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Reliability Estimation for the Inverse Weibull Distribution Under Adaptive Type-II Progressive Hybrid Censoring: Comparative Study

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Abstract The aim of this study is to investigate different methods of estimating the stress-strength reliability parameter, $\theta = P(Y < X)$, when the strength (X) and the stress (Y) are independent random variables taken from the inverse Weibull distribution (IWD), with the same shape parameter and different scale parameters. Based on adaptive Type-II hybrid progressive censored samples, we employ classical and Bayesian approaches. In the classical approach, we use the maximum likelihood estimator (MLE), the approximate maximum likelihood estimator (AMLE), and the least squares estimator (LSE). In contrast, the Bayesian approach utilizes symmetric and asymmetric loss functions. Due to the absence of explicit forms for Bayes estimators, we propose using Lindley's approximation method for computing the Bayes estimators. We compare these estimators using extensive simulations and two criteria: the bias and the mean square error (MSE). Finally, two real-life data examples based on breakdown times of an insulated fluid and the survival times of Head and Neck Cancer patients are provided for illustrations. It was evident based on our results that the Bayesian estimation methods surpassed at estimating the reliability under the adaptive Type-II progressive hybrid censoring for the IWD. Moreover, the results of our real-life examples corroborate those of the simulation and support that the IWD is a suitable fit for both examples.

Keywords Reliability, Inverse Weibull distribution, Adaptive Type-II progressive hybrid censoring, Approximate maximum likelihood estimation

AMS 2010 subject classifications 62E10, 62N01, 62N02, 62G30

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1. Introduction

In recent years, manufacturers have been under increasing pressure to produce high-quality products and reduce production costs and times. With global competition growing, reliability study becomes more vital. Lifetime testing, structural reliability, and machine maintenance are the foundations of reliability estimation, prediction, and optimization. A well-known measure of reliability is the stress-strength model, $\theta = P(Y < X)$, where X and Y are random variables. For this model, the reliability θ of the system is the probability that the system can cope with the stresses imposed upon it.

The reliability of aircraft windshields is an example of both aerodynamics and mechanical engineering. The windshields consist of several layers of materials to withstand extreme temperatures and pressure. Therefore, to maintain a regular performance of aircraft, it is vital to know the probability of windshield failure at different stages of the windshield life (after 1000, 2000..., etc., of flight hours).

Considering that there is a reasonable estimate of windshield reliability by defining stress as the temperature and/or pressure differential and strength as the thickness and/or composition of the windshield layers, it will be possible to make a rational judgment regarding whether or not windshields will need to be repaired or replaced.

θ is not only a stress-strength model but also provides a general measure of the differences between two populations and is undoubtedly applicable in various fields. For example, in clinical studies, we may compare

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the efficacy of two drugs to measure X, the patient's life expectancy when treated with one medication, and Y, the patient's life expectancy when treated with another drug.

Life-testing experiments are notorious for their challenges of controlling the test time and conserving the experimental units while estimating efficiently. However, it is possible to solve this problem by stopping the experiment before all units fail by utilizing censoring strategies carried out by removing active units from the experiment. During these experiments, units may be lost or removed for different reasons, and this is where the importance of progressive censoring comes into play in which units are removed in a life test experiment at some predetermined or random time points during the experiment.

Many models of progressive censoring have been discussed throughout the years. The majority of these models can be traced back to one of two sources: progressive Type-I censoring, which terminates the experiment after a prefixed time point, or progressive Type-II censoring, which terminates the experiment after a prefixed number of observed failures. Both censoring schemes give the experimenter more flexibility by allowing the removal of test units at non-terminal time points during the experiment.

In progressive Type-I censoring, the total time of the experiment is predetermined, and the censoring occurs at prefixed time points $T_1, T_2, \ldots, T_r$. In addition, a prefixed number of active units are removed during the experiment at the end of each specified time interval, making the number of observed failure lifetimes random. Hence, in Type-I progressive censoring, one might observe a few, if any, failures when units under the test have long lifetimes.

In progressive Type-II censoring scheme, only m units are completely observed until failure out of n units placed on a life-test. When the first failure occurs, R_1 active units are removed from the $n-1$ remaining units. After the second failure, R_2 active units are removed from the $n - R_1 - 2$ remaining units. Lastly, at the m-th failure, all the remaining $n - R_1 - R_2 \cdots - R_{m-1}$ units are removed and the experiment is terminated. Since the time of the experiment is random, when units undertaking the life test have long life times, it results in a long test duration, which is considered a disadvantage for progressive Type-II censoring.

Two progressive hybrid censoring schemes were proposed by Kundu and Joarder (2006) by stopping a progressive Type-II censoring experiment at time T^*. In Type-I progressive hybrid censoring scheme, $T^* = min(X_{m:n}, T)$, where we may have fewer than m observations. In Type-II progressive hybrid censoring scheme, $T^* = max(X_{m:n}, T)$, where we may have at least m observations but a long test duration.

In real-life experiments, having a fixed censoring scheme may not be convenient because the censoring scheme may change, intentionally or unintentionally, during the experiment. Therefore, Ng et al. (2009) proposed a newer model (see Figure 1) that allows changing the censoring scheme during the experiment. Such a model is called the adaptive Type-II progressive hybrid censoring, in which a threshold time T is used to switch from the initially planned censoring scheme to a modified one.

In a sample of size n, where m failures will be observed, and after a threshold time T, the censoring number R_j $(j = max(i; X_{i:m:n} < T))$ will adaptively change based on the previous failure times as well as the censored samples before the j-th failure. That is, when the first observed failure time that exceeds the threshold time T, the applied censoring scheme will be changed to $R^* = (R_1, \ldots, R_j, 0, \ldots, 0, n - m - \sum_{i=1}^{j} R_j)$. The initially planned progressive censoring scheme is used as long as the failures occur before time T (see Figure 1(a)). Otherwise, when time T occurs before the m-th failure, no units are withdrawn after time T except for the time of the m-th failure where all remaining surviving units are removed (see Figure 1(b)). By setting $T = \infty$ and $T = 0$, we get Type-II progressive censoring and Type-II censoring, respectively.

Failure times of units under a life-testing experiment are assumed to be identically distributed and follow a lifetime distribution. One of the most widely used lifetime distributions to model progressive censoring schemes is the Weibull distribution (WD). If a random variable T follows the WD with a shape parameter α and a scale parameter β, then the probability density function (PDF) is given by

$$f(t; \alpha, \beta) = \alpha\beta e^{-\beta t^{\alpha}} t^{-1+\alpha} \qquad t > 0, \alpha, \beta > 0, \tag{1}$$

and the hazard function (HF), which measures the probability of failure of a unit at a given time, is given by

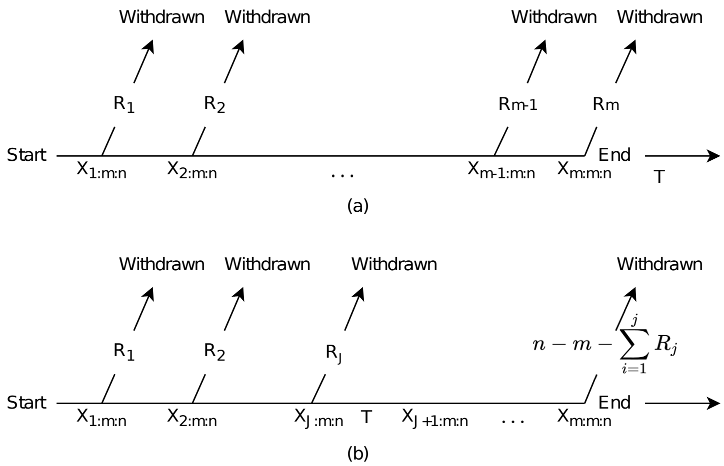

Figure 1. Adaptive Type-II progressive hybrid censoring model as proposed by Ng et al. (2009). (a) Experiment ends before time T. (b) Experiment ends after time T.

