On a Closed-loop Supply Chain with Graded Returns

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Abstract We use optimal control theory to determine the optimal manufacturing, remanufacturing, and disposal rates in a closed-loop supply chain. The returned items are of different quality levels. The firm grades the returned items according to their quality. Each class of returned items is remanufactured and stocked separately. Also, all items are subject to deterioration and the deterioration rate depends on the class. Finally, each class of items is sold to a different segment of customers. An illustrative example is presented along with a sensitivity analysis on some of the system parameters.

Keywords Manufacturing, Remanufacturing, Quality, Grading, Optimal Control, Tracking

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1. Introduction

A product return occurs when a customer returns previously purchased merchandise to a retailer for a refund, exchange, or store credit. Returned items are either remanufactured or disposed off. Remanufacturing involves repairing or replacing worn out or obsolete components and modules of a product to specifications of the original manufacturing process.

There is an impressive amount of research work on remanufacturing. There is also an immense amount of literature reviews on this topic. The last year alone has seen no less than at least twenty reviews ([1, 2, 4, 5, 6, 11, 12, 13, 14, 19, 20, 21, 22, 23, 25, 29, 31, 32, 33, 34]) and counting. This is due to the fact that remanufacturing has many facets and involves many processes.

One of the aspects that attracted the attention of researchers if the residual quality of the returned product and its effect on the overall performance of the firm.

Ferguson et al. [10] consider a firm that classifies returns according to their quality level. Using a stochastic dynamic program formulation, the firm determines during each period the amount to remanufacture for each quality level and the amount of inventory to carry over for future periods for returns and finished remanufactured products.

Incorporating service level constraints to the same model, Souza et al. [26] use a GI/G/1 queueing network to determine the optimal product mix. They also use simulation to investigate some dispatching heuristics.

Kang and Hong [17] propose to disassemble returned products at variable levels into remanufacturable parts. They use a mixed integer linear programming model to derive the optimal disassembly plan.

Multiple types of remanufacturable products are also considered by Zhou et al. [35]. They assume a stochastic demand for serviceable products and a periodic-review policy. They obtain a simple form for the optimal manufacturing, remanufacturing, and disposal rates.
In the context of heterogeneous qualities of used product returns, two inventory systems, remanufacture-to-stock (RTS) and remanufacture-to-order (RTO), are studied by Jia [15]. Then, he shows that a hybrid system that switches between RTS and RTO in the product lifecycle outperforms the RTS and RTO frameworks. Li et al. [18] also investigate a model with uncertain quality returns in RTO and RTS systems.

In Tao et al. [28], the yield from the remanufacturing of returns of different quality levels is random. The model is studied in discrete time and both demand rate and return rate are stochastic. They use stochastic dynamic programming to obtain the optimal ordering/remanufacturing policy.

Returns with only two quality levels are considered by Cai et al. [3] and Zhu et al. [36]. While Cai et al. use stochastic dynamic programming to derive the optimal acquisition pricing and production policy, Zhu et al. use mixed integer nonlinear programming to optimize the production sequence, production cycle length, shipment frequency, and shipment batch sizes.

Xiong et al. [30] assume that the condition of returned products is a continuous random variable. They model the problem as a continuous-time Markov decision process. They prove the convexity of the objective function and the uniqueness of the optimal price policy.

Sun et al. [27] are interested in the lot sizing problem and in scheduling the manufacturing and the remanufacturing sequences when returns divided into different quality grades. They assume a constant demand rate.

Farahani [7] and Farahani et al. [8] assume that the inventory of returns has a finite capacity. Demand is constant and quality of returns is random. They use a continuous-time Markov chain model and the matrix-geometric technique to study this system.

Government participation and supply chain coordination are incorporated by Feng et al. [9] in a model where returned products have different qualities. They use a game-theoretic approach to determine the optimal pricing policy.

