A weighted full-Newton step primal-dual interior point algorithm for convex quadratic optimization

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Abstract In this paper, a new weighted short-step primal-dual interior point algorithm for convex quadratic optimization (CQO) problems is presented. The algorithm uses at each interior point iteration only full-Newton steps and the strategy of the central path to obtain an $\epsilon$-approximate solution of CQO. This algorithm yields the best currently well-known theoretical iteration bound, namely, $O(\sqrt{n} \log \frac{n}{\epsilon})$ which is as good as the bound for the linear optimization analogue.

Keywords Convex quadratic optimization; weighted interior point methods; short-step primal-dual algorithms; complexity of algorithms

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1. Introduction

Consider the quadratic optimization (QO) problem in standard format:

$$(P) \quad \min_x \left\{ c^T x + \frac{1}{2} x^T Q x : A x = b, \ x \geq 0 \right\}$$

and its dual problem

$$(D) \quad \max_{x, \ y, \ z} \left\{ b^T y - \frac{1}{2} x^T Q x : A^T y + z - Q x = c, \ z \geq 0 \right\},$$

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where $Q$ is a given $(n \times n)$ real symmetric matrix, $A$ is a given $(m \times n)$ real matrix with rank $(A) = m$, $c \in \mathbb{R}^n$, $b \in \mathbb{R}^m$, $x \in \mathbb{R}^n$, $z \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$. The QO problems have many important applications in optimization and mathematical programming problems.

There are a variety of solution approaches for CQO which have been studied intensively. Among them, the interior-point methods (IPMs) gained more attention than others methods. Feasible primal-dual path-following methods are the most attractive methods of IPMs [7, 9]. Their derived algorithms achieved important results such as polynomial complexity and numerical efficiency. These algorithms trace approximately the so-called central-path which is a curve that lies in the feasible region of the considered problem and they reach an optimal solution of it. However, in practice these methods don’t always find a strictly feasible centered point to starting their derived algorithms. So, it is worth analyzing other cases when the starting points are not centered. Thus leads to the concept of Target-Following IPMs introduced by Jansen et al.,[6] as a generalization of the classical path-following methods. These methods are based on the observation that with every algorithm which follows the central-path we associate a target sequence on the central-path. Weighted path-following methods can be viewed as a particular case of it. These methods were studied extensively by many authors [3, 4, 5, 7, 8] for Linear optimization (LO) and linear complementarity problem (LCP). Recently, Achache and Khebchache [1], introduced a new weighted method for monotone LCP where the complexity of the corresponding short-step algorithm is $O(\sqrt{n} \log \frac{\epsilon}{\delta})$. Motivated by their work, we propose a new weighted primal-dual path-following algorithm for solving CQO. The algorithm uses at each interior point iteration only weighted full-Newton steps and the strategy of the central path to get an $\epsilon$-approximate solution of CQO. We prove that the short-step algorithm has the following iteration bound $O(\sqrt{n} \log \frac{\epsilon}{\delta})$ which is as good as the bound for LO [3, 7, 8], CQO [1, 3] and LCP [2, 7], analogue. The algorithm has advantages that no line searches is needed and it can start with a suitable starting point not necessarily centered.

The rest of the paper is built as follows. In Section 2, the weighted-path and the search direction are presented. The generic weighted primal-dual path-following algorithm for CQO is also stated. In Section 3, the analysis of the algorithm and the iteration bound with short-step method are presented. Finally, a conclusion and future remarks follow in Section 4.

