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On Local Antimagic $b-$Coloring of Graphs: New Notion

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Abstract Let $G = (V, E)$ be a simple, connected, and undirected graph. Given that a map $f : E(G) \longrightarrow \{1, 2, 3, \dots, |E(G)|\}$. We define a vertex weight of $v \in V$ as $w(v) = \Sigma_{e \in E(v)} f(e)$ where $E(v)$ is the set of edges incident to v. The bijection f is said to be a local antimagic labeling if, for any two adjacent vertices, their vertex weights must be distinct. Additionally, a b-coloring of a graph is a proper k-coloring of the vertices of G where, within each color class, there exists a vertex that has neighbors in all the other $k - 1$ color classes. Suppose we assign color on each vertex by the vertex weight $w(v)$ such that it induces a graph coloring satisfying the $b-$coloring property. In that case, this concept falls into a local antimagic $b-$coloring of the graph. The local antimagic b-chromatic number, denoted as $\varphi_{la}(G)$, represents the highest number of colors used in any coloring produced by a local antimagic b-coloring of G. In this study, we initiate the study of the $b-$chromatic number of G and the exact values of $\varphi_{la}(G)$ of certain classes of graph families.

Keywords Local antimagic $b-$coloring; local antimagic $b-$chromatic number; specific families of graphs

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1. Introduction

In graph theory, a graph is a mathematical structure consisting of two main components, there are a set of vertices and a set of edges. All graphs discussed in this paper are nontrivial, finite, simple, undirected, and connected. For a definition of the graph and terms used in this paper, see [1, 2]. Graph theory has many applications in various fields. Here are some examples of the main applications in graph theory, namely: social networks, computer networks, road and route mapping, computer graphics, social sciences, transportation systems, molecular biology, product recommendations, financial analysis, circuit network mapping, face recognition and image processing, geographic information systems, scheduling, delivery design, supply chain management, cryptography, machine learning, transit-oriented development for smart city, and others [3]. In this study, points in a network are called vertices, and the connections between points are called edges. Moreover, when we assign colors on the vertices or edges such that no two adjacent vertices or incidence edges have the same colors, this problem falls into the concept of graph coloring, which has a wide range of applications especially in scheduling problems. There are three types of graph coloring, namely vertex coloring, edge coloring, and face coloring; see Kristiana *et al.* [4] for a detailed definition. According to Bhavanari *et al.* [5], face coloring includes coloring the face components of a graph such that no two adjacent faces have the same color. The chromatic number of a graph G, denoted as $\chi(G)$, is the fewest colors required to create a properly colored graph G.

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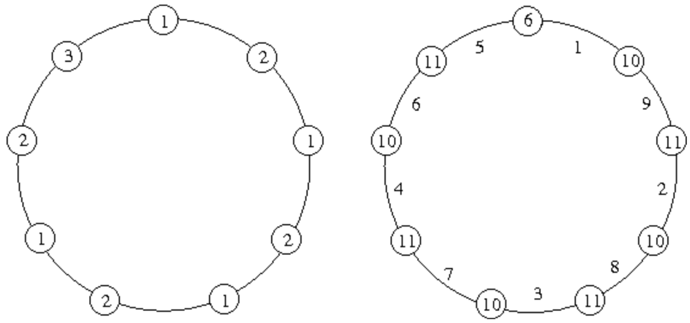
Furthermore, there are many natural extensions of proper graph coloring, one of them is a $b-$coloring of the graph. A b-coloring of a graph refers to a proper k-coloring of G where each color class includes a vertex that has neighbors in all the other $k-1$ color classes. The b-chromatic number of a graph G, denoted as $\varphi(G)$, is the maximum k for which G can be b-colored with k colors. It is easy to understand that $\chi(G) \leq \varphi(G)$. Many researchers have studied $b-$coloring, for instance Jakovac and Klavzar [6] determined the $b-$chromatic number of cubic graphs. Irving and Manlove [7] found the $b-$chromatic number of trees. Javadi and Omoomi [8, 9] determined the $b-$chromatic number of Kneser graphs and the Cartesian product of paths and cycles with complete graphs and the cartesian product of two complete graphs. Kok and Sudev [10] determined the $b-$chromatic number of linear Jaco graph, Ornated graph, Rasta graph, Chithra family graphs, and set graph. Diego and Gella [11] found the $b-$chromatic number of the center graph, middle graph, and total graph of the bistar graph. The other results regarding $b-$coloring can be seen in [12]-[30]. An example of $b-$coloring on a graph can be seen in Figure 1 (a).



(a) (b)

Figure 1. (a) b-coloring on C_9, (b) the local antimagic coloring on C_9.

