STATISTICS, OPTIMIZATION AND INFORMATION COMPUTING
Stat., Optim. Inf. Comput., Vol. **13**, April 2025, pp 1444–1456.
Published online in International Academic Press (www.IAPress.org)

Some Results of Generalized Extropy Measure and Its Application

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Abstract Taking into account the importance of extropy (see Lad et al. 2015), and its various generalizations, in the present communication we consider and study the generalized extropy of order α and type β based on Varma's (Varma, 1966) information measure for both discrete and continuous random variables. The dynamic versions (residual and past, both) of the proposed generalized extropy measure have also been presented. At the end, the interval generalized extropy measure and an application of the proposed generalized extropy measure are also presented.

Keywords Extropy measure, Residual and past extropy measure, Generalized extropy measure, Interval extropy.

DOI: 10.19139/soic-2310-5070-2075

1. Introduction

Information theory is the mathematical examination of the ideas, parameters, and laws regulating the transmission of messages across communication systems. The building block of information theory is Shannon entropy [18]. To quantify the information Shannon gave a numerical measure of information, also called measure of uncertainty or the entropy measure. Afterwards, study of entropy-based random systems has seen a noticeable increase with applications in a variety of sectors. Various characterizations and generalizations and their residual and past versions have been examined by many researchers [3], and a non-additive measure of Tsallis entropy [20], [16], and [21].

Varma entropy of two parameters is essential as a measure of complexity and uncertainty to explain many chaotic systems and is directly applicable in several fields including physics, electronics, and engineering. More results and applications of generalized entropy measure of order α and type β have been studied by many researchers, refer to [7], [11], [9], [6], [10], [1], and [19].

It is interesting to learn that the complementary dual of the Shannon entropy measure exists and has some common properties. This new measure of uncertainty has been introduced by [12] and is known as extropy. Although there are some mathematical analogies between the two measures, extropy typically has different uses and interpretations than entropy. Since the practical use of extropy is still in its infancy, it has the potential to have intriguing applications in information theory in the future. Moreover, the Extropy measure has been extensively employed in numerous domains, see [4]. lifetime distribution [8]. The concept of extropy of order statistics and record value. Also, the residual extropy of order statistics has been studied by [15]. The residual extropy of k-record values has been studied by [5], and [17]. Testing symmetry based on the extropy of record values has been studied by [22]. Recently, the Complementary dual of Renyi entropy has been studied by [13]. The complementary dual of generalized entropy measures based on order α and type β is still an open issue.

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ISSN 2310-5070 (online) ISSN 2311-004X (print)

Extropy is a cornerstone for measuring information uncertainty. While, some probability distributions are nottraceable in such cases density-based measures have limitations. Extropy of order α and type β acts as a generalization, offering more flexibility in capturing different aspects of uncertainty. In this paper, we studied generalized extropy measures of order α and type β for both discrete and continuous random variables. Also, dynamic versions of generalized extropy with their inter-relationship have been examined.
The structure of the paper is as follows: Preliminaries are given in Section 2, including Shannon entropy, Renyi entropy, and complementary duals of these entropies, Varma entropy measure. In Section 3, a generalized extropy measure for discrete random variable (random variable) is introduced and also some theorems for generalized extropy have been studied. Some numerical examples and graphical representations are also been given in the same section. Dynamic versions of generalized extropy measures are studied in section 4. In Section 5, a generalized extropy measure for a continuous random variable has been introduced and investigated the maximum generalized extropy measure. In Section 5, interval generalized extropy is studied. In Section 6, a letter-catching experiment problem as an application part is given, and Section 7 concludes this paper.

2. Preliminaries

For a given observable space X with finite discrete possible values $\{x_1, x_2 \cdots x_n\}$ and corresponding probability mass function (p.m.f) $P = \{p_1, p_2 \ldots p_n\}$, the measure of uncertainty given by Shannon [18] is

$$H(P) = -\sum_{i=1}^{n} p_i \, \log \, p_i, \; \forall \, 0 \leq p_i \leq 1, \; \sum_{i=1}^{n} p_i = 1. \tag{1}$$

A generalization of measure (**??**) for order α and type β is Varma entropy (Varma, 1966), defined as follows:

$$H_{\alpha}^{\beta}(P) = \frac{1}{\beta - \alpha} \log \sum_{i=1}^{n} (p_i)^{\alpha+\beta-1}, \; \forall \, \beta - 1 \leq \alpha < \beta. \tag{2}$$

The generalized measure (2) for discrete uniform distribution is given as

$$H_{\alpha}^{\beta}(P_U) = \frac{1}{\beta - \alpha} \log \, (n)^{2-\alpha-\beta}. \tag{3}$$

By using complementary dual to Shannon entropy, a novel method of measuring uncertainty that shares some characteristics with Shannon's entropy is introduced by [12] and it is defined as follows:

$$J(X) = -\sum_{i=1}^{n} (1 - p_i) \, \log \, (1 - p_i). \tag{4}$$

Recently, the complementary dual of Renyi entropy and its conditional and joint versions have been studied by [13].