The notation used in this paper is as follows. $\mathbb{R}^n$ denotes the space of $n$-dimensional real vectors and $\mathbb{R}_{++}^n$ is the set of all positive vectors of $\mathbb{R}^n$. Given $x, z \in \mathbb{R}_{++}^n$, their Hadamard product is $xz = (x_1z_1, \ldots, x_nz_n)^T$. The expressions $\|u\| = \sqrt{u^Tu}$ and $\|u\|_\infty = \max_i |u_i|$ denote the Euclidean and the maximum norms for a vector $u$, respectively. Let $x, z \in \mathbb{R}_{++}^n$, $\sqrt{x} = (\sqrt{x_1}, \ldots, \sqrt{x_n})^T$, $x^{-1} = (x_1^{-1}, \ldots, x_n^{-1})^T$ and $\frac{x}{z} = (\frac{x_1}{z_1}, \ldots, \frac{x_n}{z_n})^T$. Let $g(x)$ and $f(x)$, be two positive real valued functions, then $g(x) = O(f(x)) \Leftrightarrow g(x) \leq \delta f(x)$.
\[ kf(x) \text{ for some positive constant } k. \] Finally, the vector of all ones and the identity matrix are denoted by \( e \) and \( I \), respectively.

2. The weighted-path and the search direction

Throughout the paper, we make the following assumptions for QO.

**Assumption 1.** Interior Point Condition (IPC). There exists a triplet of vectors \((x^0, y^0, z^0)\) such that:

\[
Ax^0 = b, \quad x^0 > 0, \quad A^Ty + z^0 - Qx^0 = c, \quad z^0 > 0.
\]

**Assumption 2.** Positive semidefiniteness. The matrix \( Q \) is positive semidefinite, i.e., for all \( v \in \mathbb{R}^n \), \( v^T Q v \geq 0 \).

Finding an approximate solution of \((P)\) and \((D)\) is equivalent to solving the following system of optimality conditions for \((P)\) and \((D)\):

\[
\begin{align*}
Ax &= b, \quad x \geq 0, \\
A^Ty + z - Qx &= c, \quad z \geq 0, \\
xz &= 0.
\end{align*}
\]

The basic idea behind weighted primal-dual interior-point algorithm is to replace the third equation (complementarity condition) in (1) by the parametrized equation \( xz = w \) with \( w \) is a positive vector in \( \mathbb{R}^n \). Thus, we consider the following perturbed system:

\[
\begin{align*}
Ax &= b, \quad x \geq 0, \\
A^Ty + z - Qx &= c, \quad z \geq 0, \\
xz &= w.
\end{align*}
\]

Under **Assumption 1** and **Assumption 2**, the system (2) has a unique solution denoted by \((x(w), y(w), z(w))\) for all \( w > 0 \) [2]. The set

\[
\{(x(w), y(w), z(w)) : w > 0\}
\]

is called the weighted-path of problems \((P)\) and \((D)\). If \( w \) goes to zero, then the limit of the weighted-path exists and since the limit point satisfies the complementarity condition, the limit yields an optimal solution for CQO. This limiting property of the weighted-path leads to the main idea of the iterative primal-dual methods for solving (2).

**Remark 2.1**

If \( w = \mu e \) with \( \mu > 0 \), then the weighted-path coincides with the classical central-path.
Now, we proceed to describe a weighted full-Newton step produced by the algorithm for a given \( w > 0 \). Applying Newton’s method for \( (2) \) for a given feasible point \((x, y, z)\) then the Newton direction \((\Delta x, \Delta y, \Delta z)\) at this point is the unique solution of the following linear system of equations:

\[
\begin{pmatrix}
A & 0 & 0 \\
-Q & A^T & I \\
Z & 0 & X
\end{pmatrix}
\begin{pmatrix}
\Delta x \\
\Delta y \\
\Delta z
\end{pmatrix} =
\begin{pmatrix}
0 \\
0 \\
w - Xz
\end{pmatrix},
\]

(3)

where \( X := \text{diag}(x) \), \( Z := \text{diag}(z) \).