Moreover, we have also studied the graph labeling of the graph, see [31]-[33]. Let $G(V, E)$ be a simple graph, and let f is a map $f : E(G) \to \{1, 2, \ldots, |E(G)|\}$. Arumugam *et al.* defined local antimagic labeling [34]. Consider a graph $G = (V(G), E(G))$ with no isolated vertices of order n and size m. A local antimagic labeling is a bijective function from the edge set to the natural numbers up to the number of edges, ensuring that every pair of adjacent vertices have different weights, $w(u) \neq w(v)$, where $w(u) = \sum_{e \in E(u)} f(e)$. A graph G is considered local antimagic if it has such a local antimagic labeling [40]-[43]. The author defines a local antimagic coloring as the coloring on vertices of a graph G which induced by local antimagic labeling of G. The local antimagic chromatic number, denoted by $\chi_{la}(G)$, is the smallest number of colors taken over all colorings of G induced by local antimagic labeling of G [34]-[39]. An example of local antimagic coloring on a graph can be seen in Figure 1 (b).

The study in this paper is motivated by the two interesting concepts mentioned above, namely local antimagic coloring and b-coloring. Thus, this research aims to explore a new concept, namely local antimagic b-coloring. The idea behind local antimagic b-coloring is to assign colors to each vertex based on the vertex weight $w(v)$ in such a way that it satisfies the b-coloring property. Specifically, local antimagic labeling is performed by assigning labels to each edge of the graph with natural numbers from 1 to the number of edges in such a way that every two adjacent vertices have different weights, where the weight of a vertex $w(u)$ is calculated as the sum of the labels of the edges incident to that vertex. After determining the local antimagic labeling, local antimagic b-coloring is achieved by coloring the vertices based on their weights and ensuring that each color class has a vertex adjacent to vertices in all other $k-1$ color classes. The local antimagic b-chromatic number, denoted by $\varphi_{la}(G)$, is the maximum number of colors used in the local antimagic b-coloring of the graph G.

The results of this paper include on determining the boundaries of the local antimagic $b-$chromatic number and determine the exact value of $\varphi_{la}(G)$ of G whenever G are cycle, wheel, friendship, and fan graphs. A cycle graph, denoted by C_n, is a type of graph in graph theory that consists of a single closed loop or cycle. A wheel graph, denoted by W_n, is defined as the join graph $K_1 + C_n$, where K_1 is the complete graph with one vertex and C_n is the cycle graph with n vertices. The friendship graph, denoted by $\mathcal{F}_n$, is a graph constructed by joining n copies of the cycle graph C_3 with a common vertex. A fan graph, denoted by F_n, is defined as the join graph $K_1 + P_n$, where K_1 is the complete graph of one vertex and P_n is the path graph of n vertices.

2. Bounds on Local Antimagic $b-$Chromatic Number of Graphs

In this section, we will show the bounds on local antimagic $b-$chromatic number of graphs. We start with the following lemma.

Lemma 2.1
Let G be a graph and Δ is maximum degree. The local antimagic $b-$chromatic number of G is $\chi_{la}(G) \leq \varphi_{la}(G) \leq \varphi(G) \leq \Delta + 1$.

Proof: Given that the $b-$chromatic number represents the maximum coloring of G, we can write, $\varphi_{la}(G) \leq \varphi(G)$. However, considering the $b-$coloring condition, which stipulates that $\varphi(G) \leq \Delta + 1$, we arrive at the inequality:

$$\varphi_{la}(G) \leq \varphi(G) \leq \Delta + 1 \tag{2.1}$$

Applying Brook's theorem, which states that for any undirected connected graph G, $\chi(G) \leq \Delta$, and $\chi(G) \leq \Delta + 1$ for odd cycles and complete graphs, take, $\chi(G) \leq \Delta + 1$. Also by (1), we have, $\varphi_{la}(G) \leq \Delta + 1$, leading to $\chi(G) \leq \varphi_{la}(G) \leq \varphi(G)$, since we know that, $\chi(G) \leq \varphi(G)$ and so, $\chi_{la}(G) \leq \varphi_{la}(G)$. Furthermore, from [34], it's evident that $\chi(G) \leq \chi_{la}(G)$, establishing the following inequalities:

$$\chi(G) \leq \chi_{la}(G) \leq \varphi_{la}(G) \leq \varphi(G) \tag{2.2}$$

Equations (2.1) and (2.2) delineate the bounds of the local antimagic $b-$chromatic number to be $\chi_{la}(G) \leq \varphi_{la}(G) \leq \varphi(G) \leq \Delta + 1$. □

Since the lower bound of the local antimagic $b-$coloring is determined based on the local antimagic chromatic number of the graph, then determining the local antimagic $b-$chromatic number requires the local antimagic chromatic number of the graph. The local antimagic $b-$chromatic number of the cycle, wheel, and fan graphs is determined in this article. As a result, we present the local antimagic chromatic number of these graphs, which is as follows:

1. $\chi_{la}(C_n) = 3$ for $n \geq 3$ [34].
2. $\chi_{la}(W_n) = \begin{cases} 3 & \text{, for } n \equiv 2 (mod\ 4) \\ 4 & \text{, for } n \equiv 1, 3 (mod\ 4) \end{cases}$
3. $3 \leq \chi_{la}(W_n) \leq 5$ for $n \equiv 0 (mod\ 4)$.
4. $\chi_{la}(\mathcal{F}_n) = 3$ for $n \geq 3$.
5. $\chi_{la}(F_n) = 3$ for $n \geq 3$.