In this paper, we consider a manufacturing-remanufacturing firm where returns are not of the same quality. The firm sorts the returns and grades them according to their quality. Items that cannot be remanufactured are disposed off. Different returns of different grades are remanufactured and stocked separately. They are then sold to different segments of customers. Therefore, the serviceable items and the different classes of returns all have different (dynamic) demand rates. Also, all items are subject to deterioration and the deterioration rates depend on the type of items. While most previous works study their systems in discrete time, we study our system in continuous time. Since the model is dynamic, optimal control theory is employed to obtain the optimal manufacturing, remanufacturing, and disposal rates. The model is described next and solved in the following section.

2. Model Formulation

A single product is manufactured by a firm during the planning horizon $[0, T]$. A graphical illustration of the system under study is given in Figure 1. There are three types of stocks, as described below:

New items stock: At time $t$, the inventory level is $I_0(t)$, the manufacturing rate is $P_0(t)$, and the demand rate is $D_0(t)$. While on the shelves, the produced items (also called serviceable) are subject to deterioration at rate $\theta_0$. Given the initial inventory level $I_0(0)$, the dynamics of the inventory level of serviceable items are governed by the following differential equation:

$$\frac{d}{dt} I_0(t) = P_0(t) - D_0(t) - \theta_0 I_0(t), \quad I_0(0) = I_0^0. \tag{1}$$

Returned items stock: A return product can be of any quality $n$, ($n = 1, \cdots, N$), for some positive integer $N$. Items of the same quality level are stocked separately from other items. At time $t$, for each stock $n$, ($n = 1, \cdots, N$), the inventory level is $J_n(t)$, the remanufacturing rate is $P_n(t)$, the demand rate is $D_n(t)$, and the return rate is $R_n(t)$. Items that cannot be remanufactured are disposed of (thrown away) at rate $T_n(t)$. Given the initial inventory level $J_n(0)$, the dynamics of the inventory level of quality $n$ returned items are governed by the following differential
Figure 1. A manufacturing-remanufacturing system with $N$ types of returns.

Equation:

\[
\text{(returned)} \quad \frac{d}{dt} J_n(t) = -P_n(t) - T_n(t) + R_n(t), \quad J_n(0) = J_n^0. \quad (2)
\]
Remanufactured items stock: Finally, at time \( t \), for each stock \( n, (n = 1, \cdots, N) \), the inventory level is \( I_n(t) \), the remanufacturing rate is \( P_n(t) \), the demand rate is \( D_n(t) \), and the return rate is \( R_n(t) \). While on the shelves, the remanufactured items (also called cores) are subject to deterioration at rate \( \theta_n \). Given the initial inventory level \( I_n(0) \), the dynamics of the inventory level of cores of quality type \( n \) are governed by the following differential equation:

\[
\frac{d}{dt} I_n(t) = P_n(t) - D_n(t) - \theta_n I_n(t), \quad I_n(0) = I_n^0.
\]

(remanufactured)

The system considered is dynamic and an optimal control approach seems appropriate. The variables \( I_n(t), n = 0, \cdots, N \) and \( J_n(t), n = 1, \cdots, N \) are the state variables, while the variables \( P_n(t), n = 0, \cdots, N \) and \( T_n(t), n = 1, \cdots, N \) are the control variables. We assume that the system is of the tracking type, see Sethi [24], and for each variable \( x(t) \), either state or control, we associate a target variable \( \hat{x}(t) \) with the intention of minimizing the gap \( \Delta x(t) = x(t) - \hat{x}(t) \). Note that the state target variables are given while the control target variables are calculated from the state equations (1)-(3). Also, the state equations (1)-(3) can be rewritten in terms of the shift operator \( \Delta \) as follows:

\[
\frac{d}{dt} \Delta I_0(t) = \Delta P_0(t) - \theta_0 \Delta I_0(t), \quad \Delta I_0(0) = I_0^0 = \hat{I}_0^0,
\]

(4)

\[
\frac{d}{dt} \Delta I_n(t) = \Delta P_n(t) - \theta_n I_n(t), \quad \Delta I_n(0) = I_n^0 = \hat{I}_n^0,
\]

(5)

\[
\frac{d}{dt} \Delta J_n(t) = -\Delta P_n(t) - \Delta T_n(t), \quad \Delta J_n(0) = \hat{J}_n^0 - J_n^0.
\]

(6)