Again under our assumptions and the fact that \( \text{rank}(A) = m \), the system (3) has a unique solution \((\Delta x, \Delta y, \Delta z)\). Hence, a new weighted full-Newton iteration is constructed according to:

\[
x_+ := x + \Delta x; \quad y_+ := y + \Delta y; \quad \text{and} \quad z_+ = z + \Delta z.
\]

(4)

To simplify the matters, we define the vectors:

\[
v := \sqrt{xz} \quad \text{and} \quad d := \sqrt{xz^{-1}}.
\]

The vector \(d\) uses to scale the vectors \(x\) and \(z\) to the same vector \(v\) as

\[
d^{-1}x = dz = v \tag{5}
\]

and as well as for the original directions to the scaling directions:

\[
d_x = d^{-1}\Delta x \quad \text{and} \quad d_z = d\Delta z.
\]

It follows that:

\[
x\Delta z + z\Delta x = v(d_x + d_z),
\]

(6)

and

\[
d_xd_z = \Delta x\Delta z = \Delta xQ\Delta x \geq 0,
\]

(7)

since \(Q\) is a semidefinite matrix.

Hence, by using (5), (6) and (7), the system (3) becomes:

\[
\begin{pmatrix}
\bar{A} & 0 & 0 \\
-Q & \bar{A}^T & I \\
I & 0 & I
\end{pmatrix}
\begin{pmatrix}
d_x \\
d_y \\
d_z
\end{pmatrix} =
\begin{pmatrix}
0 \\
0 \\
p_v
\end{pmatrix},
\]

(8)

where

\[
p_v = v^{-1}(w - v^2) \tag{9}
\]

and \(\bar{A} = DAD\) and \(\bar{Q} = DQD\) with \(D := \text{diag}(d)\).

In the next sub-section, we describe the generic feasible weighted primal-dual path-following algorithm to solve CQO.

2.1. The Algorithm

Similar to LO case, we define for any positive vector \( v \) and in view of (9), a norm-based proximity measure as follows:

\[
\delta(v; w) = \frac{\|p_v\|}{2\sqrt{\min(w)}} = \frac{\|v^{-1}(w - v^2)\|}{2\sqrt{\min(w)}}. \tag{10}
\]

One can easily verify that

\[
\delta(v; w) = 0 \iff v^2 = w \iff xz = w.
\]

Hence the value \( \delta(v; w) \) is to measure the distance of a point \((x, y, z)\) to the weighted-path \((x(w), y(w), z(w))\).

Let denote another measure \( \sigma_C(w) \) as follows

\[
\sigma_C(w) = \frac{\max(w)}{\min(w)}. \tag{11}
\]

The role of \( \sigma_C(w) \) is to measure the closeness of \( w \) to the central path.

Here,

\[
\min(w) = \min_i w_i
\]

and likewise

\[
\max(w) = \max_i w_i.
\]

Note that in (11), \( \sigma_C(w) \geq 1 \), with equality if \( w \) is on the central-path.

Now we are ready to describe the generic weighted path-following interior-point algorithm for CQO as follows.

**A generic weighted Primal-Dual Path-Following Algorithm for CQO**

**Input**
- A threshold parameter \( 0 < \delta < 1 \) (default \( \delta = \frac{1}{\sqrt{2}} \));
- An accuracy parameter \( \epsilon > 0 \);
- A fixed barrier update parameter \( 0 < \theta < 1 \) (default \( \theta = \frac{1}{2\sqrt{n\sigma_C(w^0)}} \));
- A starting point \((x^0, y^0, z^0)\) and \( w^0 \) such that \( \delta(x^0, z^0; w^0) \leq \frac{1}{\sqrt{2}} \);

**begin**

Set \( x := x^0; y := y^0; z := z^0; w := w^0; \)

while \( x^Tz \geq \epsilon \) do

begin

\( w := (1 - \theta)w; \)

Solve system (3) to obtain the direction \((\Delta x, \Delta y, \Delta z)\);

Update \( x := x + \Delta x, y := y + \Delta y, z := z + \Delta z; \)

end

end
In the next section, we will show that Algorithm 2.1 is well-defined for the defaults
\[ \theta = \frac{1}{2^{\sqrt{n}} \sigma_C(w)} \] and \[ \delta \leq \frac{1}{\sqrt{2}} \] and can solve CQO in polynomial-time.