3. The Local Antimagic $b-$Chromatic Number of Graphs

Theorem 3.1
If G is a graph with pendant vertex, then G does not have local antimagic $b-$coloring.

Proof: Let G be a graph with pendant vertex. The vertex weight on pendant vertex depends on its edge label. It is different from the other vertex weights. Based on the definition of local antimagic $b-$coloring, the vertex of each color class must be adjacent with at least one vertex to other color classes. The color class of pendant vertex is only adjacent with one class and it is impossible adjacent with other class. Hence, the graph G does not have local antimagic $b-$coloring. □

Corollary 3.1
All tree does not have local antimagic $b-$coloring.

Since tree is a graph which contains pendant vertex, it is clear that it does not have local antimagic $b-$coloring.

Theorem 3.2
Let C_n be a cycle graph, the local antimagic $b-$chromatic number of C_n is 3 for odd $n \geq 3$, where n is the number of vertices of cycle graph.

Proof: Let C_n be a cycle graph with vertex set $V = \{u_i : 1 \leq i \leq n\}$ and edge set $E = \{u_i u_{i+1} : 1 \leq i \leq n-1\} \cup \{u_1 u_n\}$. Since $\chi_{la}(C_n) = 3$, such that $\varphi_{la}(C_n) \geq 3$. Abiding by the condition of $b-$coloring, we can start coloring the graph from $\Delta + 1 = 3$ colors. Furthermore to show $\varphi_{la}(C_n) \leq 3$, we construct the edge labels on C_n as follows. Let $e_i = u_i u_{i+1} : 1 \leq i \leq n-1$ and $e_n = u_n u_1$.

$$g(e_i) = \frac{i+1}{2} : 1 \leq i \leq n \text{ for } i \equiv 1(mod\ 2)$$

$$g(e_i) = n + 1 - \frac{i}{2} : 2 \leq i \leq n-1 \text{ for } i \equiv 0(mod\ 2)$$

Using the edge labeling above, we can determine the vertex weight of each vertex on C_n. We can determine the number of different vertex weights. Therefore, we can obtain the number of different vertex weights as follows.

$$w_1 = \frac{n+3}{2}; w_2 = n+1; w_3 = n+2$$

In the local antimagic $b-$coloring concept, the color on each vertex is induced from the vertex weight. Table 1 shows that the colors on each vertex of C_n graph.

Table 1. The Vertex Color on C_n

Vertex u_i	Color
$i = 1$	w_1
$i \equiv 0(mod\ 2)$	w_2
$i \equiv 1(mod\ 2); i \neq 1$	w_3

The vertex coloring on the graph C_n in Table 1 shows that there is a vertex in each color class that is adjacent to at least one vertex in all other color classes. Since the number of different vertex weight on C_n is 3 and satisfies the $b-$coloring property, it concludes that $\varphi_{la}(C_n) = 3$. □

According to the labeling illustration in Figure 2, the local antimagic $b-$chromatic number in the graph C_{15} is three, namely color classes $9, 16$, and 17. The 9 color class is adjacent to the 16 color class and the 17 color class; the 16 color class is adjacent to the 9 color class and the 17 color class; and the 17 color class is adjacent to the 9 color class and the 16 color class. This color class's neighborhood satisfies the definition of the local antimagic $b-$coloring concept.

Theorem 3.3
The cycle graph of even order does not have local antimagic $b-$coloring.

Proof: When we consider the concept of local antimagic $b-$coloring for an even-cycle. Suppose that there exist three distinct weights, say w_1, w_2 and w_3. However only one color class shares adjacency with all other color classes, but the rest does not adjacent with others. It implies that even-cycles does not admit the local antimagic $b-$coloring. □

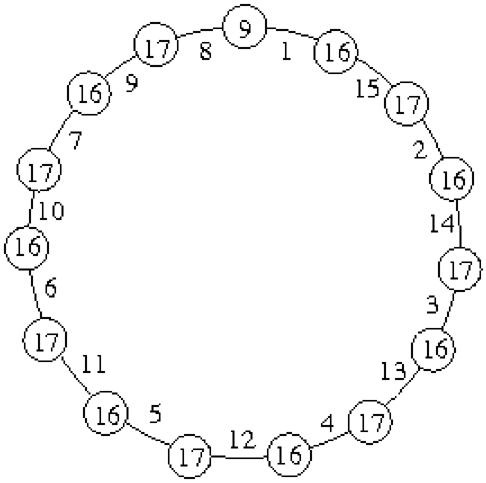


Figure 2. The Local antimagic $b-$coloring on C_{15}.