3. Generalized Extropy of Order α and Type β

In this section, we present the generalized extropy measurements for the discrete random variable. Equations (2) and (4) serve as representative forms of extropy in our study. Further, we define the generalized extropy of order α and type β for the discrete random variable based on the concept of discrete Renyi extropy [13].
For a given observable space X with finite discrete possible values $\{x_1, x_2 \cdots x_n\}$ and corresponding probability mass function $P = \{p_1, p_2 \ldots p_n\}$, we follow the concept complementary of entropy and extropy arises from the fact that the extropy of mass a function, equal a location and scale transform of the entropy of another mass function that is complementary, $q_n = \frac{(1-p_n)}{(n-1)}$ (see Lat et al. 2015). Thus the generalized extropy of order α and type β is

defined as follows:

$$\begin{aligned} J_\alpha^\beta(P) =& (n-1)\left[H_\alpha^\beta(q) - \log(n-1)\right] \\ =& (n-1)\left[H_\alpha^\beta\left(\frac{1-p_n}{n-1}\right) - log(n-1)\right] \\ =& (n-1)\left[\frac{1}{\beta-\alpha}\log\sum_{i=1}^{n}\left(\frac{1-p_i}{n-1}\right)^{\alpha+\beta-1} - \log(n-1)\right] \\ J_\alpha^\beta(P) =& \frac{-(2\beta-1)(n-1)\log(n-1) + (n-1)\log\sum_{i=1}^{n}(1-p_i)^{\alpha+\beta-1}}{\beta-\alpha}, \end{aligned} \tag{5}$$

where, $n = |X|$ is the cardinality for observable space.

Example 3.1
For a given discrete random variable $X = \{a, b\}$, with cardinality 2, corresponding probability distribution $p(a) = \frac{2}{3}$, $p(b) = \frac{1}{3}$, without loss of generality for the parameters $\alpha = 0.7$ and $\beta = 1.2$, calculated generalised extropy and Varma's entropy are equal.

$$\begin{aligned} J_\alpha^\beta(P) = H_\alpha^\beta(P) &= \frac{\log\left[(p(a))^{\alpha+\beta-1} + (p(b))^{\alpha+\beta-1}\right]}{\beta-\alpha} \\ H_\alpha^\beta(P) &= \frac{\log\left[(\frac{2}{3})^{0.9} + (\frac{1}{3})^{0.9}\right]}{0.5} = 0.05575411. \\ J_\alpha^\beta(P) &= \frac{\log\left[(\frac{1}{3})^{0.9} + (\frac{2}{3})^{0.9}\right]}{0.5} = 0.05575411. \end{aligned}$$

Example 3.2
For a given discrete uniformly distributed random variable ($|X| = 3$) $X = \{a, b, c\}$, with their corresponding probabilities $p(a) = p(b) = p(c) = \frac{1}{3}$ calculated generalized extropy of order α and type β and Varma's entropy for the parameters ($\alpha = 0.8$, $\beta = 1.1$) are

$$J_\alpha^\beta(P) = \frac{-(2\beta-1)(2) + (2)\log(3(\frac{2}{3})^{0.9})}{\beta-\alpha} = -0.28398,$$

and

$$H_\alpha^\beta(P) = \frac{\log(3)(\frac{1}{3})^{0.9}}{\beta-\alpha} = 0.100343.$$

Remark 3.1
We observe that for $|X| = 2$, generalized entropy measure (2) and extropy measure (5) have equal values.

Example 3.3
Consider a random variable with a probability mass function

$$P(X = i) = p_i = \frac{2i}{n(n+1)}, \quad i = 1, 2, ..., n,$$

substituting this in (5) and we obtain the discrete generalized extropy of order α and type β

$$J_\alpha^\beta(P) = \frac{-(2\beta-1)(n-1)\log(n-1) + (n-1)\log\sum_{i=1}^{n}(\frac{n(n+1)-2i}{n(n+1)})^{\alpha+\beta-1}}{\beta-\alpha}.$$

Example 3.4
Consider a random variable with a probability mass function

$$P(X = i) = p_i = \frac{1}{n}, \quad i = 1, 2, ..., n.$$

Then we obtain the discrete generalized extropy of order α and type β for uniform distribution

$$J_{\alpha}^{\beta}(P) = \frac{n-1}{\beta-\alpha}\left\{\log\frac{(n-1)^{\alpha-\beta}}{n^{\alpha+\beta-2}}\right\}. \tag{6}$$

For $\beta = 1$, equation (6) will reduce to $J_{\alpha}(P) = (n-1)\log\frac{n}{n-1}$, a Renyi extropy for uniform distribution; a result obtained by Liu and Xiao, (2021).

Next, we will show that the maximum of the generalized extropy measure (5) depends on its parameters.

Theorem 3.1
The generalized extropy measure of order α and β (5) may not attain their maximum when random variable follows a uniform distribution.