Since the goal is to have each variable converge towards its goal, we introduce the costs \( h_n, k_n, \pi_n, (n = 0, \cdots, N) \), and \( h_n, \kappa_n, \pi_n, (n = 1, \cdots, N) \) to penalize gaps. The objective function to minimize is then defined by:

\[
J = \frac{1}{2} \int_0^T \left\{ h_0 \Delta I_0(t)^2 + k_0 \Delta P_0(t)^2 \\
+ \sum_{n=1}^N \left[ h_n \Delta I_n(t)^2 + h_n \Delta J_n(t)^2 + k_n \Delta P_n(t)^2 + \kappa_n \Delta T_n(t)^2 \right] \right\} dt
+ \frac{1}{2} \sum_{n=1}^N \left[ p_n \Delta J_0(T)^2 + \pi_n \Delta J_n(T)^2 \right].
\]

(7)

The problem then is to determine the optimal state and control variables that minimize the objective function (7) subject to the state equations (4)-(6).

3. Model Analysis

The first step in solving the above problem is to rewrite it in matrix form. Introducing \( x(t) \) and \( u(t) \), the \((2N+1) \times 1\) state and control vectors, respectively,

\[
x(t) = \begin{bmatrix} \Delta I_0(t) & \Delta I_1(t) & \cdots & \Delta I_N(t) & \Delta J_1(t) & \Delta J_2(t) & \cdots & \Delta J_N(t) \end{bmatrix}^T,
\]

\[
u(t) = \begin{bmatrix} \Delta P_0(t) & \Delta P_1(t) & \cdots & \Delta P_N(t) & \Delta T_1(t) & \Delta T_2(t) & \cdots & \Delta T_N(t) \end{bmatrix}^T,
\]

the state equations (4)-(6) are rewritten as:

\[
\frac{d}{dt} x(t) = Ax(t) + Bu(t), \quad x(0) = x^0,
\]

(8)
where $A$ and $B$ are square matrices of dimension $(2N+1) \times (2N+1)$:

$$A = \begin{bmatrix} 0 & 1 & \cdots & N & N+1 & N+2 & \cdots & 2N \\ -\theta_0 & -\theta_1 & \cdots & -\theta_N \\ \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 1 & \cdots & N & N+1 & N+2 & \cdots & 2N \end{bmatrix},$$

and

$$B = \begin{bmatrix} 0 & 1 & \cdots & N & N+1 & N+2 & \cdots & 2N+1 \\ 0 & 1 & \cdots & N & N+1 & N+2 & \cdots & 2N+1 \end{bmatrix},$$

and $x^0$ is the initial state. Similarly, the objective function becomes

$$J = \frac{1}{2} \int_0^T \left[ \|x(t)\|_H^2 + \|u(t)\|_K^2 \right] dt + \frac{1}{2} \|x(T)\|_P^2,$$

where $\|x(t)\|_A^2 = x(t)^T Ax(t)$ for any square matrix $A$, and the matrices $H, K,$ and $P$, and diagonal matrices of dimension $(2N+1) \times (2N+1)$:

$$H = \begin{bmatrix} 0 & 1 & \cdots & N & N+1 & N+2 & \cdots & 2N \\ h_0 & h_1 & \cdots & h_N \\ \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 1 & \cdots & N & N+1 & N+2 & \cdots & 2N \end{bmatrix},$$

$$K = \begin{bmatrix} 0 & 1 & \cdots & N & N+1 & N+2 & \cdots & 2N \\ k_0 & k_1 & \cdots & k_N \\ \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 1 & \cdots & N & N+1 & N+2 & \cdots & 2N \end{bmatrix},$$

and $x^0$ is the initial state. Similarly, the objective function becomes

$$J = \frac{1}{2} \int_0^T \left[ \|x(t)\|_H^2 + \|u(t)\|_K^2 \right] dt + \frac{1}{2} \|x(T)\|_P^2,$$

where $\|x(t)\|_A^2 = x(t)^T Ax(t)$ for any square matrix $A$, and the matrices $H, K,$ and $P$, and diagonal matrices of dimension $(2N+1) \times (2N+1)$:
The solution by the maximum principle, see Sethi [24], involves the $(2N + 1) \times 1$ adjoint vector $\Lambda(t) = \begin{bmatrix} \lambda_0(t) & \lambda_1(t) & \cdots & \lambda_{2N}(t) \end{bmatrix}^\top$, and the Hamiltonian function:

\[
H = -\frac{1}{2} \left[ \|x(t)\|_H^2 + \|u(t)\|_K^2 \right] + \Lambda(t) \top \left[ Ax(t) + Bu(t) \right].
\]