Theorem 3.4
Let W_n be a wheel graph, the local antimagic $b-$chromatic number of W_n is 4 for odd $n \geq 3$, where n is the number of spokes of wheel graph.

Proof: Let W_n be a wheel graph with vertex set $V = \{a, v_i : 1 \leq i \leq n\}$ and edge set $E = \{v_iv_{i+1} : 1 \leq i \leq n-1\} \cup \{v_1v_n\} \cup \{av_i : 1 \leq i \leq n\}$. Since $\chi_{la}(W_n) = 4$, such that $\varphi_{la}(W_n) \geq 4$. Abiding by the condition of $b-$coloring, we can start coloring the graph from $\Delta + 1 = n + 1$ colors and iterating it until every color class has a dominating $b-$vertex. Furthermore to show $\varphi_{la}(W_n) \leq 4$, we construct the edge labels on W_n as follows. Let $e_i = v_iv_{i+1} : 1 \leq i \leq n-1$ and $e_n = v_nv_1$.

$$g(e_i) = \frac{i+1}{2} : 1 \leq i \leq n \text{ for } i \equiv 1(mod\,2)$$

$$g(e_i) = \frac{n+1+i}{2} : 2 \leq i \leq n-1 \text{ for } i \equiv 0(mod\,2)$$

$$g(av_1) = 2n$$

$$g(av_i) = 2n+2-i : 3 \leq i \leq n \text{ for } i \equiv 1(mod\,2)$$

$$g(av_i) = 2n-i : 2 \leq i \leq n-1 \text{ for } i \equiv 0(mod\,2)$$

Using the edge labeling above, we can determine the vertex weight of each vertex on W_n. The number of various vertex weights may be determined. Therefore, we can obtain the number of different vertex weights as follows.

$$w_1 = \frac{5n+3}{2}; w_2 = \frac{5n+1}{2}; w_3 = \frac{5n+5}{2}; w_4 = \frac{3n^2+n}{2}$$

In the local antimagic $b-$coloring concept, the color on each vertex is induced from the vertex weight. Table 2 shows that the colors on each vertex of W_n graph.

From the vertex coloring on the graph W_n in Table 2, it is evident that within each color class, there is a vertex that is connected to at least one vertex in every other color class. Given that W_n has 3 distinct vertex weights and meets the criteria for b-coloring, it follows that $\varphi_{la}(W_n) = 4$. □

Table 2. The Vertex Color on W_n

Vertex	Color
a	w_4
v_1	w_1
$v_i, i \equiv 0(mod\ 2)$	w_2
$v_i, i \equiv 1(mod\ 2); i \neq 1$	w_3

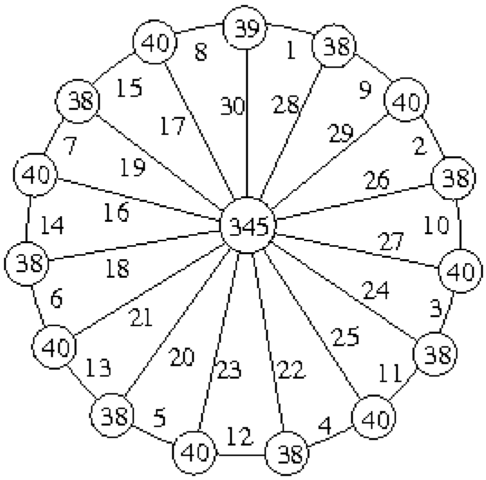


Figure 3. The Local antimagic $b-$coloring on W_{15}.

According to the labeling illustration in Figure 3, the local antimagic $b-$chromatic number in the graph W_{15} is 4, namely color classes $38, 39, 40$, and 345. The 38 color class is adjacent to the 39 color class, the 40 color class, and the 345 color class; the 39 color class is adjacent to the 38 color class, the 40 color class, and the 345 color class; the 40 color class is adjacent to the 38 color class, the 39 color class, and the 345 color class, and the 345 color class is adjacent to the 38 color class, the 39 color class, and the 40 color class. This color class's neighborhood satisfies the definition of the local antimagic $b-$coloring concept.

Theorem 3.5
The wheel graph of even order does not have local antimagic $b-$coloring.