Proof
Let us consider the Langrage function under equation (5).

$$L = \frac{-(2\beta-1)(n-1)\log(n-1) + (n-1)\log\sum_{i=1}^{n}(1-p_i)^{\alpha+\beta-1}}{\beta-\alpha} + \lambda(\sum_{i=1}^{n}p_i - 1). \tag{7}$$

Calculate the gradient and stationary point

$$\begin{aligned}\frac{\partial L}{\partial p_i} =& \frac{n-1}{\beta-\alpha}\left(\frac{1}{\sum_{i=1}^{i=n}(1-p_i)^{\alpha+\beta-1}}\right)\frac{\partial}{\partial p_i}\left\{\sum_{i=1}^{n}(1-p_i)^{\alpha+\beta-1}\right\} + \lambda = 0\\ =& \frac{(n-1)(\alpha+\beta-1)}{(\beta-\alpha)\sum_{i=1}^{i=n}(1-p_i)^{\alpha+\beta-1}}(1-p_i)^{\alpha+\beta-2} + \lambda = 0\\ \Rightarrow& \frac{(n-1)(\alpha+\beta-1)}{(\beta-\alpha)}(1-p_i)^{\alpha+\beta-2} = \lambda\sum_{i=1}^{i=n}(1-p_i)^{\alpha+\beta-1}\\ \Rightarrow& (1-p_i)^{\alpha+\beta-2} = \frac{(1-p_i)^{\alpha+\beta-2}}{(n-1)(\alpha+\beta-1)}(\beta-\alpha)\lambda\sum_{i=1}^{i=n}(1-p_i)^{\alpha+\beta-1}\end{aligned} \tag{8}$$

The right-hand side of the equation (8) is constant, so $(1-p_1)^{\alpha+\beta-2} = (1-p_2)^{\alpha+\beta-2} = ...(1-p_n)^{\alpha+\beta-2}$. It provides

$$p_1 = p_2 = \ldots = p_n = \frac{1}{n}.$$

After substituting the above values in equation (5), we get

$$J_{\alpha}^{\beta}(P) = \frac{-(\beta-\alpha)(n-1)\log(n-1) + (n-1)(2-\alpha-\beta)\log n}{\beta-\alpha}. \tag{9}$$

The equation (9) is not free from the parameters. Hence proved the result. □

Example 3.5
Generalized extropy (5) for a uniformly distributed random variable is given as

$$J_{\alpha}^{\beta}(P_U) = \left(\frac{n-1}{\beta-\alpha}\right)\{(2-\alpha-\beta)\log n - (\beta-\alpha)\log(n-1)\}. \tag{10}$$

The following example shows that the maximum value for the generalized extropy measure (5) depends on parameters.

Example 3.6
For a given uniformly distributed discrete random variable $X = \{a, b\}$, with probabilities $p(a) = p(b) = \frac{1}{2}$, without loss of generality for $(\alpha = 0.7,\ \beta = 1.2)$ and $(\alpha = 0.6,\ \beta = 1.5)$, values of the calculated generalized extropy (5) are 0.60206 and -0.03345 respectively.

The figure 1 shows that the generalized extropy measure of order α and type β for $|X| = 2$ attains its maximum or minimum values when random variables follow a uniform distribution. For $\alpha + \beta < 2$ or > 2, $J_\alpha^\beta(P_u)$ attains its maximum and minimum values, respectively.

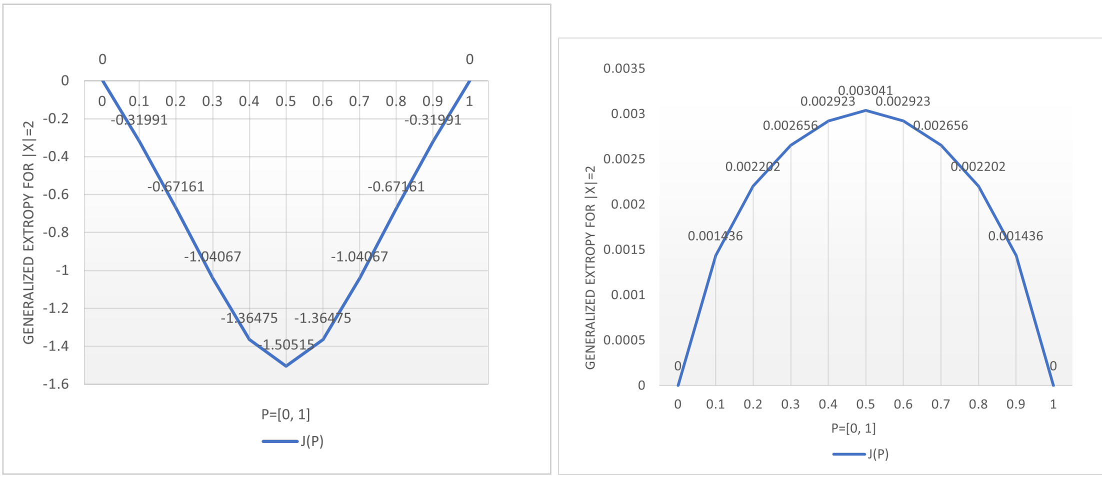

Figure 1. Values of $J_\alpha^\beta(P)$ for $(\alpha = 2, \beta = 2.5)$ *and* $(\alpha = 0.5, \beta = 1.49)$ for $|X| = 2$.

The following theorem compares Varma's entropy and generalized extropy of order α and type β for uniform distribution.