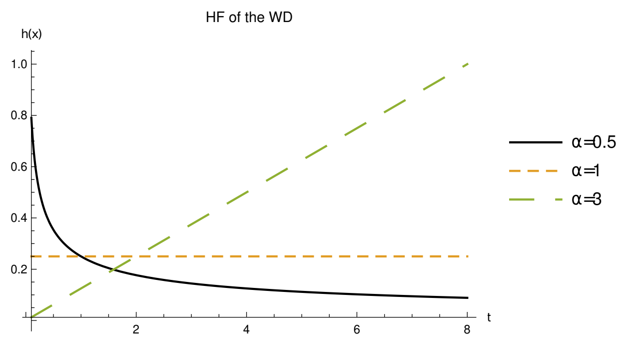

Figure 2. HF of the WD. The HF of the WD with scale parameter $\beta = 4$ and different values for the shape parameter α.

$$h(t; \alpha, \beta) = \alpha\beta^{-\alpha}t^{-1+\alpha} \qquad t > 0. \tag{2}$$

The HF of the WD given in (2) cannot be used to model life time data with a bathtub shaped hazard function, since it is increasing, decreasing, or constant as shown in Figure 2. This is considered a drawback for the WD.
The Inverse Weibull distribution (IWD), also known as the Frechet distribution (Johnson et al. (1994)), is used to model a variety of failure characteristics such as infant mortality, useful life, and wear-out periods (Kim and Han, 2010).

The HF of the IWD given in (4), is uni-modal (see Figure 3). Having a uni-modal hazard function is essential in many practical situations where the risk increases and then decreases as the study continues, like the process of recovery after a patient undergoes a surgery.

If $X = \frac{1}{T}$, then X follows the IWD with PDF, cumulative distribution function (CDF), and HF given by (3), (4), and (5), respectively.

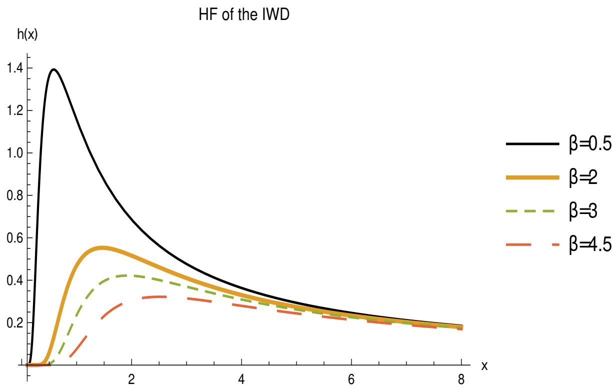

Figure 3. HF of the IWD. HF of the IWD with a shape parameter $\alpha = 1.5$ and different values for the scale parameter β.

$$
\begin{aligned}
f(x;\alpha,\beta) &= \alpha\beta e^{-\beta x^{-\alpha}} x^{-1-\alpha} \qquad x>0, \alpha, \beta > 0, & (3)\\
F(x;\alpha,\beta) &= e^{-\beta x^{-\alpha}} \qquad x>0, & (4)\\
h(x;\alpha,\beta) &= \frac{\alpha\beta x^{-\alpha-1}}{e^{\beta x^{-\alpha}}-1} & (5)
\end{aligned}
$$

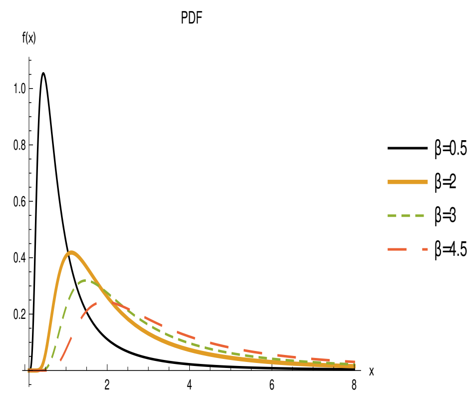

Figure 4. PDF of the IWD when $\alpha = 1.5$

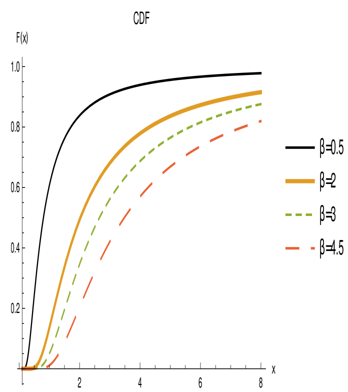

Figure 5. CDF of the IWD when $\alpha = 1.5$

Many studies have considered the IWD under progressive censoring; for example, Marusic et al. (2010), estimated the unknown parameters of the three-parameter IWD and, as a result, obtained a theorem on the existence of the least squares estimates, Musleh and Helu (2014) considered statistical inferences about the unknown parameters of the IWD based on progressively Type-II censoring using. For recent references see Shawky and Khan (2022).

Ubiquitous applications on the adaptive Type-II progressive hybrid censoring can be found in the literature due to its importance and efficacy in rel-life applications. Most recently, Asadi, Saeid, et al. (2022) implemented an adaptive Type-II progressive hybrid censored accelerated life test for the average diameter of virus containing micro droplets data to detect the persistence of the Virus-MD in a single cough at different time points. A chemical application is used to test lifetimes (in cycles) of sodium sulfur for twenty batteries using the XLindley distribution (Alotaibi, R., et al. (2022)). Panahi, H., and Asadi, P. (2022) used the generalized inverted exponential distribution to estimate the micro plasma spread factor under the adaptive Type-II progressive hybrid censoring. Helu and Samawi (2021) employed the adaptive Type-II hybrid progressive censoring for a data representing radar-evaluated rainfall from 52 south Florida cumulus clouds. The stress-strength model has been the subject of a considerable investigation by researchers. Birnbaum (1956) was the first to combine the stress-strength model with the Man-Whitney statistic to estimate θ in situations where X and Y are independent. Further studies have been carried out that provide point and interval estimations of θ using multiple techniques. Kotz and Pensky (2003) provided a comprehensive overview of the development of stress-strength reliability and its applications through 2003. Kumar, S. (2021) considered Chen distribution and derived UMVUEs and MLEs for the unknown parameters, hazard rate h (t) and two measures of reliability under type II censoring scheme. In recent research, Helu and Samawi (2022) used both classical and Bayesian methods to derive estimates of θ when X and Y are dependent random variables from a Bivariate Lomax distribution based on a progressive Type II censored sample. Musleh et al. (2022) derived a kernel estimate of θ and examined its properties based on progressively Type-II censored data.