Proof: The local antimagic $b-$coloring of even order of wheel graph acquires four distinct weights, say, w_1, w_2, w_3 and w_4. However, the fact shows that every color class has been failed to establish adjacency with any one of the other color class. This shows that local antimagic $b-$coloring is admitted for wheel graphs of even order. □

Theorem 3.6
If $\mathcal{F}_n$ is a friendship graph, than the local antimagic $b-$chromatic number of $\mathcal{F}_n$ is 3 for $n \geq 2$, where n is the number of C_3 in the friendship graph.

Proof: Let $\mathcal{F}_n$ be a friendship graph with vertex set $V = \{\alpha, x_i, y_i : 1 \leq i \leq n\}$ and edge set $E = \{\alpha x_i, \alpha y_i, x_i y_i : 1 \leq i \leq n\}$. Since $\chi_{la}(\mathcal{F}_n) = 3$, such that $\varphi_{la}(\mathcal{F}_n) \geq 3$. Abiding by the condition of $b-$coloring, we can start coloring the graph from $\Delta + 1 = n + 1$ colors and iterating it until every color class has a dominating $b-$vertex. Furthermore to show $\varphi_{la}(\mathcal{F}_n) \leq 3$, we construct the edge labels on $\mathcal{F}_n$ as follows.

$$g(\alpha x_i) = i : 1 \leq i \leq n$$

$$g(\alpha y_i) = n + i : 1 \leq i \leq n$$

$$g(x_i y_i) = 3n + 1 - i : 1 \leq i \leq n$$

From the edge labeling above, we can determine the vertex weight of each vertex on $\mathcal{F}_n$. We can determine the number of different vertex weights. Therefore, we can obtain the number of different vertex weights as follows.

$$w_1 = 3n + 1; w_2 = 4n + 1; w_3 = n(2n + 1) = 2n^2 + n$$

In the local antimagic $b-$coloring concept, the color on each vertex is induced from the vertex weight. Table 3 shows that the colors on each vertex of $\mathcal{F}_n$ graph.

Table 3. The Vertex Color on $\mathcal{F}_n$

Vertex	Color
α	w_3
x_i	w_1
y_i	w_2

From the vertex coloring on the graph $\mathcal{F}_n$ in Table 3, it is evident that within each color class, there is a vertex adjacent to at least one vertex in every other color class. Since $\mathcal{F}_n$ has 3 distinct vertex weights and fulfills the b-coloring criteria, it follows that $\varphi_{la}(\mathcal{F}_n) = 3$. □

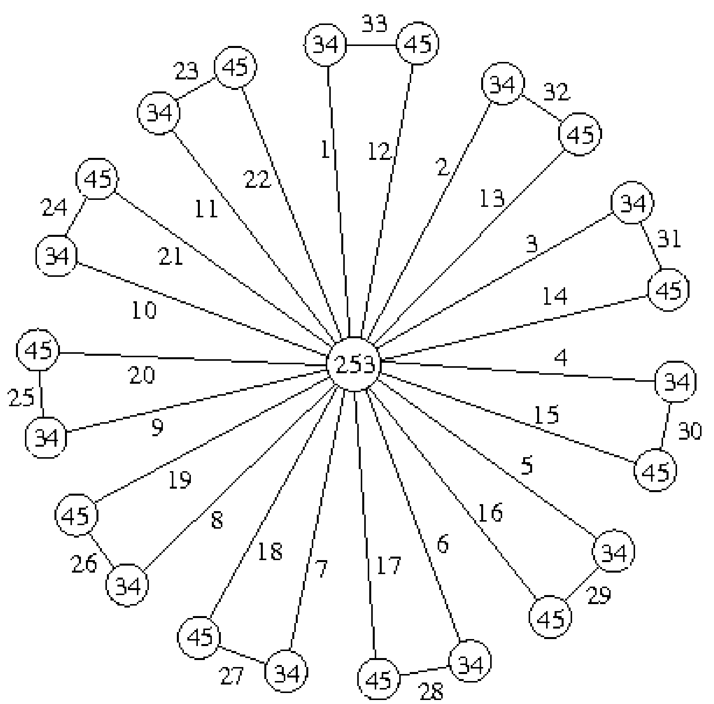


Figure 4. The Local antimagic $b-$coloring on $\mathcal{F}_{11}$.

According to the labeling illustration in Figure 4, the local antimagic $b-$chromatic number in the graph $\mathcal{F}_{11}$ is three, namely color classes $34, 45$, and 253. The 34 color class is adjacent to the 45 color class and the 253 color class; the 45 color class is adjacent to the 34 color class and the 253 color class; and the 253 color class is adjacent to the 34 color class and the 45 color class. This color class's neighborhood satisfies the definition of the local antimagic $b-$coloring concept.

Theorem 3.7
Let F_n be a fan graph, the local antimagic $b-$chromatic number of F_n is 4 for $n \geq 3$ and $n \equiv 1, 3(mod\, 6)$, where n is the number of spokes of F_n.