Theorem 3.2
For cardinality greater than 2 and uniformly distributed random variable, the Varma entropy (1) is always greater than or equal to the generalized extropy (5)

$$H_\alpha^\beta(P_U) \geq J_\alpha^\beta(P_U),\ \forall\, n \geq 2.$$

Proof
We will prove this theorem in two parts: in part 1, we will conclude the result for $n = 2$, and in part II, we will prove the result for $n \geq 3$.
Part-1 In remark (5) result has been already shown for $|X| = 2$. Part-II for n$\geq$ 3, Subtracting (7) from (3) we get,

$$[H_\alpha^\beta(P_U)] - [J_\alpha^\beta(P_U)] = (n-1)\log(n-1) - (n-2)\left(\frac{2-(\alpha+\beta)}{\beta-\alpha}\right)\log n. \tag{11}$$

If $\alpha + \beta = 2$, then (11) gives $H_\alpha^\beta(P_U) - J_\alpha^\beta(P_U) = (n-1)\log(n-1) = +ve\ \forall\, n \geq 3$.
If, $\alpha + \beta < (>)2,$ then (11) reduces to

$$H_\alpha^\beta(P_U) - J_\alpha^\beta(P_U) = \log\left(\frac{(n-1)^{(n-1)}}{n^{\frac{(n-2)(2-(\alpha+\beta))}{\beta-\alpha}}}\right),$$

since, $\frac{(n-1)^{(n-1)}}{n^{\frac{(n-2)(2-(\alpha+\beta))}{\beta-\alpha}}} > 1,\ \forall n \geq 3$. Hence the result is also true for $n \geq 3$. □

Table 1. Value of $H_\alpha^\beta(P_U)\ and\ J_\alpha^\beta(P_U)$ for different the permeters when cardinality changes

$\|X\|$	β	α	$H_\alpha^\beta(P_U)$	$J_\alpha^\beta(P_U)$
for, $n = 2$,	1.1	0.8	0.100343	0.100343
	1.49	0.59	−0.02676	−0.02676
	1.5	0.6	−0.3345	−0.3345
	1.5	0.5	0	0
for, $n = 3$,	1.1	0.8	−0.15904	−0.92014
	1.49	0.59	−0.04241	−0.68688
	1.5	0.6	−0.05301	0.70809
	1.5	0.5	0	−0.60206
for, n=4,	1.1	0.8	0.2000687	−0.8293
	1.49	0.59	−0.05352	−1.59199
	1.5	0.6	−0.0669	−1.63205
.	1.5	0.5	0	−1.43136
for, n=5,	1.1	0.8	0.23299	−1.47628
	1.49	0.59	−0.06213	−2.65676
	1.5	0.6	−0.07766	−2.65676
	1.5	0.5	0	−2.4082
for, n=6,	1.1	0.8	0.80071	−2.19793
	1.49	0.59	−0.06917	−3.807
	1.5	0.6	0.08646	−3.92716
	1.5	0.5	0	−3.4948
for, n=7,	1.1	0.8	0.281699	−2.97871
	1.49	0.59	−0.07512	−5.11963
	1.5	0.6	−0.0939	−5.23231
	1.5	0.5	0	−4.6689

For uniformly distributed random variables, Table 1 clarifies the relationship between Varma's entropy and the generalized extropy measure for various cardinalities. We observe that the values of the generalized extropy measure of order α and type β for a uniformly distributed discrete random variable decrease as cardinality increases. But Varma's entropy behavior conflicts with it.

Keeping α and β fixed and changing n from 2 to 7 it shows the discrepancy between the values of the generalized extropy measure and Varma's entropy measure as shown in Figure 2.

The Varma entropy is always zero for $\alpha + \beta = 2$, but not for the generalized extropy measure, as Figure 3 illustrates.

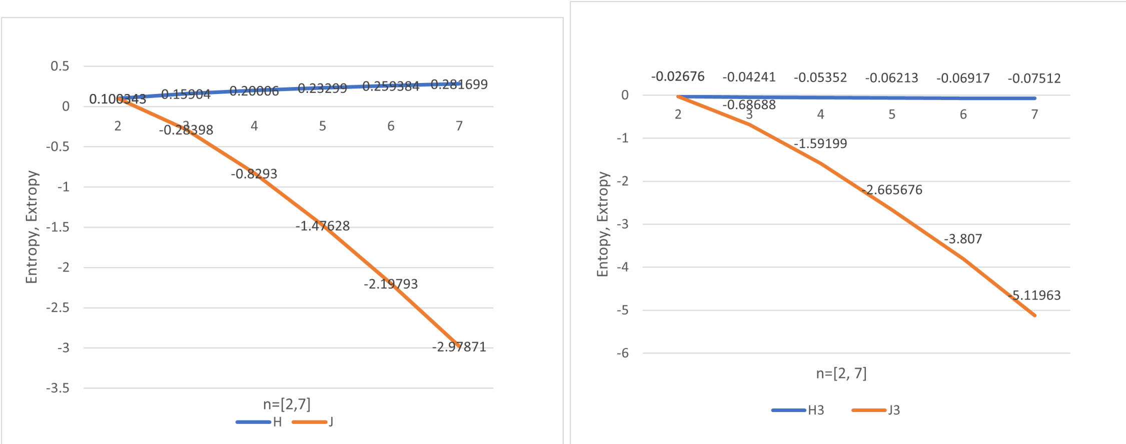

Figure 2. $H_\alpha^\beta(P_U)$ and $J_\alpha^\beta(P_U)$ at $(\alpha = 0.8,\ \beta = 1.1)$ *and* $(\alpha = 0.59,\ \beta = 1.49)$ for $|X| = [2, 7]$.