Although reliability is applicable to many real-life situations, not many estimate it with the adaptive Type-II progressive hybrid censoring under the IWD and hence the main objective of this study is to compare classical and Bayesian approaches to estimating $\theta = P(Y < X)$, where X and Y are two independent IWD variables under adaptive hybrid progressive censoring with the same shape parameter and different scale parameters. It is observed that the MLEs can not be obtained in a closed form. In this case, we suggest using the AMLE. We derive the AMLE by expanding the normal equations using Taylor approximation method. Although the MLE appears to be the most popular method from a theoretical perspective, the least square method is computationally easier to handle and provides simple closed-form solutions for the estimates (Hossain and Zimmer (2003)). Further, we consider the Bayesian estimates based on squared error (SEL) and LINEX loss functions as examples of symmetric and asymmetric loss functions, respectively. Bayesian methods have the disadvantage that they cannot be expressed explicitly, so rather than applying numerical techniques, an approximation method, such as the Lindley approximation, is utilized.

This article is organized as follows: In Section 2, the classical estimation methods for θ are derived, namely MLE, AMLE, and LSE. In Section 3, the Bayes estimates are provided. In Section 4, we present the simulation study and results. In Section 5, two real-life data examples are provided. Finally, conclusions and recommendations are presented in Section 6.

2. Classical estimation methods

2.1. Maximum likelihood estimation

Suppose $X \sim IWD(\alpha, \beta_1)$ and $Y \sim IWD(\alpha, \beta_2)$ are two independent random variables representing the strength and stress components, respectively. Then it can be easily seen that

$$\theta = P(Y < X) = \frac{\beta_1}{\beta_1 + \beta_2} \tag{6}$$

Let $\mathbf{X} = X_{1:m_1:n_1} < X_{2:m_1:n_1} < ... < X_{m_1:m_1:n_1}$ be an adaptive Type-II hybrid progressive censoring sample from $IWD(\alpha, \beta_1)$ under the censoring scheme $\{n_1, m_1, R_1, ..., R_{J_1}, 0, ..., 0, R_{m_1} = n_1 - m_1 - \sum_{i=1}^{J_1} R_i\}$ such that $X_{J_1:m_1:n_1} < T_1 < X_{J_1+1:m_1:n_1}$. And, $\mathbf{Y} = \{Y_{1:m_2:n_2} < Y_{2:m_2:n_2} < ... < Y_{m_2:m_2:n_2}\}$ be an adaptive Type-II hybrid progressive censoring sample from $IWD(\alpha, \beta_2)$ under the scheme $\{n_2, m_2, S_1, ..., S_{J_2}, 0, ..., 0, S_{m_2} = n_2 - m_2 - \sum_{i=1}^{J_2} S_i\}$ such that $Y_{J_{2:m_2:n_2}} < T_2 < Y_{J_2+1:m_2:n_2}$. For simplicity, let $X_i = X_{i:m_1:n_1}$ and $Y_i = Y_{i:m_2:n_2}$. Then the joint likelihood function of the adaptive Type-II hybrid progressively censored sample (see Balakrishnan and Cramer, 2014) can be written as

$$L(\alpha,\beta_1,\beta_2|\mathbf{X},\mathbf{Y}) = C_1C_2[1-F_1(x_{m_1})]^{R_{m_1}}\prod_{i=1}^{m_1} f_1(x_i)\prod_{i=1}^{J_1}[1-F_1(x_i)]^{R_i}$$
$$[1-F_2(y_{m_2})]^{S_{m_2}}\prod_{i=1}^{m_2} f_2(y_i)\prod_{i=1}^{J_2}[1-F_2(y_i)]^{S_i}, \tag{7}$$

where,

$$C_1 = n_1(n_1-R_1-1)(n_1-R_1-R_2-2)...(n_1-R_1-R_2-...-R_{m_1-1}-m_1+1)$$

$$C_2 = n_2(n_2-S_1-1)(n_2-S_1-S_2-2)...(n_2-S_1-S_2-...-S_{m_2-1}-m_2+1)$$

$$\begin{aligned}
f_1(x;\alpha,\beta_1) &= \alpha\beta_1 e^{-\beta_1 x^{-\alpha}}x^{-1-\alpha} \qquad x>0,\alpha,\beta_1>0. && (8)\\
F_1(x;\alpha,\beta_1) &= e^{-\beta_1 x^{-\alpha}} \qquad x>0 && (9)\\
f_2(y;\alpha,\beta_2) &= \alpha\beta_2 e^{-\beta_2 y^{-\alpha}}y^{-1-\alpha} \qquad y>0,\alpha,\beta_2>0 && (10)\\
F_2(y;\alpha,\beta_2) &= e^{-\beta_2 y^{-\alpha}} \qquad y>0 && (11)
\end{aligned}$$

After substituting Eqs. 8 - 11 into Eq. 7, then taking the log-likelihood function, we get the following:

$$\begin{aligned}
l &= \ln C_1 + \ln C_2 + (m_1+m_2)\ln\alpha + m_1\ln\beta_1 + m_2\ln\beta_2\\
&+ R_{m_1}\ln(1-e^{-\beta_1 x_{m_1}^{-\alpha}}) - (1+\alpha)\sum_{i=1}^{m_1} ln(x_i) - \beta_1\sum_{i=1}^{m_1} x_i^{-\alpha}\\
&+ \sum_{i=1}^{J_1} R_i\ln(1-e^{-\beta_1 x_i^{-\alpha}}) + S_{m_2}\ln(1-e^{-\beta_2 y_{m_2}^{-\alpha}})\\
&- (1+\alpha)\sum_{i=1}^{m_2}\ln(Y_i) - \beta_2\sum_{i=1}^{m_2} y_i^{-\alpha} + \sum_{i=1}^{J_2} s_i\ln(1-e^{-\beta_2 y_i^{-\alpha}}).
\end{aligned} \tag{12}$$

If the shape parameter α is known, the MLEs of the parameters β_1 and β_2 can be obtained by deriving (12) with respect to β_1 and β_2 and equating the normal equations to 0 as follows:

$$\frac{\partial l}{\partial\beta_1} = \frac{m_1}{\beta_1} + \frac{e^{-\beta_1 x_{m_1}^{-\alpha}}R_{m_1}x_{m_1}^{-\alpha}}{1-e^{-\beta_1 x_{m_1}^{-\alpha}}} - \sum_{i=1}^{m_1} x_i^{-\alpha} + \sum_{i=1}^{J_1}\frac{e^{-\beta 1 x_i^{-\alpha}}R_i x_i^{-\alpha}}{1-e^{-\beta_1 x_i^{-\alpha}}} = 0, \tag{13}$$

$$\frac{\partial l}{\partial\beta_2} = \frac{m_2}{\beta_2} + \frac{e^{-\beta_2 y_{m_2}^{-\alpha}}S_{m_2}y_{m_2}^{-\alpha}}{1-e^{-\beta_2 y_{m_2}^{-\alpha}}} - \sum_{i=1}^{m_2} y_i^{-\alpha} + \sum_{i=1}^{J_2}\frac{e^{-\beta 2 y_i^{-\alpha}}S_i y_i^{-\alpha}}{1-e^{-\beta_2 y_i^{-\alpha}}} = 0. \tag{14}$$

It is noted that Eqs. (13) and (14) do not yield explicit forms. Therefore, we apply numerical methods using Mathematica 12 to find $\hat{\beta}_{1_{MLE}}$ and $\hat{\beta}_{2_{MLE}}$ and hence, $\hat{\theta}_{MLE}$.