3. Complexity analysis

In the next lemma, we state some useful technical results that will be used later in
the analysis of the algorithm.

Lemma 3.1
Let \((d_x, d_z)\) be a solution of (8) and suppose \(w > 0\). If \(\delta := \delta(v; w)\). Then, one has
\[ 0 \leq d_x^T d_z \leq 2\delta^2 \min(w), \] (12)
and
\[ \|d_x d_z\|_\infty \leq \delta^2 \min(w) \text{ and } \|d_x d_z\| \leq \sqrt{2}\delta^2 \min(w). \] (13)

Proof: Since \(0 \leq d_x^T d_z\), the statement in (12) follows immediately from the
following equality:
\[ \|d_x\|^2 + \|d_z\|^2 + 2d_x^T d_z = \|d_x + d_z\|^2 = \|p_v\|^2 = 4\delta^2 \min(w). \]
For (13), (see Lemma C.4 in [7]), since
\[ \|d_x d_z\|_\infty \leq \frac{1}{4} \|p_v\|^2 \text{ and } \|d_x d_z\| \leq \frac{1}{2\sqrt{2}} \|p_v\|^2. \]
This completes the proof.

The following lemma shows that the feasibility of the weighted full-Newton step
under the condition \(\delta := \delta(v; w) < 1\).

Lemma 3.2
Let \((x, z)\) be a strictly feasible primal-dual point. Then \(x_+ = x + \Delta x > 0\) and \(z_+ = y + \Delta z > 0\) if and only if \(w + d_x d_z > 0\).

Proof: For the first statement we have,
\[
x_+ z_+ = (x + \Delta x)(z + \Delta z)
= xz + x\Delta z + z\Delta x + \Delta x \Delta z
= xz + (w - xz) + \Delta x \Delta z
= w + \Delta x \Delta z.
\]
Then from equation in (7), we have,
\[
x_+ z_+ = w + \Delta x \Delta z
= w + d_x d_z.
\]
If the full-Newton step is strictly feasible $x_+ > 0$ and $z_+ > 0$ then $x_+z_+ > 0$ and so $w + d_x d_z > 0$.

To show that $x_+$ and $z_+$ are positive, we introduce a step length $\alpha \in [0, 1]$ and we define

$$x^\alpha = x + \alpha \Delta x, \quad z^\alpha = z + \alpha \Delta z.$$ 

So $x^0 = x, x^1 = x_+$ and similar notations for $z$, hence $x^0 z^0 > x z > 0$. We have,

$$x^\alpha z^\alpha = (x + \alpha \Delta x)(z + \alpha \Delta z) = xz + \alpha(x \Delta z + z \Delta x) + \alpha^2 \Delta x \Delta z.$$ 

Now by using (6), we get

$$x^\alpha z^\alpha > xz + \alpha(w - xz) + \alpha^2 \Delta x \Delta z.$$ 

We assume that $w + d_x d_z > 0$, we deduce that $w + \Delta x \Delta z > 0$ which equivalent to $\Delta x \Delta z > -w$. Substitution we obtain

$$x^\alpha z^\alpha > xz + \alpha(w - xz) - \alpha^2 w$$

$$= (1 - \alpha)xz + (\alpha - \alpha^2)w$$

$$= (1 - \alpha)xz + \alpha(1 - \alpha)w.$$ 

Since $xz$ and $w$ are positive it follows that $x^\alpha z^\alpha > 0$ for $\alpha \in [0, 1]$. Hence, none of the entries of $x^\alpha$ and $z^\alpha$ vanish for $\alpha \in [0, 1]$. Since $x^0$ and $z^0$ are positive, this implies that $x^\alpha > 0$ and $z^\alpha > 0$ for $\alpha \in [0, 1]$. Hence, by continuity argument, the vectors $x^\alpha$ and $z^\alpha$ must be positive which proves that $x_+$ and $z_+$ are positive. This completes the proof.