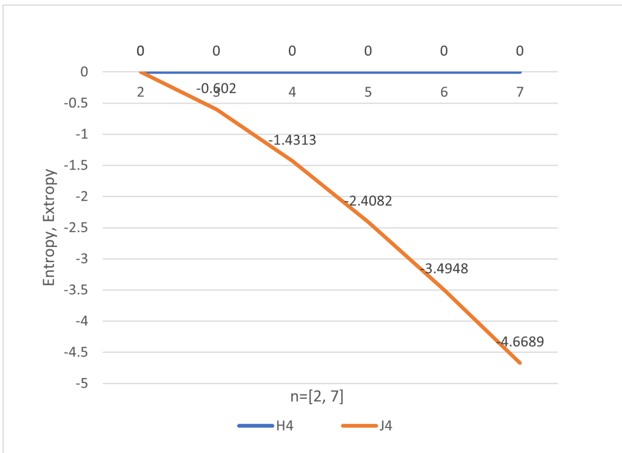

Figure 3. $H_\alpha^\beta(P_U)$ and $J_\alpha^\beta(P_U)$ at $(\alpha = 0.5,\ \beta = 1.5)$.

4. Dynamic Versions of Generalized Extropy $J_\alpha^\beta(P)$.

The generalized extropy measure (5) is useless for a system that has survived for a unit of time. Since uncertainty and diversity fluctuate over time in actual events, dynamic extropy measures account for dynamic changes in uncertainty and consider both the system's past and residual state, leading to better information. In this section, we will study dynamic versions of generalized extropy $J_\alpha^\beta(P)$.

If X is a discrete supported random variable $[1, n]$ and $X_t = (X - t|X > tis)$ be residual lifetime whose probability mass function $P_t = \frac{p_i}{R_t}$ and $\bar{X}_t = (X|X \le t)$ be past lifetime whose probability mass function $\bar{P}_t = \frac{p_i}{F(t)}$. The discrete form of residual generalized extropy measure of order α and type β is defined as follows:

$$J_\alpha^\beta(P_t) = \frac{-(2\beta - 1)(n - t)\log(n - t) + (n - t)\log\sum_{i=t}^{i=n}(1 - (\frac{p_i}{R_t}))^{\alpha+\beta-1}}{\beta - \alpha}. \tag{12}$$

In parallel to (12), the past version of generalized extropy is defined as

$$J_\alpha^\beta(\bar{P}_t) = \frac{-(2\beta - 1)(t - 1)\log(t - 1) + (t - 1)\log\sum_{i=1}^{i=t}(1 - (\frac{p_i}{F_t}))^{\alpha+\beta-1}}{\beta - \alpha}. \tag{13}$$

For $\beta = 1$, both of the measures (12) and (13) reduce to the dynamic (residual and past) versions of Renyi extropy given as

$$J_\alpha(P_t) = \frac{-(n-t)\log(n-t) + (n-t)\log\sum_{i=t}^{i=n}(1-(\frac{p_i}{F_t}))^\alpha}{1-\alpha}.$$

and

$$J_\alpha(\bar{P}_t) = \frac{-(t-1)\log(t-1) + (t-1)\log\sum_{i=1}^{i=t}(1-(\frac{p_i}{F_t}))^\alpha}{1-\alpha}.$$

respectively. refer to Jawa, et al. (2022).

Example 4.1
Suppose that the discrete random variable X has a uniform distribution. Then, the residual and past generalized extropy (12) and (13) are given, respectively, by

$$\begin{aligned} J_\alpha^\beta(P_t) =& \frac{-(2\beta-1)(n-t)\log(n-t) + (n-t)\log\sum_{i=t}^{i=n}(1-(\frac{1}{n-t+1}))^{\alpha+\beta-1}}{\beta-\alpha} \\ =& \frac{-(2\beta-1)(n-t)\log(n-t) + (n-t)\log(n-t+1)(1-(\frac{1}{n-t+1}))^{\alpha+\beta-1}}{\beta-\alpha} \\ =& \frac{-(2\beta-1)(n-t)\log(n-t) + (n-t)\log(\frac{(n-t)^{\alpha+\beta-1}}{(n-t+1)^{\alpha+\beta-2}})}{\beta-\alpha}, \end{aligned}$$

and

$$\begin{aligned} J_\alpha^\beta(\bar{P}_t) =& \frac{-(2\beta-1)(t-1)\log(t-1) + (t-1)\log\sum_{i=1}^{i=t}(1-(\frac{1}{n-t+1}))^{\alpha+\beta-1}}{\beta-\alpha}; \\ =& \frac{-(2\beta-1)(t-1)\log t(t-1) + (t-1)\log((\frac{n-t}{n-t+1}))^{\alpha+\beta-1}}{\beta-\alpha}, \end{aligned}$$

respectively.

Example 4.2
Consider a random variable with a probability mass function

$$P(X=i) = p_i = \frac{2i}{n(n+1)}, \quad i = 1, 2, ..., n, \tag{14}$$

and survival function corresponding to X given as

$$\bar{F}(i) = 1 - F(i-1) = 1 - \frac{i(i-1)}{n(n+1)}, \tag{15}$$

using the given distribution function in (12) and (13) respectively and we obtain the discrete residual generalized extropy measure and past generalized extropy measure

$$J_\alpha^\beta(P,t) = \frac{-(2\beta-1)(n-t)\log(n-t) + (n-t)\log\sum_{i=t}^{i=n}(1-(\frac{2i}{n(n+1)}))^{\alpha+\beta-1}}{\beta-\alpha}, \tag{16}$$

and

$$J_\alpha^\beta(\bar{P}_t) = \frac{-(2\beta-1)(n-t)\log(n-t) + (n-t)\log\sum_{i=1}^{i=t}(\frac{t-i}{t}))^{\alpha+\beta-1}}{\beta-\alpha}, \tag{17}$$

respectively.