2.2. *Approximate maximum likelihood method*

As seen in equations (13) and (14), the MLEs do not provide explicit forms, thus we derive the approximate MLE. The AMLE method, which was developed by Balakrishnan (1989a), uses Taylor series expansion to the PDF of

location-scale families, but the IWD model is not a location-scale distribution. To solve this issue, we consider the transformations
$W = -\ln X$ and $W^* = -\ln Y$ to get

$$F_W(w) = 1 - e^{-\beta_1 e^{w\alpha}}, \tag{15}$$

$$F_{W^*}(w^*) = 1 - e^{-\beta_2 e^{w^*\alpha}}. \tag{16}$$

Next, let $\beta_1 = e^{-\mu_1\alpha}$, $\beta_2 = e^{-\mu_2\alpha}$ and $\alpha = \frac{1}{\sigma}$, then based on equations (15) and (16), W and W^* both have the Extreme Value Distribution (EVD), namely $EVD(\mu_1, \sigma)$ and $EVD(\mu_2, \sigma)$ respectively. Hence, the EVD is a location-scale distribution. Let $t_i = \frac{w_i - \mu_1}{\sigma}$ and $z_i = \frac{w_i^* - \mu_2}{\sigma}$ then, if $\mathbf{T} = (t_1, \dots, t_m)$ and $\mathbf{Z} = (z_1, \dots, z_k)$. The likelihood and the Log-likelihood functions (ignoring the additive constants) are given in equations (17) and (18), respectively.

$$l^*(\mathbf{T}, \mathbf{Z}|\mu_1, \mu_2, \sigma) = e^{-e^{t_{m_1}} R_{m_1}} \prod_{i=1}^{m_1} \frac{e^{t_i - e^{t_i}}}{\sigma} \prod_{i=1}^{J_1} e^{-e^{t_i} R_i} e^{-e^{z_{m_2}} S_{m_2}} \prod_{i=1}^{m_2} \frac{e^{z_i - e^{z_i}}}{\sigma} \prod_{i=1}^{J_2} e^{-e^{z_i} S_i}, \tag{17}$$

$$\begin{aligned} l^*(\mathbf{T}, \mathbf{Z}|\mu_1, \mu_2, \sigma) = {} & -R_{m_1} e^{t_{m_1}} - m_1 \ln\sigma + \sum_{i=1}^{m_1} t_i - \sum_{i=1}^{m_1} e^{t_i} - \sum_{i=1}^{J_1} R_i e^{t_i} \\ & - S_{m_2} e^{z_{m_2}} - m_2 \ln\sigma + \sum_{i=1}^{m_2} z_i - \sum_{i=1}^{m_2} e^{z_i} - \sum_{i=1}^{J_2} S_i e^{Z_i}. \end{aligned} \tag{18}$$

Taking the derivatives with respect to μ_1 and μ_2 and equating them to zero, we obtain the following:

$$\frac{\partial l^*}{\partial \mu_1} = \frac{-(m_1 - e^{t_{m_1}} R_{m_1} - \sum_{i=1}^{m_1} e^{t_i} - \sum_{i=1}^{J_1} e^{t_i} R_i)}{\sigma} = 0 \tag{19}$$

$$\frac{\partial l^*}{\partial \mu_2} = \frac{-(m_2 - e^{z_{m_2}} S_{m_2} - \sum_{i=1}^{m_2} e^{z_i} - \sum_{i=1}^{J_2} e^{z_i} S_i)}{\sigma} = 0 \tag{20}$$

Notice that Eqs. (19) and (20) do not yield explicit forms. Therefore, first-order Taylor approximation is implemented by expanding the terms e^{t_i} and e^{z_i} in the likelihood function around the points $v_i = \ln(1 - \ln(1 - q_i))$ and $v_i^{'} = \ln(1 - \ln(1 - q_i^{'}))$, respectively.
Where,

$$q_i = 1 - \prod_{j=m_1-i+1}^{m_1} \frac{j + \sum_{l=m_1-j+1}^{m_1} R_l}{(1 + j + \sum_{l=m_1-j+1}^{m_1} R_l)}, \qquad i = 1, 2 \dots, m_1 \tag{21}$$

$$q_i^{'} = 1 - \prod_{j=m_2-i+1}^{m_2} \frac{j + \sum_{l=m_2-j+1}^{m_2} S_l}{(1 + j + \sum_{l=m_2-j+1}^{m_2} S_l)}, \qquad i = 1, 2 \dots, m_2 \tag{22}$$

Next, expand $g_1(t_i) = e^{t_i}$ and $g_2(z_i) = e^{z_i}$ around the points v_i and $v_i^{'}$ using first-order Taylor expansions to get

$$\begin{aligned} & g_1(t_i) \simeq g_1(v_i) + g_1^{'}(v_i)(t_i - v_i) \\ & e^{t_i} \simeq \gamma_i + \phi_i t_i, \end{aligned} \tag{23}$$

$$
\begin{aligned}
&g_2(z_i) \simeq g_2(v_i^{'}) + g_2^{'}(v_i^{'})(z_i - v_i^{'}) \\
&e^{z_i} \simeq \tilde{\gamma_i} + \tilde{\phi}_i z_i.
\end{aligned} \tag{24}
$$

Where, $\gamma_i = e^{v_i}(1 - v_i), \phi_i = e^{v_i}$ and $\tilde{\gamma_i} = e^{v_i^{'}}(1 - v_i^{'}), \tilde{\phi_i} = e^{v_i^{'}}$, using equations (23) and (24), the log-likelihood equations (19) and (20) are approximated as follows:

$$
\frac{\partial l^*}{\partial \mu_1} \simeq -\frac{(m_1 - R_{m_1}(\gamma_{m_1} + \phi_{m_1} t_{m_1}) - \sum_{i=1}^{m_1}(\gamma_i + \phi_i t_i) - \sum_{i=1}^{J_1}(\gamma_i + \phi_i t_i) R_i)}{\sigma} = 0 \tag{25}
$$

$$
\frac{\partial l^*}{\partial \mu_2} \simeq -\frac{(k - S_{m_2}(\tilde{\gamma_{m_2}} + \tilde{\phi_{m_2}} z_{m_2}) - \sum_{i=1}^{m_2}(\tilde{\gamma_i} + \tilde{\phi}_i z_i) - \sum_{i=1}^{J_2}(\tilde{\gamma_i} + \tilde{\phi}_i z_i) S_i)}{\sigma} = 0 \tag{26}
$$