**Lemma 3.3**

If $\delta := \delta(v; w) < 1$. Then, the primal-dual full-Newton step is strictly feasible, i.e., $x_+ > 0$ and $z_+ > 0$.

**Proof:** In Lemma 3.2, we have seen that:

$$x_+z_+ > 0 \text{ if } w + d_x d_z > 0.$$ 

So $w + d_x d_z > 0$ holds if

$$w_i + (d_x)_i(d_z)_i > 0, \text{ for all } i.$$ 

We have

$$w_i + (d_x)_i(d_z)_i \geq w_i - |(d_x)_i(d_z)_i| \geq \min(w) - \|d_x d_z\|_\infty \text{ for all } i.$$ 

Now, according to (13), Lemma 3.1, it follows that:

$$\min(w) - \|d_x d_z\|_\infty > \min(w)(1 - \delta^2).$$
Thus $w + d_x d_z > 0$ holds if $\delta < 1$. This completes the proof.

For convenience, we may write

$$v_+ = \sqrt{x_+ z_+}.$$

**Lemma 3.4**

If $\delta < 1$. Then

$$\|v_+^{-1}\| \leq \frac{1}{\sqrt{\min(w)(1 - \delta^2)}}.$$

Proof: It follows straightforwardly from Lemma 3.3 and since

$$v_+^{-2} = \frac{e}{w + d_x d_z}.$$

In the next lemma, we show the influence of a weighted full-Newton step on the proximity measure.

**Lemma 3.5**

If $\delta < 1$. Then

$$\delta_+ := \delta(v_+; w) \leq \frac{\delta^2}{\sqrt{2(1 - \delta^2)}}.$$ 

Proof: By definition, we have,

$$\delta_+ = \frac{1}{2\sqrt{\min(w)}} \|v_+^{-1}(w - v_+^2)\| \leq \frac{1}{2\sqrt{\min(w)}} \|v_+^{-1}\| \|w - v_+^2\| .$$

But $w - v_+^2 = -d_x d_z$ and $v_+^{-1} = \frac{e}{\sqrt{w + d_x d_z}}$, then by Lemmas 3.1 and 3.4, we have,

$$\delta_+ = \frac{1}{2\sqrt{\min(w)}} \left\| \frac{d_x d_z}{\sqrt{w + d_x d_z}} \right\| \leq \frac{1}{2\sqrt{\min(w)}} \frac{\sqrt{\min(w)} \delta^2}{\sqrt{\min(w) - \|d_x d_z\|_{\infty}}} \leq \frac{1}{2\sqrt{\min(w)}} \frac{\sqrt{\min(w)} \delta^2}{\sqrt{\min(w)(1 - \delta^2)}} \leq \frac{\delta^2}{\sqrt{2}(1 - \delta^2)} .$$

This completes the proof.
Corollary 3.1
If $\delta < 1$. Then $\delta_+ \leq \delta^2$ which indicates the convergence quadratic of the proximity when iterations are closed to the path. In addition if $\delta \leq \frac{1}{\sqrt{2}}$, then $\delta_+ \leq \frac{1}{2}$.

In the next lemma, we discuss the influence on the proximity measure of the update barrier parameter $w_+ = (1 - \theta) w$ on the Newton process along the weighted-path.

Lemma 3.6
If $\delta(w; v) < 1$ and $w_+ = (1 - \theta) w$ where $0 < \theta < 1$. Then
\[
\delta(v_+; w_+) \leq \frac{\theta}{2\sqrt{1 - \theta\sqrt{1 - \delta^2}}} \sqrt{n\sigma_C(w)} + \frac{1}{\sqrt{2(1 - \theta)}} \delta_+.
\]
In addition, if $\delta \leq \frac{1}{\sqrt{2}}, \theta = \frac{1}{2\sqrt{n\sigma_C(w)}}$ and $n \geq 3$, then we have,
\[
\delta(v_+; w_+) \leq \frac{1}{\sqrt{2}}.
\]