Proof: Let F_n be a fan graph with vertex set $V = \{\alpha, u_i : 1 \leq i \leq n\}$ and edge set $E = \{\alpha u_i : 1 \leq i \leq n\} \cup \{u_i u_{i+1} : 1 \leq i \leq n-1\}$. Since $\chi_{la}(F_n) = 4$, such that $\varphi_{la}(F_n) \geq 4$. Abiding by the condition of $b-$coloring, we can start coloring the graph from $\Delta + 1 = 2n + 1$ colors and iterating it until every color class has a dominating $b-$vertex. Furthermore to show $\varphi_{la}(F_n) \leq 4$, we construct the edge labels on F_n as follows.

$$g(u_i u_{i+1}) = \frac{i}{2} : 2 \leq i \leq n, i \equiv 0(mod\, 2)$$

$$g(u_i u_{i+1}) = \frac{n+i}{2} : 1 \leq i \leq n, i \equiv 1(mod\, 2)$$

Case 1. For $n \equiv 1(mod\, 6)$

$$g(\alpha u_i) = 2n - i - 3 : 1 \leq i \leq n-1, i \equiv 1(mod\, 3)$$

$$g(\alpha u_i) = 2n - i - 1 : 2 \leq i \leq n-1, i \equiv 2(mod\, 3)$$

$$g(\alpha u_i) = 2n - i + 1 : 3 \leq i \leq n-1, i \equiv 0(mod\, 3)$$

$$g(\alpha u_n) = 2n - 1$$

From the edge labeling above, we can determine the vertex weight of each vertex on F_n. We can determine the number of different vertex weights. Therefore, we can obtain the number of different vertex weights as follows.

$$w_1 = \frac{5n-7}{2}; w_2 = \frac{5n-3}{2}; w_3 = \frac{5n+1}{2}; w_4 = \frac{3n^2-n}{2}$$

In the local antimagic $b-$coloring concept, the color on each vertex is induced from the vertex weight. Table 4 shows the colors on each vertex of the F_n graph.

Table 4. The Vertex Color on F_n

Vertex	Color
α	w_4
$u_i : i \equiv 1(mod\, 3)$	w_1
$u_i : i \equiv 2(mod\, 3)$	w_2
$u_i : i \equiv 0(mod\, 3)$	w_3

Case 2. For $n \equiv 3(mod\, 6)$

$$g(\alpha u_i) = 2n - i - 3 : 3 \leq i \leq n-1, i \equiv 0(mod\, 3)$$

$$g(\alpha u_i) = 2n - i - 1 : 4 \leq i \leq n-1, i \equiv 1(mod\, 3)$$

$$g(\alpha u_i) = 2n - i + 1 : 5 \leq i \leq n-1, i \equiv 2(mod\, 3)$$

$$g(\alpha u_1) = 2n - 2$$

$$g(\alpha u_2) = 2n - 1$$

$$g(\alpha u_n) = 2n - 3$$

Using the edge labeling shown above, we can determine the vertex weight of each vertex on F_n. We can determine the number of different vertex weights. Therefore, we can obtain the number of different vertex weights as follows.

$$w_1 = \frac{5n-3}{2}; w_2 = \frac{5n+1}{2}; w_3 = \frac{5n-7}{2}; w_4 = \frac{3n^2-n}{2}$$

In the local antimagic $b-$coloring concept, the color on each vertex is induced from the vertex weight. Table 5 shows the colors on each vertex of the F_n graph.

Table 5. The Vertex Color on F_n

Vertex	Color
α	w_4
$u_i : i \equiv 1(mod\, 3)$	w_1
$u_i : i \equiv 2(mod\, 3)$	w_2
$u_i : i \equiv 0(mod\, 3)$	w_3

From the vertex coloring on the graph F_n in Table 4 and Table 5, it is apparent that within each color class, there is a vertex that is adjacent to at least one vertex in every other color class. Given that F_n has 4 distinct vertex weights and meets the b-coloring criteria, it follows that $\varphi_{la}(F_n) = 4$. □

