribution has a constant hazard rate while the exponentiated exponential distribution's hazard rate depends upon γ, a constant hazard rate if $\gamma = 1$, an increasing hazard rate if $\gamma > 1$ and a decreasing hazard rate if $\gamma < 1$. Furthermore, its pdf is unimodal on $(0, \infty)$ with mode at $x = \ln \gamma/\lambda$. Then the quantile version of pdf of m^{th} record for F^{γ}-distributions is given by

$$\text{f̧}_m(w) = \frac{\gamma w^{\gamma-1}\{-\ln(1-w^{\gamma})\}^{m-1}}{\Gamma(m)q(w)}. \tag{1}$$

Quantile based extropy of m^{th} upper record for F^{γ}-distributions is given by

$$\text{Ļ}(U_m) = -\frac{1}{2}\int_0^1 \{\text{f̧}_m(w)\}^2 q(w)dw,$$

which can be written using (5.1) as

$$\text{Ļ}(U_m) = -\frac{1}{2}\frac{\gamma^2}{\{\Gamma(m)\}^2}\int_0^1 \frac{w^{2(\gamma-1)}(-\ln(1-w^{\gamma}))^{2(m-1)}}{q(w)}dw. \tag{2}$$

When $\gamma = 1$, the above equation reduces to the quantile-based extropy of m^{th} record statistics.

When $\gamma < 1$

When $0 < \gamma < 1$, the factor $w^{2(\gamma-1)}$ behaves like a negative power of w as $w \to 0^+$, since

$$2(\gamma-1) < 0 \quad \Rightarrow \quad w^{2(\gamma-1)} \to \infty.$$

Thus the integrand becomes more singular near the lower bound $w = 0$. In this case the integral is more likely to diverge unless $q(w)$ compensates the divergence. Therefore,

$$\text{Ļ}(U_m) \text{ increases (or diverges) as } \gamma \text{ decreases below } 1.$$

When $\gamma > 1$

For $\gamma > 1$, we have

$$2(\gamma-1) > 0,$$

hence

$$w^{2(\gamma-1)} \to 0 \quad \text{as } w \to 0^+.$$

This suppresses the integrand near the origin, leading to a regularizing effect and reducing the overall value of the integral. Consequently,

$$\text{Ļ}(U_m) \text{ decreases and typically remains finite as } \gamma > 1.$$

5.1. $\text{Ļ}(U_m)$ *for Expontiated Exponential Distribution*

The quantile function for the exponentiated exponential distribution is $Q(w) = -\frac{1}{\lambda}\log(1-w)$. Then measure $\text{Ļ}(U_m)$ for expontiated exponential distributionis defined as

$$\mathcal{L}_E(U_m) = -\frac{\lambda\gamma^2}{2\{\Gamma(m)\}^2}\int_0^1 u^{2(\alpha-1)+1}\big(-\log(1-u^{\alpha})\big)^{2(\alpha-1)}\,du$$

Make the substitution $t = u^{\alpha}$. Then $u = t^{1/\alpha}$, $du = \dfrac{1}{\alpha}t^{1/\alpha-1}\,dt$, and

$$u^{2(\alpha-1)+1} = u^{2\alpha-1} = t^{(2\alpha-1)/\alpha}.$$

Hence, the integral becomes

$$\int_0^1 u^{2\alpha-1}\big(-\log(1-u^\alpha)\big)^{2(\alpha-1)}\,du = \frac{1}{\alpha}\int_0^1 t\big(-\log(1-t)\big)^{2(\alpha-1)}\,dt.$$

Set $s = 2(\alpha-1)$. Using the identity

$$\int_0^1 t^{a-1}\big(-\log(1-t)\big)^{s}\,dt = \Gamma(s+1)\sum_{k=0}^{\infty}\binom{a-1}{k}\frac{(-1)^k}{(k+1)^{s+1}},$$

with $a-1=1$ (so $a=2$) we obtain (noting $\binom{1}{k}=0$ for $k\geq 2$)

$$\int_0^1 t\big(-\log(1-t)\big)^{s}\,dt = \Gamma(s+1)\left(\frac{1}{1^{\,s+1}}-\frac{1}{2^{\,s+1}}\right) = \Gamma(s+1)\Big(1-2^{-(s+1)}\Big).$$

Therefore (with $s+1=2\alpha-1$ and $\Gamma(s+1)=\Gamma(2\alpha-1)$) the integral equals

$$\frac{1}{\alpha}\,\Gamma(2\alpha-1)\Big(1-2^{-(2\alpha-1)}\Big),$$

and we obtained

$$\mathcal{L}_E(U_m) = -\frac{\lambda\gamma^2}{2\{\Gamma(m)\}^2}\cdot\frac{1}{\alpha}\,\Gamma(2\alpha-1)\Big(1-2^{-(2\alpha-1)}\Big)$$

.