Remark 4.1
For $\beta = 1$, both of the measures (16) and (17) reduce to the dynamic (residual and past) versions of Renyi extropy for the same distribution

$$J(P,t) = -\sum_{i=t}^{i=n}\left(1 - \frac{2i}{n(n+1) - t(t-1)}\right)\log\left(1 - \frac{2i}{n(n+1) - t(t-1)}\right),$$

and

$$J_{\alpha}^{\beta}(\bar{P}_t) = -\sum_{i=1}^{i=t}\left(\frac{t-i}{t}\right)\log\left(\frac{t-i}{t}\right)$$

respectively.

5. Continuous Generalized Extropy

In this section, we introduce the continuous generalized extropy based on continuous distribution lifetime. Continuous extropy is a more theoretical, concept used to quantify the information content or uncertainty associated with a continuous probability distribution. The extropy of the continuous random variable X supported on R defined as

$$J(X) = -\int_R (1 - f(x))\log(1 - f(x))dx. \tag{18}$$

Let X be a continuous random variable having p.d.f, $f(x)$ with support in $[a, b]$, $-\infty < a < b < \infty$ and $b - a \neq 1$. Then, analogous to (5) continuous generalized extropy is proposed as

$$CJ_{\alpha}^{\beta}(X) = \frac{-(2\beta-1)(b-a-1)\log(b-a-1) + (b-a-1)\log\int_a^b (1-f(x))^{\alpha+\beta-1}dx}{\beta-\alpha} \tag{19}$$

Example 5.1
Suppose that the continuous random variable X has uniform distribution over $[a, b]$, provided that $b - a \neq 1$. Then, continuous generalized extropy (20) is defined as

$$\begin{aligned}CJ_{\alpha}^{\beta}(X) &= \frac{-(2\beta-1)(b-a-1)\log(b-a-1) + (b-a-1)\log\int_a^b (1-\frac{1}{b-a})^{\alpha+\beta-1}dx}{\beta-\alpha},\\ &= \frac{-(2\beta-1)(b-a-1)\log(b-a-1) + (b-a-1)\log\int_a^b (\frac{b-a-1}{b-a})^{\alpha+\beta-1}dx}{\beta-\alpha},\\ &= (b-a-1)\left(\frac{(\alpha-\beta)\log(b-a-1) - (\alpha+\beta-2)\log(b-a)}{\beta-\alpha}\right),\end{aligned}$$

where $b - a \neq 1$.

5.0.1. The Maximum Generalized Extropy Measure Provided that X is a continuous random variable supported in $[a, b]$, $-\infty < a < b < \infty$. Then we show that X has the maximum continuous generalized extropy if and only if it follows the continuous uniform distribution. As,

$$CJ_{\alpha}^{\beta}(X) = \frac{-(2\beta-1)(b-a-1)\log(b-a-1) + (b-a-1)\log\int_a^b (1-f(x))^{\alpha+\beta-1}dx}{1-\alpha},$$

subjected to constraints

$$\int_a^b f(x)dx = 1. \tag{20}$$

We can obtain the maximization of Continuous generalized extropy using the Lagrange multipliers method as follows:

$$L(X) = \frac{1}{\beta-\alpha}[-(2\beta-1)(b-a-1)\log(b-a-1) + (b-a-1)\log\int_a^b (1-f(x))^{\alpha+\beta-1}dx.$$

Differentiating $L(X)$ w.r.t. $f(x)$, then equating to zero we obtain

$$\frac{dL(X)}{df(x)} = \frac{1}{\beta-\alpha}\left[\frac{-(b-a-1)(\alpha+\beta-1)(1-f(x))^{\alpha+\beta-2}}{\int_a^b (1-f(x))^{\alpha+\beta-1}dx}\right] + \lambda = 0$$

$$f(x) = 1 - \left[\frac{\lambda(\beta-\alpha)}{(b-a-1)(\alpha+\beta-1)}\int_a^b (1-f(x))^{\alpha+\beta-1}dx\right]^{\frac{1}{\alpha+\beta-2}}.$$

Now putting this value in equation 20, we get

$$\int_a^b 1 - \left[\frac{\lambda(\beta-\alpha)}{(b-a-1)(\alpha+\beta-1)}\int_a^b (1-f(x))^{\alpha+\beta-1}dx\right]^{\frac{1}{\alpha+\beta-2}} dx = 1 \tag{21}$$

$$\lambda = \frac{(b-a-1)(\alpha+\beta-1)}{(\beta-\alpha)\int_a^b (1-f(x))^{\alpha+\beta-1}dx}\left(1-\frac{1}{b-a}\right)^{\alpha+\beta-2} \tag{22}$$

now putting this value in equation (5.3), we get $f(x) = \frac{1}{b-a}$, which is a pdf of uniform distribution.