By solving equations (25) and (26), we get

$$
\hat{\mu}_{1_{\text{AMLE}}} = A_1 + B_1 \sigma \tag{27}
$$

$$
\hat{\mu}_{2_{\text{AMLE}}} = A_2 + B_2 \sigma \tag{28}
$$

where,

$$
A_1 = \frac{R_{m_1}\phi_{m_1} w_{m_1} + \sum_{i=1}^{m_1} \phi_i w_i + \sum_{i=1}^{J1} R_i \phi_i w_i}{R_{m_1}\phi_{m_1} + \sum_{i=1}^{m_1} \phi_i + \sum_{i=1}^{J_1} R_i \phi_i}, \; B_1 = \frac{m_1 - (R_{m_1}\gamma_{m_1}) - \sum_{i=1}^{m_1} \gamma_i - \sum_{i=1}^{J_1} R_i \gamma_i}{R_{m_1}\phi_{m_1} + \sum_{i=1}^{m_1} \phi_i + \sum_{i=1}^{J_1} R_i \phi_i}
$$

$$
A_2 = \frac{S_{m_2}\tilde{\phi}_{m_2} w_i^* + \sum_{i=1}^{m_2} \tilde{\phi}_i w_i^* + \sum_{i=1}^{J_2} S_i \tilde{\phi}_i w_i^*}{S_{m_2}\tilde{\phi}_{m_2} + \sum_{i=1}^{m_2} \tilde{\phi}_i + \sum_{i=1}^{J_2} S_i \tilde{\phi}_i}, \; B_2 = \frac{m_2 - (S_{m_2}\tilde{\gamma}_{m_2}) - \sum_{i=1}^{m_2} \tilde{\gamma}_i - \sum_{i=1}^{J_2} S_i \tilde{\gamma}_i}{S_{m_2}\tilde{\phi}_{m_2} + \sum_{i=1}^{m_2} \tilde{\phi}_i + \sum_{i=1}^{J_2} S_i \tilde{\phi}_i}
$$

Hence, AMLEs of β_1 and β_2 are used to obtain the AMLE of θ. Where,

$$
\hat{\beta}_{1_{\text{AMLE}}} = e^{-\alpha \hat{\mu}_{1_{\text{AMLE}}}}, \; \hat{\beta}_{2_{\text{AMLE}}} = e^{-\alpha \hat{\mu}_{2_{\text{AMLE}}}}, \; \hat{\theta}_{AMLE} = \frac{\hat{\beta}_{1_{\text{AMLE}}}}{\hat{\beta}_{1_{\text{AMLE}}} + \hat{\beta}_{2_{AMLE}}}.
$$

2.3. Least squares method

Using the same settings as in Section 2, we aim to use a combination of non-parametric and parametric CDFs to find the LSE (see Marusic et al. (2010)).

For $\mathbf{X} = \{X_1, X_2, \ldots, X_{m_1}\}$, the LSE of β_1 is obtained by minimizing $\sum_{i=1}^{m_1}(F_1(X_i) - \hat{F}_1(X_i))^2$, such that $\hat{F}_1(X_i)$ is the non-parametric distribution function proposed by Cacciari & Montanari(1987), which is given by Eq. 29.

$$
\hat{F}_1(X_j) = \frac{K_j - 0.5}{n_1 + 0.25}, \tag{29}
$$

where,

$$
K_j = K_{j-1} + \Delta, \qquad j = 1, 2, ..., m_1 \quad and \quad K_0 = 0,
$$

and

$$
\Delta = \frac{n_1 + 1 - K_{j-1}}{(m_1 - j + 2 + R_j + R_{j+1} + ... + R_{m_1})}.
$$

$F_1(X_i)$ is the parametric distribution function proposed by Kim & Han (2010) which is given by

$$F_1(X_j) = 1 - l_{j-1} \sum_{i=1}^{j} \frac{a_{i,j}}{r_i} (1 - F(x_j))^{r_i}, \qquad j = 1, 2, ..., m_1, \tag{30}$$

where,
where,

$$\begin{array}{rl} l_{j-1} = \prod_{u=1}^{j} r_u, & 1 \leq j \leq m_1, \\ r_i = m_1 - i + 1 + \sum_{u=1}^{m_1} R_u, & 1 \leq i \leq m_1, \\ a_{i,j} = \prod_{u=1, u \neq i}^{j} \frac{1}{r_u - r_i}, & 1 \leq i \leq j \leq m_1. \end{array}$$

Similarly, for $\mathbf{Y} = \{Y_1, Y_2, \ldots, Y_{m_2}\}$. Upon obtaining the least squares estimates $\hat{\beta}_{1_{\text{LSE}}}$ and $\hat{\beta}_{2_{\text{LSE}}}$, the LSE of θ can be obtained as follows

$$\hat{\theta}_{\text{LSE}} = \frac{\hat{\beta}_{1_{\text{LSE}}}}{\hat{\beta}_{1_{\text{LSE}}} + \hat{\beta}_{2_{\text{LSE}}}}. \tag{31}$$

3. Bayesian estimation

In this section, we derive the Bayes estimate of θ using symmetric and asymmetric loss functions. A commonly used loss function is the squared error loss function (SEL).

$$L(\hat{\gamma}, \gamma) = (\hat{\gamma} - \gamma)^2, \tag{32}$$

The Bayesian estimate under (32) is the posterior mean. Given by $\hat{\gamma}_{SEL} = E_\pi \gamma$. The SEL is widely used in Bayesian inference due to its computational simplicity. It is a symmetric loss function that is equally adverse to overestimation and underestimation. Practically, however, this is not a very useful criterion. In estimating reliability and failure rate functions, an overestimation causes more damage than an underestimation. For example, Feynman (1987) argued that in the Challenger disaster, management may have overestimated the average life or reliability of solid-fuel rocket boosters. To resolve such situation, asymmetrical loss functions are more appropriate. Varian (1975) introduces the LINEX loss function (Linear- Exponential) in response to the criticism of the SEL. The LINEX loss function is defined as follows:

$$L(\hat{\gamma}, \gamma) = \exp(\lambda(\hat{\gamma} - \gamma)) - \lambda(\hat{\gamma} - \gamma) - 1, \ \lambda \neq 0. \tag{33}$$

The magnitude of λ reflects the degree of symmetry while the sign of λ reflects the direction of symmetry. Zellner (1986) obtained the Bayesian estimator under LINEX loss function by minimizing the posterior expected loss, provided that $E_\pi(e^{-\lambda\theta})$ exists and finite.

$$\widehat{\gamma}_{LIN} = -\frac{1}{\lambda} \ln E_\pi(e^{-\lambda\gamma}). \tag{34}$$

The LINEX loss function is suitable for situations where overestimation may lead to serious consequences, and it is known for its flexibility and popularity. A common feature of lifetime distributions with a shape parameter is that the Bayes estimators cannot be expressed in closed forms. We suggest using Lindley's approximation to derive the Bayes estimators the reliability model θ.

3.1. Lindley's approximation method

It is assumed that the shape parameter α is known and β_1 and β_2 have two independent gamma priors

$$\pi(\beta_1) = \frac{b^a}{\Gamma(a)} \beta_1^{a-1} e^{-b\beta_1}, \qquad \pi(\beta_2) = \frac{d^c}{\Gamma(c)} \beta_2^{c-1} e^{-d\beta_2}, \tag{35}$$

where a,b,c,d are non-negative known parameters. The joint prior distribution is given by

$$\pi(\beta_1,\beta_2)=\frac{b^a d^c}{\Gamma(a)\Gamma(c)}\beta_1^{a-1}\beta_2^{c-1}e^{-(b\beta_1+d\beta_2)}, \tag{36}$$

Based on the likelihood function (12) and equation (36), the joint posterior distribution, given the adaptive censored data described in Section 2, is as follows

$$\pi(\beta_1,\beta_2|\mathbf{X},\mathbf{Y})=\frac{L(\alpha,\beta_1,\beta_2|\mathbf{X},\mathbf{Y})\pi(\beta_1,\beta_2)}{\int_0^\infty\int_0^\infty L(\alpha,\beta_1,\beta_2|\mathbf{X},\mathbf{Y})\pi(\beta_1,\beta_2)d\beta_2 d\beta_1}. \tag{37}$$