5.2. $\c{L}(U_m)$ *for Expontiated Uniform Distribution*

The quantile function for the exponentiated uniform distribution is $Q(w)=b+w(b-a),\, b>a$. Then measure $\c{L}(U_m)$ for expontiated uniform distributionis defined as

$$\mathcal{L}_U(U_m) = \frac{-\gamma^2}{2(b-a)}\frac{1}{\{\Gamma(m)\}^2}\int_0^1 w^{2(\gamma-1)}\left(-\ln\left(1-w^\gamma\right)\right)^{2(\gamma-1)}\,dw$$

Let $t=w^\gamma \Rightarrow w=t^{1/\gamma},\; dw=\frac{1}{\gamma}t^{1/\gamma-1}dt$. Then

$$\int_0^1 w^{2(\gamma-1)}\left(-\ln\left(1-w^\gamma\right)\right)^{2(\gamma-1)}dw = \frac{1}{\gamma}\int_0^1 t^{(\gamma-1)/\gamma}\left(-\ln(1-t)\right)^{2(\gamma-1)}\,dt.$$

Let

$$a-1=\frac{\gamma-1}{\gamma} \quad\Longrightarrow\quad a=\frac{2\gamma-1}{\gamma}, \qquad s=2(\gamma-1).$$

Using the known identity

$$\int_0^1 t^{a-1}\big(-\ln(1-t)\big)^{s}\,dt = \Gamma(s+1)\sum_{k=0}^{\infty}\binom{a-1}{k}\frac{(-1)^k}{(k+1)^{\,s+1}},$$

we obtain

$$I = \frac{1}{\gamma}\Gamma(2\gamma-1)\sum_{k=0}^{\infty}\binom{\frac{\gamma-1}{\gamma}}{k}\frac{(-1)^k}{(k+1)^{\,2\gamma-1}}.$$

Therefore,

$$\mathcal{L}_U(U_m) = \frac{-\gamma^2}{2(b-a)}\frac{\Gamma(2\gamma-1)}{\gamma\{\Gamma(m)\}^2}\sum_{k=0}^{\infty}\binom{\frac{\gamma-1}{\gamma}}{k}\frac{(-1)^k}{(k+1)^{\,2\gamma-1}}$$

with the generalized binomial coefficient

$$\binom{\alpha}{k} = \frac{\Gamma(\alpha+1)}{\Gamma(k+1)\Gamma(\alpha-k+1)}.$$

5.3. $Ļ(U_m)$ *for Expontiated Weibull Distribution*

The quantile function for the exponentiated weibull distribution is $Q(w) = \left[\frac{\ln(1-w)}{-\lambda}\right]^{\frac{1}{\beta}}$, where $\lambda,\ \beta > 0$. Then, the measure $Ļ(U_m)$ for the exponentiated Weibull distribution is defined as

$$\mathcal{L}_W(U_m) = -\frac{\gamma^2}{2\beta\lambda^{1/\beta}}\,\frac{1}{\{\Gamma(m)\}^2}\int_0^1 w^{2(\gamma-1)}\big(-\ln(1-w^\gamma)\big)^{2(\gamma-1)}\,dw$$

Make the substitution $t = w^\gamma$. Then $w = t^{1/\gamma}$ and $dw = \dfrac{1}{\gamma}t^{1/\gamma-1}\,dt$, so

$$\int_0^1 w^{2(\gamma-1)}\big(-\ln(1-w^\gamma)\big)^{2(\gamma-1)}\,dw = \frac{1}{\gamma}\int_0^1 t^{(\gamma-1)/\gamma}\big(-\ln(1-t)\big)^{2(\gamma-1)}\,dt.$$

Set

$$a-1 = \frac{\gamma-1}{\gamma} \quad\Rightarrow\quad a = \frac{2\gamma-1}{\gamma}, \qquad s = 2(\gamma-1).$$

Using the identity

$$\int_0^1 t^{a-1}\big(-\ln(1-t)\big)^s\,dt = \Gamma(s+1)\sum_{k=0}^{\infty}\binom{a-1}{k}\frac{(-1)^k}{(k+1)^{s+1}},$$

with a and s as above, we obtain

$$\int_0^1 w^{2(\gamma-1)}\big(-\ln(1-w^\gamma)\big)^{2(\gamma-1)}\,dw = \frac{1}{\gamma}\,\Gamma(2\gamma-1)\sum_{k=0}^{\infty}\binom{\frac{\gamma-1}{\gamma}}{k}\frac{(-1)^k}{(k+1)^{\,2\gamma-1}}.$$