6. Interval Generalized Extropy of Order α and Type β

Overall, interval extropy offers a valuable tool for delving deeper into the uncertainty associated with events within specific timeframes. Unlike extropy [12], which deals with the overall uncertainty in a system, interval extropy focuses on the uncertainty within a specific time interval. This makes it particularly relevant in reliability analysis and related fields where understanding the remaining lifespan or failure time is critical. In the next section, we study the interval generalized extropy of order α and type β. If a continuous random variable is supported in $[t_1, t_2]$, then corresponding extropy measure can be define as

$$J(X) = -\int_{t_1}^{t_2} (1-f(x))\log(1-f(x))dx \tag{23}$$

The doubly truncated random variable $(X|t_1 < X < t_2)$ represents the lifetime of a unit which fails between t_1 and t_2 where $(t_1, t_2) \in D = \left\{(u, v) \in R_+^2 : F(u) < F(v)\right\}$, then interval (doubly truncated) generalized extropy measure can be defined as follow:

$$CIJ(X) = -\int_{t_1}^{t_2}\left(1-\frac{f(x)}{F(t_2)-F(t_1)}\right)\log\left(1-\frac{f(x)}{F(t_2)-F(t_1)}\right)dx \tag{24}$$

Analogus to (5), we proposed doubly truncated generalized extropy of order α and type β which is defined as follow:

$$CIJ_\alpha^\beta(X) = \frac{-(2\beta-1)(t_2-t_1-1)\log(t_2-t_1-1) + (t_2-t_1-1)\log\int_{t_1}^{t_2}\left(1-\frac{f(x)}{F(t_2)-F(t_1)}\right)^{\alpha+\beta-1}dx}{\beta-\alpha}. \tag{25}$$

Measure (25) is an extension of interval extropy and is known as "*doubly truncated generalized extropy of order α and type β*".

Proposition 6.1. Providing that X is a continuous random variable. Then, from (24) and (25), we have

$$\lim_{\alpha\Rightarrow 1,\, \beta=1} CIJ_{\alpha}^{\beta}(X) = CIJ(X),$$

Proof

Using (25), with applying L'Hospital rule, we get

$$\begin{aligned}
\lim_{\alpha\Rightarrow 1,\, \beta=1} CIJ_{\alpha}^{\beta}(X) &= \lim_{\alpha\to 1} \frac{-(t_2-t_1-1)\log(t_2-t_1-1) + (t_2-t_1-1)\log \int_{t_1}^{t_2} \left(1-\frac{f(x)}{F(t_2)-F(t_1)}\right)^{\alpha} dx}{1-\alpha} \\
&= \lim_{\alpha\to 1} \frac{(t_2-t_1-1)\int_{t_1}^{t_2} \frac{d}{d\alpha}\left(1-\frac{f(x)}{F(t_2)-F(t_1)}\right)^{\alpha} dx}{-\int_{t_1}^{t_2}\left(1-\frac{f(x)}{F(t_2)-F(t_1)}\right)^{\alpha} dx} \\
&= \lim_{\alpha\to 1} \frac{(t_2-t_1-1)\int_{t_1}^{t_2} \left(1-\frac{f(x)}{F(t_2)-F(t_1)}\right)^{\alpha} \log\left(1-\frac{f(x)}{F(t_2)-F(t_1)}\right) dx}{-\int_{t_1}^{t_2}\left(1-\frac{f(x)}{F(t_2)-F(t_1)}\right)^{\alpha} dx} \\
&= -\int_{t_1}^{t_2} \left(1-\frac{f(x)}{F(t_2)-F(t_1)}\right) \log\left(1-\frac{f(x)}{F(t_2)-F(t_1)}\right) dx \\
&= CIJ(X).
\end{aligned}$$

□

Remark 6.1

If $\beta = 1$ and $\alpha = 2$, doubly truncated generalized extropy (25) of order α and type β have a relationship with Interval extropy measure $IJ(t_1, t_2)$ (Buono et al. 2021).

Proof

Putting $\beta = 1$ and $\alpha = 2$ in equation (25) we get,

$$\begin{aligned}
CIJ_{\alpha}^{\beta}(X) =&(t_2-t_1-1)\log(t_2-t_1-1) - (t_2-t_1-1)\log \int_{t_1}^{t_2} \left(1-\frac{f(x)}{F(t_2)-F(t_1)}\right)^{2} dx \\
=&(t_2-t_1-1)\left[\log(t_2-t_1-1) - \log \int_{t_1}^{t_2} \left(1+\frac{f^2(x)}{(F(t_2)-F(t_1))^2} - \frac{2f(x)}{F(t_2)-F(t_1)}\right) dx\right] \\
=&(t_2-t_1-1)\left[\log(t_2-t_1-1) - \log\left((t_2-t_1) - 2IJ(t_2-t_1) - 2\right)\right]
\end{aligned} \tag{26}$$