Therefore, the Bayes estimators of any function of $\beta_1$andβ_2 say $u(\beta_1,\beta_2)$ are the posterior expected value of $u(\beta_1,\beta_2)$ and is given by

$$\hat{u}=E_\pi(u(\beta_1,\beta_2)|\mathbf{X},\mathbf{Y})=\frac{\int_0^\infty\int_0^\infty u(\beta_1,\beta_2)e^{(\rho(\beta_1,\beta_2))}e^{l(\mathbf{X},\mathbf{Y}|\beta_1,\beta_2)}d\beta_2 d\beta_1}{\int_0^\infty\int_0^\infty e^{\rho(\beta_1,\beta_2)}e^{l(\mathbf{X},\mathbf{Y}|\beta_1,\beta_2)}d\beta_2 d\beta_1} \tag{38}$$

where, $\rho(\beta_1,\beta_2)=\ln\pi(\beta_1,\beta_2)$ and, $l(\mathbf{X},\mathbf{Y}|\alpha,\beta_1,\beta_2)=\ln L(\mathbf{X},\mathbf{Y}|\alpha,\beta_1,\beta_2)$.

It can be noticed that $\hat{u}$ cannot be simplified in a closed form. Hence, Lindley (1980) provided asymptotic solution. Lindley's approximation method which is used to find an asymptotic solution for the ratio of two integrals often encountered for various lifetime distributions in Bayesian estimation (Sharma et al. (2015)). Thus, Eq. 38 is reduced to the following expression:

$$\begin{aligned}\hat{u}\simeq& u(\hat{\beta}_{1_{MLE}},\hat{\beta}_{2_{MLE}})+\frac{1}{2}((\hat{u}_{\beta_1\beta_1}+2\hat{u}_{\beta_1}\hat{\rho}_{\beta_1})\hat{\sigma}_{\beta_1\beta_1}+(\hat{u}_{\beta_2\beta_2}+2\hat{u}_{\beta_2}\hat{\rho}_{\beta_2}\hat{\sigma}_{\beta_2\beta_2}))\\&+\frac{1}{2}(\hat{u}_{\beta_1}(\hat{\sigma}_{\beta_1\beta_1})^2\hat{l}_{\beta_1\beta_1\beta_1}+\hat{u}_{\beta_2}(\hat{\sigma}_{\beta_2\beta_2})^2\hat{l}_{\beta_2\beta_2\beta_2}),\end{aligned} \tag{39}$$

where, $\hat{\beta}_1$ and $\hat{\beta}_2$ are the MLEs of β_1 and β_2 respectively,

$$\hat{u}_{\beta_1\beta_1}=\frac{\partial^2 u(\beta_1,\beta_2)}{\partial\beta_1\partial\beta_1}|_{(\hat{\beta}_1,\hat{\beta}_2)},\quad \hat{\rho}_{\hat{\beta}_1}=\frac{(a-1)}{\hat{\beta}_1}-b,\quad \hat{\rho}_{\beta_2}=\frac{(c-1)}{\hat{\beta}_2}-d.$$

Other expressions can be defined similarly (see the Appendix).

3.2. *Squared error loss function*

If $u(\beta_1,\beta_2)=\beta_1$, then $u_{\beta_1}=1$, $u_{\beta_1}=u_{\beta_1\beta_1}=u_{\beta_1\beta_2}=u_{\beta_2\beta_2}=u_{\beta_2\beta_1}=0$, and

$$\hat{\beta}_{1_{SEL}}=\hat{\beta}_1+\hat{\rho}_{\beta_1}\hat{\sigma}_{\beta_1\beta_1}+\frac{1}{2}(\hat{\sigma}_{\beta_1\beta_1})^2\hat{l}_{\beta_1\beta_1\beta_1} \tag{40}$$

Similarly, if $u(\beta_1,\beta_2)=\beta_2$, $u_{\beta_2}=1$, then $u_{\beta_2}=u_{\beta_2\beta_2}=u_{\beta_2\beta_1}=u_{\beta_1\beta_1}=u_{\beta_1\beta_2}=0$, and

$$\hat{\beta}_{2_{SEL}}=\hat{\beta}_2+\hat{\rho}_{\beta_2}\hat{\sigma}_{\beta_2\beta_2}+\frac{1}{2}(\hat{\sigma}_{\beta_2\beta_2})^2\hat{l}_{\beta_2\beta_2\beta_2} \tag{41}$$

Hence,

$$\hat{\theta}_{\text{SEL}}=\frac{\hat{\beta}_{1_{SEL}}}{\hat{\beta}_{1_{SEL}}+\hat{\beta}_{2_{SEL}}}. \tag{42}$$

3.3. LINEX loss function

If $u(\beta_1,\beta_2)=e^{-\lambda\beta_1}$, $u_{\beta_1}=-\lambda e^{-\lambda\beta_1}$, $u_{\beta_1\beta_1}=-\lambda^2 e^{-\lambda\beta_1}$, $u_{\beta_2}=u_{\beta_2\beta_2}=u_{\beta_2\beta_1}=u_{\beta_1\beta_2}=0$. Then,

$$\begin{aligned}\hat{\beta}_{1_{LINEX}} &= \frac{-1}{\lambda}\ln\left[u(\hat{\beta}_1,\hat{\beta}_2)+\frac{1}{2}(\hat{u}_{\beta_1\beta_1}+2\rho_{\beta_1}\hat{u}_{\beta_1})\hat{\sigma}_{\beta_1\beta_1}+\frac{1}{2}(\hat{u}_{\beta_1}\hat{\sigma}^2_{\beta_1\beta_1}\hat{l}_{\beta_1\beta_1\beta_1})\right]\\ &= \frac{-1}{\lambda}\ln[e^{-\lambda\hat{\beta}_1}+\frac{1}{2}(\lambda^2 e^{-\lambda\hat{\beta}_1}-2\hat{\rho}_{\beta_1}\lambda e^{-\lambda\hat{\beta}_1})\hat{\sigma}_{\beta_1\hat{\beta}_1}-\frac{1}{2}(\lambda e^{-\lambda\hat{\beta}_1}\hat{\sigma}^2_{\beta_1\hat{\beta}_1}\hat{l}_{\beta_1\beta_1\beta_1})]\end{aligned} \tag{43}$$

Proceeding similarly, if $u(\beta_1,\beta_2)=e^{-\lambda\beta_2}$, $u_{\beta_2}=-\lambda e^{-\lambda\beta_2}$, $u_{\beta_2\beta_2}=-\lambda^2 e^{-\lambda\beta_2}$,$u_{\beta_1}=u_{\beta_1\beta_1}=u_{\beta_1\beta_2}=u_{\beta_2\beta_1}=0$. Then,

$$\begin{aligned}\hat{\beta}_{2_{LINEX}} &= \frac{-1}{\lambda}\ln\left[u(\hat{\beta}_1,\hat{\beta}_2)+\frac{1}{2}(\hat{u}_{\beta_2\beta_2}+2\rho_{\beta_2}\hat{u}_{\beta_2})\hat{\sigma}_{\beta_2\beta_2}+\frac{1}{2}(\hat{u}_{\beta_2}\hat{\sigma}^2_{\beta_2\beta_2}\hat{l}_{\beta_2\beta_2\beta_2})\right]\\ &= \frac{-1}{\lambda}\ln[e^{-\lambda\hat{\beta}_2}+\frac{1}{2}(\lambda e^{-\lambda\hat{\beta}_2}-2\hat{\rho}_{\beta_2}\lambda e^{-\lambda\hat{\beta}_2})\hat{\sigma}_{\beta_2\hat{\beta}_2}-\frac{1}{2}(\lambda e^{-\lambda\hat{\beta}_2}\hat{\sigma}^2_{\beta_2\hat{\beta}_2}\hat{l}_{\beta_2\beta_2\beta_2})]\end{aligned} \tag{44}$$

Hence,

$$\hat{\theta}_{\text{LINEX}}=\frac{\hat{\beta}_{1_{LINEX}}}{\hat{\beta}_{1_{LINEX}}+\hat{\beta}_{2_{LINEX}}}. \tag{45}$$

4. Simulation Study and Conclusions

4.1. Simulation Criteria

In this section, we undertake a simulation study to test the performance of the various estimation methods that we have discussed previously.