The control equation $H_u = 0$ yields the optimal control vector

\[
u(t) = -K^{-1}B^\top \Lambda(t). \tag{9}
\]

Substituting the control vector (9) in the state equation (8) to obtain

\[
\frac{dx}{dt} = Ax(t) - BK^{-1}B^\top \Lambda(t), \quad x(0) = x^0.
\]

The adjoint equation $H_x = -\frac{d}{dt}\Lambda(t)$ is easily found to be

\[
\frac{d}{dt}\Lambda(t) = -Hx(t) - A^\top \Lambda(t), \quad \Lambda(T) = Px(T).
\]

Introduce the vector $Z(t) = \begin{bmatrix} x(t) & \Lambda(t) \end{bmatrix}^\top$. Then, the above two equations are equivalent to the vector-matrix equation

\[
\frac{d}{dt}Z(t) = \Phi Z(t),
\]

where

\[
\Phi = \begin{bmatrix} A & -BK^{-1}B^\top \\ -H & -A^\top \end{bmatrix}.
\]

The solution of this differential system has the following form

\[
Z(t) = \varphi(t)Z(0), \tag{10}
\]

where $\varphi(t)$ and $Z(0)$ need to be determined. Starting with $\varphi(t)$, let $m_i$, $(i = 1, \cdots, 4N + 2)$ denote the eigenvalues of the matrix $\Phi$ and let $Y$ denote the matrix whose columns are the corresponding eigenvectors. Then,

\[
\varphi(t) = \sum_{i=1}^{4} Y(:,i)Y^{-1}(i,:)e^{m_i t},
\]

where $Y(:,i)$ is the $i$th column of $Y$ and $Y^{-1}(i,:)$ is the $i$th row of $Y^{-1}$. To determine $Z(0)$, we partition appropriately the matrix $\varphi(t)$ and write the solution (10) at $t = T$, $Z(T) = \varphi(T)Z(0)$ which can be rewritten as

\[
\begin{bmatrix} x(T) \\ \Lambda(T) \end{bmatrix} = \begin{bmatrix} \varphi_1(T) & \varphi_2(T) \\ \varphi_3(T) & \varphi_4(T) \end{bmatrix} \begin{bmatrix} x(0) \\ \Lambda(0) \end{bmatrix}.
\]
Using the terminal condition $\Lambda(T) = Px(T)$, we readily have

$$\Lambda(0) = \left[ P\varphi_2(T) - \varphi_4(T) \right]^{-1} \left[ \varphi_3(T) - P\varphi_1(T) \right] x(0).$$

Now (10) yields the optimal state vector

$$x(t) = \varphi_1(t)x(0) + \varphi_2(t)\Lambda(0)$$

$$= \left\{ \varphi_1(t) + \varphi_2(t) \left[ P\varphi_2(T) - \varphi_4(T) \right]^{-1} \left[ \varphi_3(T) - P\varphi_1(T) \right] \right\} x(0),$$

and the optimal adjoint vector

$$\Lambda(t) = \varphi_3(t)x(0) + \varphi_4(t)\Lambda(0)$$

$$= \left\{ \varphi_3(t) + \varphi_4(t) \left[ P\varphi_2(T) - \varphi_4(T) \right]^{-1} \left[ \varphi_3(T) - P\varphi_1(T) \right] \right\} x(0)$$

Using (9) yields the optimal control vector

$$u(t) = -K^{-1}B^T \left\{ \varphi_3(t) + \varphi_4(t) \left[ P\varphi_2(T) - \varphi_4(T) \right]^{-1} \left[ \varphi_3(T) - P\varphi_1(T) \right] \right\} x(0).$$