□

7. Application

A letter-catching experiment is used to investigate the connection between generalized extropy and Information theory. Hypothesize that each of the four buckets contains eight letters. "AAAAAAAA" are letters in the 1st bucket, "AAAABBCD" are the letters in 2nd bucket, "AAABBBCD" are the letters in 3rd bucket, and "AABBCCDD" are the letters in 4th bucket. Easily, someone can observe entropy in 1st bucket have less uncertainty. But the difference between the 2nd, 3rd, and 4th is not easy just by observation. Renyi extropy, Varma entropy, and generalized extropy are calculated for each bucket, probability distributions corresponding

to buckets are $P_1 = [1,0,0,0]$ $P_2 = [1/2,1/4,1/8,1/8]$ $P_3 = [3/8,3/8,1/8,1/8]$ and $P_4 = [1/4,1/4,1/4,1/4]$ respectively.
Calculated Renyi extropy for the parameter $\alpha = 2$ is, $J_\alpha(P_1) = 0,\ J_\alpha(P_2) = 0.7406, \dot{J}_\alpha(P_3) = 0.7808,\ and\, J_\alpha(P_4) = 0.8630$.
Calculated Varma entropy and generalized extropy for the parameters $\alpha = 2, \beta = 2.5$, are
$H_\alpha^\beta(P_1) = 0,\ H_\alpha^\beta(P_2) = -2.0212,\ H_\alpha^\beta(P_3) = -2.3613,\ H_\alpha^\beta(P_4) = -3.0103$.

$$J_\alpha^\beta(P_1) = \frac{-12\log 3 + 3\log 3}{0.5} = -8.58812$$

$$J_\alpha^\beta(P_2) = \frac{-12\log 3 + 3\log[(1/2)^{3.5} + (3/4)^{3.5} + (7/8)^{3.5} + (7/8)^{3.5}]}{0.5} = -9.92588,$$

$$J_\alpha^\beta(P_3) = \frac{-12\log 3 + 3\log[(5/8)^{3.5} + (5/8)^{3.5} + (7/8)^{3.5} + (7/8)^{3.5}]}{0.5} = -10.16291,$$

and

$$J_\alpha^\beta(P_4) = \frac{-12\log 3 + 3\log[(3/4)^{3.5} + (3/4)^{3.5} + (3/4)^{3.5} + (3/4)^{3.5}]}{0.5} = -10.46226.$$

Calculated generalized extropy for the parameter $\alpha = 0.8,\ \beta = 1.1$ are

$$J_\alpha^\beta(P_1) = \frac{-3.3\log 3 + 3\log(3)}{0.3} = -0.477121,$$

$$J_\alpha^\beta(P_2) = \frac{-3.3\log 3 + 3\log[(1/2)^{0.9} + (3/4)^{0.9} + (7/8)^{0.9} + (7/8)^{0.9}]}{0.3} = -0.36099,$$

$$J_\alpha^\beta(P_3) = \frac{-3.3\log 3 + 3\log[(5/8)^{0.9} + (5/8)^{0.9} + (7/8)^{0.9} + (7/8)^{0.9}]}{0.5} = -0.35764,$$

and

$$J_\alpha^\beta(P_4) = \frac{-12\log 3 + 3\log[(3/4)^{0.9} + (3/4)^{0.9} + (3/4)^{0.9} + (3/4)^{0.9}]}{0.5} = -0.352182.$$

Some observations and calculated results are as follows:

- For the parameter $\alpha = 2$, Renyi extropy is observed as $J_\alpha(P_1) <\ J_\alpha(P_2) < \dot{J}_\alpha(P_3) <\ J_\alpha(P_4)$.
- For the parameters $\alpha = 2,\ and\, \beta = 2.5$ such that $\alpha + \beta > 2$ Varma entropy and generalized extropy are observed as follow,

  $$H_\alpha^\beta(P_1) > H_\alpha^\beta(P_2) > H_\alpha^\beta(P_3) > H_\alpha^\beta(P_4),$$

  and

  $$J_\alpha^\beta(P_1) > J_\alpha^\beta(P_2) > J_\alpha^\beta(P_3) > J_\alpha^\beta(P_4)$$

  .
- But for the parameter $\alpha = 0.8, and\, \beta = 1.1$ such that $\alpha + \beta < 2$, generalized extropy is observed as

  $$J_\alpha^\beta(P_1) < J_\alpha^\beta(P_2) < J_\alpha^\beta(P_3) < J_\alpha^\beta(P_4)$$

  .

8. Conclusion

The concept of extropy has garnered significant recognition within the realm of information theory. Recently, the generalized extropy measure has excellent performance in uncertainty measurement. In this article, the proposed generalized extropy of order α and type β has been studied. At the same time, interval generalized extropy are studied. A few numerical examples, theorems, and proofs are given in addition. A brief overview of some of this paper's key results has been given below.

- Generalised extropy degenerates into Renyi extropy if the parameter β equal to 1.
- For $|X| = 2$, the generalized extropy and the Varma entropy exhibit equality.
- For $n = 2$, and $\alpha + \beta > (<) 2$, $J_{\alpha}^{\beta}(P)$ and $H_{\alpha}^{\beta}(P)$ will be negative and positive respectively.
- For $n \geq 2$, and $\alpha + \beta = 2$, $H_{\alpha}^{\beta}(P)$ is always zero. It is not for $J_{\alpha}^{\beta}(P)$.
- The value of uniformly distributed generalized extropy measure of order α and type β for discrete random variable decreases as cardinality increases.
- Difference between Varma entropy and generalized extropy of order α and type β increases as cardinality increases.

When assessing distributions without a trackable distribution, the generalised extropy measure (3.1) has limitations. The quantile function can be used to define a probability distribution. In certain situations, quantile-based generalized extropy measurements may prove beneficial for further research.

Acknowledgments

The authors are thankful to an anonymous referee for careful reading and useful comments that led to an improvement in the presentation of this article.

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