Proof: Let $\delta(v_+; w_+)$ and $w_+ = (1 - \theta) w$ with $0 < \theta < 1$. Then, by definition we have,
\[
\delta(v_+; w_+) = \frac{1}{2\sqrt{\min(w_+)} \||v_+^{-1}(w_+ - v_+^2)\||}
\]
\[
= \frac{1}{2\sqrt{1 - \theta\sqrt{\min(w)}} \||v_+^{-1}(w_+ - v_+^2)\||}
\]
\[
= \frac{1}{2\sqrt{1 - \theta\sqrt{\min(w)}} \||v_+^{-1}(w_+ - w + w - v_+^2)\||}
\]
\[
\leq \frac{1}{2\sqrt{1 - \theta\sqrt{\min(w)}} \left(||v_+^{-1}|| \left(||w_+ - w|| + ||w - v_+^2||\right)\right)}.
\]
Now since $w - v_+^2 = -d_x d_z$ and $w_+ - w = -\theta w$ and by Lemmas 3.1 and 3.4 and with the fact that $||w|| \leq \sqrt{n} ||w||_\infty$, we get,
\[
\delta(v_+; w_+) \leq \frac{1}{2\sqrt{1 - \theta\min(w)\sqrt{1 - \delta^2}}} \left(||\theta w|| + ||d_x d_z||\right)
\]
\[
\leq \frac{1}{2\sqrt{1 - \theta\min(w)\sqrt{1 - \delta^2}}} \left(||\theta w|| + \min(w)\theta^2\right)
\]
\[
\leq \frac{\theta \|w\|}{2\sqrt{1 - \theta\min(w)\sqrt{1 - \delta^2}}} + \frac{\theta^2}{2\sqrt{1 - \theta\sqrt{1 - \delta^2}}}
\]
\[
\leq \frac{\theta \sqrt{n} \|w\|_\infty}{2\sqrt{1 - \theta\min(w)\sqrt{1 - \delta^2}}} + \frac{\theta^2}{2\sqrt{1 - \theta\sqrt{1 - \delta^2}}}
\]
\[
= \frac{\theta \sqrt{n} \max(w)}{2\sqrt{1 - \theta\min(w)\sqrt{1 - \delta^2}}} + \frac{\theta^2}{2\sqrt{1 - \theta\sqrt{1 - \delta^2}}}.
\]
Using Lemma 3.5 and (11), we have,
\[
\delta(v_+; w_+) \leq \frac{\theta \sqrt{n} \sigma_C(w)}{2\sqrt{1-\theta}} \sqrt{1-\theta^2} + \frac{\delta}{\sqrt{2(1-\theta)}}.
\]
If \( \theta = \frac{1}{2 \sqrt{n} \sigma_C(w)} \), and observe that \( \sigma_C(w) \geq 1 \), and for \( n \geq 3 \), then \( \theta \leq \frac{1}{4} \). Furthermore, if \( \delta \leq \frac{1}{\sqrt{2}} \), then from Corollary 3.1, \( \delta_+ \leq \frac{1}{2} \). Finally, the above inequalities yield \( \delta(v_+; w_+) \leq \frac{1}{\sqrt{2}} \). This completes the proof. \( \Box \)

Note that, in all the iterates produced by Algorithm 2.1, we have \( \sigma_C(w) = \sigma_C(w_0) \). Thus, we deduce from Lemma 3.6 that for the default \( \theta = \frac{1}{2 \sqrt{n} \sigma_C(w_0)} \), the conditions \( x, y > 0 \) and \( \delta(v_+; w_+) \leq \frac{1}{\sqrt{2}} \) are maintained during the algorithm. Thus, confirms that Algorithm 2.1, is well-defined.

The upper bound of the duality gap after a weighted full-Newton step is presented in the following lemma.