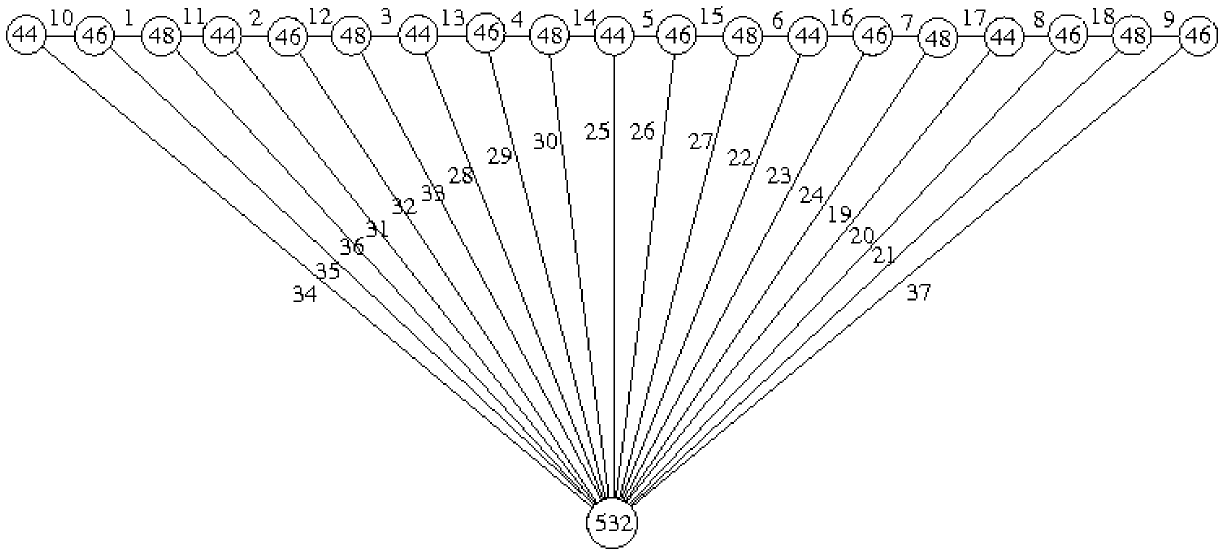


Figure 5. The Local antimagic $b-$coloring on F_{19}.

According to the labeling illustration in Figure 5, the local antimagic $b-$chromatic number in the graph F_{19} is 4, namely color classes $44, 46, 48$, and 532. The 44 color class is adjacent to the 46 color class, the 48 color class, and the 532 color class; the 46 color class is adjacent to the 44 color class, the 48 color class, and the 532 color class; the 48 color class is adjacent to the 44 color class, the 46 color class, and the 532 color class, and the 532 color class is adjacent to the 44 color class, the 46 color class, and the 48 color class. This color class's neighborhood satisfies the definition of the local antimagic $b-$coloring concept.

Theorem 3.8
Let $Amal(C_n, v, m)$ be an amalgamation of the cycle graph, for even $n \geq 2$ and $m \geq 2$ the local antimagic $b-$chromatic number of $Amal(C_n, v, m)$ is 3.

Proof: Let $Amal(C_n, v, m)$ be an amalgamation of the cycle graph with vertex set $V = \{\alpha, u_{i,j} : 1 \leq i \leq n, 1 \leq j \leq m\}$ and edge set $E = \{\alpha u_{1,j}, \alpha u_{n,j}, u_{i,j}u_{i+1,j} : 1 \leq i \leq n-2, 1 \leq j \leq m\}$. Since $\chi_{la}(Amal(C_n, v, m)) = \chi_{la}(C_n) = 3$, such that $\varphi_{la}(Amal(C_n, v, m)) \geq 3$.

Furthermore, to show $\varphi_{la}(Amal(C_n, v, m)) \leq 3$, we construct the edge labels on $Amal(C_n, v, m)$ as follows.

$$f(\alpha u_{1,j}) = j : \text{ if } 1 \leq j \leq m$$

$$f(u_{i,j}u_{i+1,j}) = m\frac{i}{2} + j : \text{if } i \equiv 0(mod\ 2),\ 2 \leq i \leq n-2, 1 \leq j \leq m$$

$$f(\alpha u_{n,j}) = \frac{mn}{2} + j : \text{ if } 1 \leq j \leq m$$

$$f(u_{i,j}u_{i+1,j}) = mn - j + \frac{i+1}{2} : \text{if } i \equiv 1(mod\ 2)\ 1 \leq i \leq n-1, 1 \leq j \leq m$$

Using the edge labeling above, we can determine the vertex weight of each vertex on $Amal(C_n, v, m)$. We can determine the number of different vertex weights. Therefore, we can obtain the number of different vertex weights as follows.

$$w(\alpha) = m^2 + m + \frac{m^2 n}{2}$$

$$f(w_{i,j}) = \begin{cases} mn + m + 1, & \text{if } i \equiv 1(\text{mod}2) \\ mn + 2m + 1, & \text{if } i \equiv 0(\text{mod}2) \end{cases}$$

In the local antimagic $b-$coloring concept, the color on each vertex is induced from the vertex weight. From the vertex weights, We know that there are three colors such that it means $\varphi_{la}(Amal(C_n, v, m)) \leq 3$. Since we have $\varphi_{la}(Amal(C_n, v, m)) \geq 3$ and $\varphi_{la}(Amal(C_n, v, m)) \leq 3$, it concludes that $\varphi_{la}(Amal(C_n, v, m)) = 3$. □