Therefore the evaluated expression is

$$\mathcal{L}_W(U_m) = -\frac{\gamma}{2\beta\lambda^{1/\beta}}\,\frac{\Gamma(2\gamma-1)}{\{\Gamma(m)\}^2}\sum_{k=0}^{\infty}\binom{\frac{\gamma-1}{\gamma}}{k}\frac{(-1)^k}{(k+1)^{\,2\gamma-1}}$$

5.4. $Ļ(U_m)$ *for Expontiated Generalized Rayleigh(GR) Distribution*

The quantile function for the expontiated generalized rayleigh(GR) distribution is $Q(w) = \left[\frac{\ln(1-w)}{-\lambda}\right]^{\frac{1}{2}}$, where $\lambda > 0$. Then measure $Ļ(U_m)$ for expontiated generalized rayleigh(GR) distribution is defined as

$$\mathcal{L}_G(U_m) = -\frac{\gamma^2}{2\beta\lambda^{1/2}}\,\frac{1}{\{\Gamma(m)\}^2}\int_0^1 w^{2(\gamma-1)}\big(-\ln(1-w^\gamma)\big)^{2(\gamma-1)}\,dw.$$

Make the substitution $t = w^\gamma$. Then $w = t^{1/\gamma}$ and $dw = \dfrac{1}{\gamma}t^{1/\gamma-1}\,dt$, so

$$\int_0^1 w^{2(\gamma-1)}\big(-\ln(1-w^\gamma)\big)^{2(\gamma-1)}\,dw = \frac{1}{\gamma}\int_0^1 t^{(\gamma-1)/\gamma}\big(-\ln(1-t)\big)^{2(\gamma-1)}\,dt.$$

Set

$$a-1 = \frac{\gamma-1}{\gamma} \quad\Rightarrow\quad a = \frac{2\gamma-1}{\gamma}, \qquad s = 2(\gamma-1).$$

Using the identity

$$\int_0^1 t^{a-1}\big(-\ln(1-t)\big)^s\,dt = \Gamma(s+1)\sum_{k=0}^{\infty}\binom{a-1}{k}\frac{(-1)^k}{(k+1)^{s+1}},$$

with the above a and s, we get

$$\int_0^1 w^{2(\gamma-1)}\big(-\ln(1-w^\gamma)\big)^{2(\gamma-1)}\,dw = \frac{1}{\gamma}\,\Gamma(2\gamma-1)\sum_{k=0}^{\infty}\binom{\frac{\gamma-1}{\gamma}}{k}\frac{(-1)^k}{(k+1)^{\,2\gamma-1}}.$$

Therefore, we obtained

$$\mathcal{L}_G(U_m) = -\frac{\gamma}{2\beta\lambda^{1/2}}\,\frac{\Gamma(2\gamma-1)}{\{\Gamma(m)\}^2}\sum_{k=0}^{\infty}\binom{\frac{\gamma-1}{\gamma}}{k}\frac{(-1)^k}{(k+1)^{\,2\gamma-1}}$$

5.5. $\L(U_m)$ *for Expontiated Power Distribution*

The quantile function for the exponentiated power distribution is $Q(w) = \beta w^{\frac{1}{c}}$, where $\lambda,\ \beta > 0$. Then the measure $\L(U_m)$ for the exponentiated power distribution is defined as

$$\mathcal{L}_{EP}(U_m) = -\frac{c\alpha^2}{2\beta}\,\frac{1}{\{\Gamma(m)\}^2}\int_0^1 w^{\frac{c-1}{c}}\big(-\ln(1-w)\big)^{2(\gamma-1)}\,dw$$

Using the standard identity

$$\int_0^1 t^{a-1}\big(-\ln(1-t)\big)^{s}\,dt = \Gamma(s+1)\sum_{k=0}^{\infty}\binom{a-1}{k}\frac{(-1)^k}{(k+1)^{s+1}},$$

set

$$a-1 = \frac{c-1}{c} \quad\Rightarrow\quad a = \frac{2c-1}{c}, \qquad s = 2(\gamma-1).$$