**Lemma 3.7**

Let \( \delta := \delta(v; w) \leq \frac{1}{\sqrt{2}} \) and \( x_+ = x + \Delta x \) and \( z_+ = z + \Delta z \). Then the duality gap satisfies:

\[
x_+^T z_+ \leq (n + 1) \max(w).
\]

**Proof:** By Lemma 3.2, we have seen that

\[
x_+ z_+ = w + d_x d_z.
\]

Hence

\[
e^T(x_+ z_+) = e^T w + e^T d_x d_z = e^T w + d_x^T d_z.
\]

According to (13), Lemma 3.1 and \( \delta \leq \frac{1}{\sqrt{2}} \), we deduce that:

\[
x_+^T z_+ \leq e^T w + 2\delta^2 \min(w), \leq e^T w + \min(w).
\]

Now, since \( e^T w \leq n \max(w) \), we get

\[
x_+^T z_+ \leq (n + 1) \max(w).
\]

This completes the proof. \( \Box \)

The following lemma gives an upper bound for the total number of iterations produced by Algorithm 2.1.

Lemma 3.8
Let $x^{k+1}$ and $z^{k+1}$ be the $(k+1)$-th iteration produced by the Algorithm 2.1, with $w := w^k$. Then

$$(x^{k+1})^T z^{k+1} \leq \epsilon$$

if

$$k \geq \left\lceil \frac{1}{\theta} \log \frac{2n \max(w^0)}{\epsilon} \right\rceil.$$

Proof: By Lemma 3.7, it follows that:

$$(x^{k+1})^T z^{k+1} \leq (n + 1) \max(w^k)$$

with

$$w^k = (1 - \theta) w^{k-1} = (1 - \theta)^k w^0.$$ 

Then, we have

$$(x^{k+1})^T z^{k+1} \leq (1 - \theta)^k (n + 1) \max(w^0) \leq (1 - \theta)^k 2n \max(w^0),$$

since $n + 1 \leq 2n$ for all $n \geq 1$.

Thus the inequality $(x^{k+1})^T z^{k+1} \leq \epsilon$ is satisfied if

$$(1 - \theta)^k 2n \max(w^0) \leq \epsilon.$$

Now taking logarithms, we may write

$$k \log(1 - \theta) \leq \log \epsilon - \log 2n \max(w^0)$$

and since $-\log(1 - \theta) \geq \theta$ for $0 < \theta < 1$, then the inequality holds if

$$k \theta \geq \log \frac{2n \max(w^0)}{\epsilon}.$$

This completes the proof.

$\square$

Theorem 3.1
Suppose that $x^0$ and $z^0$ are strictly feasible starting point for CQO, $w^0 = \frac{x^0 z^0}{2 \max(x^0 z^0)}$, and such that $\delta(x^0 z^0; w^0) \leq \frac{1}{\sqrt{2}}$ for $n \geq 3$. If $\theta = \frac{1}{2 \sqrt{n \sigma_C(w^0)}}$ then, Algorithm 2.1, requires at most $O\left(\sqrt{n} \sigma_C(w^0) \log \frac{\epsilon}{2} \right)$ iterations to obtain an $\epsilon$-approximate solution of CQO.

In particular, if $w^0 = \frac{1}{2} e$, then Algorithm 2.1, requires at most $O\left(\sqrt{n} \log \frac{n}{\epsilon} \right)$ iterations which is the currently best known iteration bound for short-update methods.

Proof: By taking the value of $\theta$ and $w^0$ in Lemma 3.8, the result follows straightforwardly. This completes the proof. $\square$
4. Conclusion and future remarks

In this paper, we have presented a weighted full-Newton step path-following method for CQO. At each interior point iteration, only full-Newton steps are used. The favorable polynomial complexity bound for the algorithm with short-step method is deserved, namely, $O(\sqrt{n} \log \frac{n}{\epsilon})$ which is as good as LO case. Finally, the numerical implementation of this algorithm remains to be investigated.

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