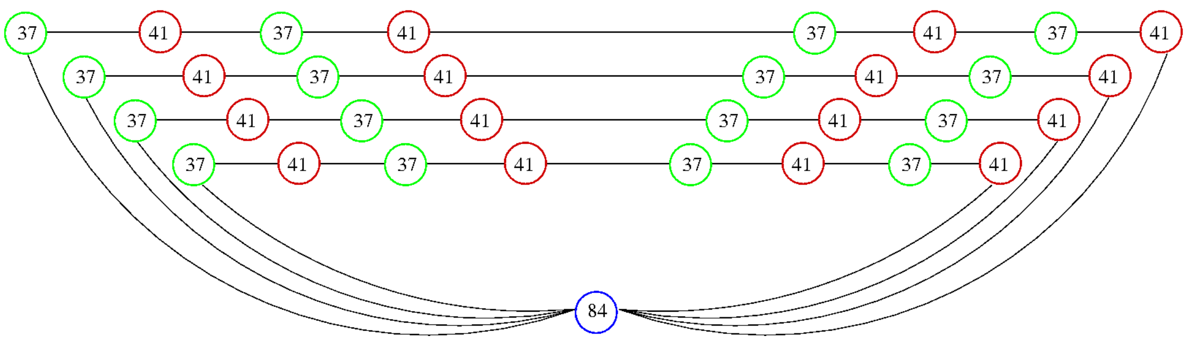


Figure 6. Local antimagic $b-$coloring on graph $Amal(C_9, 1, 4)$.

4. Discussion

This study introduces a new concept called local antimagic b-coloring, inspired by local antimagic coloring and b-coloring. In local antimagic b-coloring, each vertex is assigned a color based on its vertex weight $w(v)$ to satisfy the b-coloring property. Vertex weights are determined by labeling each edge so that adjacent vertices have different weights. Then, vertices are colored based on these weights, ensuring each color class has a vertex adjacent to vertices in all other $k-1$ color classes. The local antimagic b-chromatic number, denoted by $\varphi_{la}(G)$, represents the maximum number of colors used.

Local antimagic b-coloring is closely related to various graph coloring and labeling techniques extensively studied in graph theory. Total coloring, strong coloring, harmonic coloring, and strong or total labeling are techniques that provide context to the proposed concept.

Total coloring, introduced by Behzad and Vizing, extends vertex coloring to both vertices and edges of a graph, ensuring adjacent vertices and edges receive different colors. Strong coloring, introduced by Faudree et al., focuses on coloring edges so that every two edges incident to the same vertex have different colors. Harmonic coloring, introduced by Chartrand et al., assigns colors to vertices such that adjacent vertices have distinct color sums under certain conditions. These coloring techniques explore different aspects of graph structures and have been applied in various theoretical and practical scenarios.

Additionally, strong and total labeling assign labels to vertices or edges of a graph with specific properties. Strong labeling ensures that adjacent vertices have distinct label sums, while total labeling extends this concept to include edges as well. Understanding these related graph coloring and labeling techniques provides valuable insights into the development and applications of local antimagic b-coloring. By building upon and relating to these existing techniques, local antimagic b-coloring contributes to the broader landscape of graph theory and combinatorial optimization.

While still emerging, this concept may find applications in various fields. For instance, in communication networks, graph coloring is used for efficient channel allocation. Local antimagic b-coloring could aid in resource allocation considering specific constraints. Additionally, in scheduling tasks with dependencies, this concept could be applied to schedule tasks more efficiently. Furthermore, the application of local antimagic b-coloring can be utilized in the development of precision agriculture by leveraging the property of b-coloring, which involves using maximum coloring with the condition that each color class is adjacent to other color classes. By considering these aspects, the study not only contributes to the development of graph theory but also opens doors for practical applications in various fields such as network optimization and scheduling algorithms.

5. Concluding Remarks

We have determined several graphs that admit local antimagic b-coloring and have shown the exact values of their local antimagic b-chromatic numbers. The results are as follows: (i) $\varphi_{la}(C_n) = 3$ for $n \geq 3$, (ii) $\varphi_{la}(W_n) = 4$ for $n \geq 3$, (iii) $\varphi_{la}(\mathcal{F}n) = 3$ for $n \geq 2$, and (iv) $\varphi la(F_n) = 4$ for $n \geq 3$. Since we are just beginning to study this problem and it is a new concept combining antimagic labeling and b-coloring, it opens up many wide-ranging problems for further research.

As future research directions, we propose the following open problems:

- Determine the exact values of the local antimagic b-chromatic number for specific classes of graphs.
- Characterize the existence of local antimagic b-coloring for any graph and obtain the best lower bounds.

By further exploring this concept, we hope to expand the understanding of graph coloring and open new avenues for research in graph theory.

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