Thus,

$$\int_0^1 w^{\frac{c-1}{c}}\big(-\ln(1-w)\big)^{2(\gamma-1)}\,dw = \Gamma(2\gamma-1)\sum_{k=0}^{\infty}\binom{\frac{c-1}{c}}{k}\frac{(-1)^k}{(k+1)^{\,2\gamma-1}}.$$

Therefore the evaluated expression becomes

$$\mathcal{L}_{EP}(U_m) = -\frac{c\alpha^2}{2\beta}\,\frac{\Gamma(2\gamma-1)}{\{\Gamma(m)\}^2}\sum_{k=0}^{\infty}\binom{\frac{c-1}{c}}{k}\frac{(-1)^k}{(k+1)^{\,2\gamma-1}}$$

Quantile-based extropy of m^{th} record for F^γ family distributions is summarized in Table 3.

6. Conclusion

The quantile-based extropy measures possess some unique properties than its distribution function approach. The quantile-based extropy of record statistics has several advantages its dynamic versions provide a linear equation that presents a unique hazard function corresponding to the extropy measure of m^{th} upper record statistics but $J(f_m; t)$ may or may not determine hazard function uniquely. Quite simple in cases where the distribution function is not tractable, while the quantile function has a simpler form.

Conflict of Interest

The corresponding author declares that there is no conflict of interest on behalf of all authors.

Distribution	$Q(w)$ (Baseline Distribution)	Ļ(U_m)
Exponentiated Exponential	$-\lambda^{-1}\ln(1-w)$	Ļ$_E(U_m) = -\frac{\lambda\gamma^2}{2\{\Gamma(m)\}^2}\int_0^1 u^{2(\alpha-1)+1}(-\log(1-u^\alpha))^{2(\alpha-1)}du$
Exponentiated Uniform	$b+u(b-a)$	$\mathrm{L}_U(U_m) = \frac{-\gamma^2}{2(b-a)}\frac{\Gamma(2\gamma-1)}{\gamma\{\Gamma(m)\}^2}\sum_{k=0}^{\infty}\binom{\frac{\gamma-1}{\gamma}}{k}\frac{(-1)^k}{(k+1)^{2\gamma-1}}$
Exponentiated Weibull	$\left[\frac{\ln(1-w)}{-\lambda}\right]^{\frac{1}{\beta}}$	$\mathcal{L}_W(U_m) = -\frac{\gamma}{2\beta\lambda^{1/\beta}}\frac{\Gamma(2\gamma-1)}{\{\Gamma(m)\}^2}\sum_{k=0}^{\infty}\binom{\frac{\gamma-1}{\gamma}}{k}\frac{(-1)^k}{(k+1)^{2\gamma-1}}$
Generalized Rayleigh(GR)	$\left[\frac{\ln(1-w)}{-\lambda}\right]^{\frac{1}{2}}$	$\mathcal{L}_G(U_m) = -\frac{\gamma}{2\beta\lambda^{1/2}}\frac{\Gamma(2\gamma-1)}{\{\Gamma(m)\}^2}\sum_{k=0}^{\infty}\binom{\frac{\gamma-1}{\gamma}}{k}\frac{(-1)^k}{(k+1)^{2\gamma-1}}$
Power (EP)	$\beta w^{\frac{1}{c}}$	$\mathcal{L}_{EP}(U_m) = -\frac{c\alpha^2}{2\beta}\frac{\Gamma(2\gamma-1)}{\{\Gamma(m)\}^2}\sum_{k=0}^{\infty}\binom{\frac{c-1}{c}}{k}\frac{(-1)^k}{(k+1)^{2\gamma-1}}$
Exponentiated Log Logistic	$\frac{w}{\theta(1-w)}^{\frac{1}{b}}$	$\mathcal{L}_{LG}(U_m) = \frac{b\,\gamma^2}{2\,\theta^{1/b}\,\{\Gamma(m)\}^2\left(2\gamma-\frac{1}{b}\right)}$
Exponentiated Govindarajulu	$\theta+\sigma\{(\beta+1)w^\beta-\beta w^{\beta+1}\}$	$-\frac{1}{2}\frac{\gamma^2}{\sigma\beta}\frac{1}{\{\Gamma(m)\}^2}\int_0^1 w^{2\gamma-\beta+1}(-\ln(1-w^\gamma))^{2(\gamma-1)}dw$ $+\frac{1}{2}\frac{\gamma^2}{\sigma\beta}\frac{1}{\{\Gamma(m)\}^2}\int_0^1 w^{2\gamma-\beta+2}(-\ln(1-w^\gamma))^{2(\gamma-1)}dw$

Table 3 Quantile-based extropy (2) for F^γ family distributions.

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