1. Values of β_1 and β_2 are generated from π_1 and π_2 as given in equations (35) using pre-specified values of a, b, c, and d. See Yadav et al.(2018).
2. Generate two independent Type-II censored samples $X_1,X_2,...,X_{m_1}$ and $Y_1,Y_2,...,Y_{m_2}$ from the IWD distribution with shape parameter α and scale parameters β_1 and β_2 respectively with censoring schemes $(R_1,R_2,\ldots,R_{m_1})$ and $(S_1,S_2,\ldots,S_{m_2})$.
3. Determine the values of J_1 and J_2 , such that $X_{J_1}<T_1<X_{J_1+1}$ and $Y_{J_2}<T_2<Y_{J_2}$, then remove $X_{J_1+2},\ldots,X_{m_1}$ and $Y_{J_2+2},\ldots,Y_{m_2}$.
4. Generate the first $m-j_1-1$ order statistics from the truncated distribution $\frac{f(x)}{1-F(x_{J_1}+1)}$ as $X_{J_1+2},\ldots,X_{m_1}$ and the censoring scheme will change to $(R_1,\ldots,R_{J_1},0,\ldots,0,R_{m_1}=n_1-m_1-\sum_{i=1}^{J_1}R_i)$. Similarly, we generate $Y_{J_2+2},\ldots,Y_{m_2}$ with modified censoring scheme $(S_1,\ldots,S_{J_2},0,\ldots,0,S_{m_2}=n_2-m_2-\sum_{i=1}^{J_2}S_i)$.
5. Obtain the MLE, AMLE, LSE, and Bayes estimates of the model parameters using iterative process when $\lambda=1$ and $\lambda=-1$.
6. Define the Bias and the MSE of the estimates of (θ) as follows:
   - Bias = $\left|\bar{\theta}_i-\theta_{exact}\right|$, where $\bar{\theta}_i$ is the average of the 3000 values of $\bar{\theta}_i$ for each one of the estimators.
   - MSE =$\frac{\sum_{i=1}^{3000}(\hat{\theta}_i-\theta_{exact})^2}{3000}$

In this study, the simulation has been performed by considering the shape parameter $\alpha=1.5$, and three cases for the hyper parameters: $(a,b,c,d)=(2,1,3,1)$; $(3,1,2,1)$; $(4.5,1,1,2)$. The resulting values of θ_{exact} are approximately 0.4, 0.6, and 0.9 respectively. Three main stopping times are considered: $T_1=X_{\lfloor\frac{m}{4}\rfloor}$, $T_2=X_{\lfloor\frac{4m}{5}\rfloor}$, and $T_3=X_m+2$, with the following censoring schemes (C.s):

- C.s 1: $\{n-m, 0^{*(m-1)}\}$, which is known as First-step censoring, i.e., n-m units are removed just after the first failure
- C.s 2: $\{0^{*(m-1)}, n-m\}$, which is known as Right censoring, i.e., n-m units are removed after the last failure
- C.s 3: $\{\frac{n-m}{2}, 0^{*(m-2)}, \frac{n-m}{2}\}$ When removals take place at the beginning and at the end of the experiment, i.e., $\frac{n-m}{2}$ units are withdrawn just after the first failure and after the last failure

The sample sizes of the strength and stress components are chosen to be $n = n_1 = n_2 = \{20, 40, 60, 100\}$ and $m = m_1 = m_2 = \{4, 10\}$ when n=20, $\{12, 20\}$ when n=40, $\{18, 30\}$ when n=60, and $\{20, 50\}$ when n=100. Results are summarized in tables 1-6 as follows: Tables 1-3 provide the estimates at three stopping times and three censoring schemes when $\theta = 0.4$. Similarly for Tables 4-6 when $\theta = 0.9$.

4.2. Simulation Analysis and Results

Results are summarized as follows:

- The AMLE performs better under the third censoring scheme in most cases in terms of MSE.
- Bayes estimates perform better than all other estimates in general.
- When comparing Bayes estimates it is noted that the LINEX loss function with $\lambda = 1$ performs better than the SEL.
- In general, Bias and MSE of the calculated estimates decreases as effective sample sizes increases.
- It is noted that most estimates perform better under the second censoring scheme when removals take place after the m-th failure.
- It is noted that all estimators perform best when $\theta = 0.9$

Table 1. Bias and MSE of θ with $T1 = X_{\frac{m}{4}}$ for different censoring schemes when $\Theta = 0.4$