4. Numerical Example

Consider a manufacturing-remanufacturing firm with $N = 1$, that is all returns are of the same quality. The planning horizon has length $T = 15$. The initial inventory level are $I_0(0) = 15$ (serviceable items), $I_1(0) = 10$ (remanufactured items), and $J_1(0) = 5$ (returned items). Serviceable items deteriorate at rate $\theta_0 = 0.01$ while remanufactured items deteriorate at rate $\theta_1 = 0.02$. The units costs for a variable to deviate from its goal are given by $h_0 = 3$, $h_1 = 4$, $k_1 = 5$, $k_0 = 10$, $k_1 = 15$, $\kappa_1 = 20$, $p_0 = 100$, $p_1 = 150$, $\pi_0 = 200$. Implementing the results of the previous yields the optimal deviations depicted in Figure 2. Since all deviations converge to zero, this means that, by the end of the planning horizon, each variable will reach its goal, as desired. The minimum cost of this policy is $J^* = 1448.90$.

We now assume $N = 3$ so that returns have three quality levels. Again the planning horizon has length $T = 15$. The initial inventory level are $I_0(0) = 20$ (serviceable items), $I_1(0) = 15$ (remanufactured items of quality level 1), $I_2(0) = 10$ (remanufactured items of quality level 2), $I_3(0) = 5$ (remanufactured

![Figure 2. N = 1. Optimal deviations of the state variables (left) and optimal deviations of the control variables (right).](image-url)
items of quality level 3), \( J_1(0) = 15 \) (returned items of quality level 1), \( J_2(0) = 10 \) (returned items of quality level 2), \( J_3(0) = 5 \) (returned items of quality level 3). Serviceable items deteriorate at rate \( \theta_0 = 0.01 \) while remanufactured items of quality levels 1, 2, 3 deteriorate at rate \( \theta_1 = 0.02, \theta_2 = 0.03, \) and \( \theta_3 = 0.04 \), respectively. The units costs for a variable to deviate from its goal are given by \( h_0 = 3, h_1 = 4, h_2 = 5, h_3 = 6, h_1 = 5, h_2 = 10, h_3 = 15, k_0 = 20, k_1 = 25, k_2 = 30, k_3 = 35, k_1 = 40, k_2 = 45, k_3 = 50, p_0 = 100, p_1 = 150, p_2 = 200, p_3 = 250, \pi_1 = 300, \pi_2 = 350, \pi_3 = 400 \). Implementing the results of the previous yields the optimal deviations depicted in Figure 3. As in the previous case, all deviations converge to zero, which means that, by the end of the planning horizon, each variable will reach its goal, as desired. The minimum cost of this policy is \( J^* = 8935.30 \).

Sensitivity analysis can be conducted on the system parameter to gain insight about the system. Keeping all parameters constant except for the planning horizon length, we obtained the graph of the optimal objective function value depicted in Figure 4 (left). As \( T \) increases, \( J^* \) decreases sharply in the beginning to become almost constant. The minimum of \( J^* \) is attained when \( T = 23 \).

Next we kept all the parameters constant and varied the initial inventory level of serviceable items. The graph in Figure 4 (middle) shows that \( J^* \) increases steadily as \( I_0(0) \) increases.

Finally, we kept all the parameters constant and varied the deterioration rate of serviceable items. The graph in Figure 4 (right) shows that \( J^* \) decreases steadily as \( \theta_0 \) increases.

Figure 3. \( N = 3 \). Optimal deviations of the state variables (left) and optimal deviations of the control variables (right).

Figure 4. \( N = 3 \). Optimal objective function value as a function of planning horizon length (left), initial inventory level of serviceable items (middle), and deterioration rate of serviceable items (right).

5. Summary and Future Research Directions

The general model of a closed-loop supply chain has been considered in this paper. Returns are classified according to their quality level. Each class is remanufactured and stored separately. Demand rates for each class are functions of times. All items are subject to deterioration and the deterioration rate varies with the grade.
This work can be generalized in different ways. For example, we assumed constant deterioration rates. This assumption can be released and replaced with dynamic or even random rates. Similarly, the demand and returned rates have been taken to be general functions of time. This assumption can also be released and replaced by random rates. Finally, it may be worthwhile studying this model in discrete time.

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