		Bias						MSE					
		Classical			Bayes			Classical			Bayes		
n,m	C.s	MLE	AMLE	LS	SEL	LINEX		MLE	AMLE	LS	SEL	LINEX	
						$\lambda = 1$	$\lambda = -1$					$\lambda = 1$	$\lambda = -1$
20,4	1	0.00732	0.00781	0.01348	0.00147	0.00789	0.00008	0.00774	0.01564	0.02227	0.00544	0.00429	0.00669
	2	0.00262	0.00364	0.00637	0.00094	0.00586	0.00282	0.00557	0.00610	0.01308	0.00423	0.00389	0.00490
	3	0.00370	0.00129	0.00872	0.00104	0.00678	0.00266	0.00657	0.00757	0.01468	0.00470	0.00416	0.00570
20,10	1	0.00252	0.00298	0.00367	0.00375	0.00045	0.00519	0.00414	0.00488	0.00721	0.00337	0.00292	0.00376
	2	0.00015	0.00085	0.00158	0.00349	0.00048	0.00598	0.00351	0.00356	0.00723	0.00293	0.00267	0.00324
	3	0.00232	0.00329	0.00286	0.00239	0.00179	0.00451	0.00389	0.00378	0.00767	0.00319	0.00284	0.00354
40,12	1	0.00051	0.00069	0.00163	0.00471	0.00160	0.00620	0.00310	0.00476	0.00767	0.00272	0.00246	0.00294
	2	0.00026	0.00097	0.00199	0.00162	0.00073	0.00352	0.00217	0.00288	0.00498	0.00195	0.00186	0.00206
	3	0.00068	0.00096	0.00587	0.00164	0.00119	0.00361	0.00248	0.00304	0.01081	0.00218	0.00204	0.00234
40,20	1	0.00145	0.00053	0.00373	0.00245	0.00029	0.00373	0.00216	0.00222	0.00376	0.00195	0.00183	0.00205
	2	0.00177	0.00191	0.00227	0.00026	0.00165	0.00173	0.00175	0.00220	0.00433	0.00160	0.00154	0.00167
	3	0.00012	0.00065	0.00147	0.00284	0.00069	0.00435	0.00192	0.00208	0.00407	0.00176	0.00166	0.00185
60,18	1	0.00083	0.00113	0.00369	0.00249	0.00041	0.00373	0.00232	0.00284	0.00532	0.00211	0.00199	0.00222
	2	0.00091	0.00095	0.00287	0.00011	0.00143	0.00142	0.00139	0.00227	0.00371	0.00130	0.00126	0.00134
	3	0.00011	0.00049	0.00705	0.00182	0.00008	0.00330	0.00170	0.00228	0.00700	0.00156	0.00149	0.00163
60,30	1	0.00055	0.00022	0.00036	0.00231	0.00083	0.00336	0.00148	0.00138	0.00261	0.00138	0.00133	0.00143
	2	0.00039	0.00140	0.00018	0.00172	0.00041	0.00282	0.00112	0.00160	0.00244	0.00106	0.00103	0.00109
	3	0.00072	0.00014	0.00421	0.00122	0.00021	0.00235	0.00132	0.00149	0.00360	0.00124	0.00120	0.00128
100,20	1	0.00112	0.00157	0.00314	0.00151	0.00019	0.00261	0.00190	0.00265	0.00513	0.00177	0.00168	0.00184
	2	0.00083	0.00143	0.00042	0.00030	0.00135	0.00065	0.00104	0.00216	0.00135	0.00099	0.00097	0.00101
	3	0.00175	0.00065	0.00267	0.00071	0.00208	0.00046	0.00134	0.00208	0.00197	0.00126	0.00123	0.00130
100,50	1	0.00152	0.00172	0.00199	0.00038	0.00053	0.00112	0.00091	0.00080	0.00131	0.00087	0.00085	0.00088
	2	0.00011	0.00135	0.00007	0.00098	0.00022	0.00166	0.00063	0.00128	0.00089	0.00061	0.00060	0.00063
	3	0.00029	0.00021	0.00069	0.00154	0.00068	0.00228	0.00074	0.00102	0.00103	0.00071	0.00070	0.00073

Table 2. Bias and MSE of Θ with $T2 = X_{\frac{4m}{5}}$ for different censoring schemes when $\Theta = 0.4$

		Bias						MSE					
		Classical			Bayes			Classical			Bayes		
n,m	C.s	MLE	AMLE	LS	SEL	LINEX		MLE	AMLE	LS	SEL	LINEX	
						$\lambda = 1$	$\lambda = -1$					$\lambda = 1$	$\lambda = -1$
20,4	1	0.00315	0.00622	0.00626	0.00210	0.00478	0.00390	0.00699	0.01223	0.02073	0.00493	0.00384	0.00608
	2	0.00317	0.00252	0.00576	0.00134	0.00628	0.00237	0.00548	0.00622	0.01294	0.00417	0.00383	0.00482
	3	0.00484	0.00366	0.00874	0.00194	0.00768	0.00161	0.00635	0.00580	0.01348	0.00453	0.00402	0.00549
20,10	1	0.00312	0.00321	0.00492	0.00328	0.00089	0.00466	0.00440	0.00573	0.00737	0.00358	0.00311	0.00399
	2	0.00274	0.00315	0.00442	0.00081	0.00306	0.00320	0.00338	0.00376	0.00687	0.00280	0.00256	0.00308
	3	0.00367	0.00364	0.00526	0.00116	0.00296	0.00322	0.00380	0.00375	0.00748	0.00310	0.00277	0.00344
40,12	1	0.00061	0.00030	0.00312	0.00367	0.00057	0.00514	0.00315	0.00494	0.00772	0.00275	0.00249	0.00297
	2	0.00010	0.00101	0.00072	0.00127	0.00108	0.00316	0.00212	0.00313	0.00468	0.00190	0.00182	0.00201
	3	0.00159	0.00206	0.00496	0.00080	0.00200	0.00274	0.00241	0.00292	0.01005	0.00212	0.00199	0.00227
40,20	1	0.00128	0.00120	0.00093	0.00264	0.00046	0.00393	0.00222	0.00271	0.00380	0.00200	0.00188	0.00211
	2	0.00027	0.00068	0.00092	0.00169	0.00025	0.00320	0.00168	0.00222	0.00425	0.00156	0.00150	0.00164
	3	0.00079	0.00176	0.00134	0.00197	0.00017	0.00346	0.00191	0.00211	0.00431	0.00174	0.00165	0.00183
60,18	1	0.00158	0.00175	0.00344	0.00179	0.00030	0.00304	0.00230	0.00316	0.00525	0.00209	0.00197	0.00220
	2	0.00156	0.00215	0.00393	0.00050	0.00203	0.00080	0.00138	0.00232	0.00375	0.00129	0.00125	0.00133
	3	0.00157	0.00108	0.00821	0.00021	0.00166	0.00166	0.00163	0.00163	0.00735	0.00149	0.00143	0.00156
60,30	1	0.00158	0.00175	0.00344	0.00179	0.00030	0.00304	0.00230	0.00316	0.00525	0.00209	0.00197	0.00220
	2	0.00156	0.00215	0.00393	0.00050	0.00203	0.00080	0.00138	0.00232	0.00375	0.00129	0.00125	0.00133
	3	0.00157	0.00108	0.00821	0.00021	0.00166	0.00166	0.00163	0.00163	0.00735	0.00149	0.00143	0.00156
100,20	1	0.00112	0.00138	0.00164	0.00156	0.00015	0.00266	0.00189	0.00291	0.00514	0.00175	0.00167	0.00182
	2	0.00014	0.00150	0.00074	0.00038	0.00068	0.00134	0.00098	0.00213	0.00124	0.00093	0.00091	0.00095
	3	0.00131	0.00105	0.00290	0.00028	0.00166	0.00089	0.00126	0.00203	0.00197	0.00118	0.00115	0.00122
100,50	1	0.00008	0.00024	0.00026	0.00179	0.00086	0.00255	0.00092	0.00102	0.00130	0.00088	0.00086	0.00090
	2	0.00122	0.00095	0.00202	0.00033	0.00108	0.00034	0.00064	0.00128	0.00091	0.00062	0.00061	0.00063
	3	0.00115	0.00162	0.00206	0.00012	0.00072	0.00086	0.00077	0.00109	0.00106	0.00074	0.00073	0.00075

Table 3. Bias and MSE of Θ with $T3 = X_m + 2$ for different censoring schemes when $\Theta = 0.4$

		Bias						MSE					
		Classical			Bayes			Classical			Bayes		
n,m	C.s	MLE	AMLE	LS	SEL	LINEX		MLE	AMLE	LS	SEL	LINEX	
						$\lambda = 1$	$\lambda = -1$					$\lambda = 1$	$\lambda = -1$
20,4	1	0.00418	0.00823	0.00608	0.00135	0.00548	0.00305	0.00690	0.02469	0.02123	0.00488	0.00382	0.00600
	2	0.00261	0.00279	0.00577	0.00095	0.00589	0.00282	0.00541	0.00611	0.01321	0.00410	0.00377	0.00475
	3	0.00228	0.00266	0.00423	0.00029	0.00561	0.00403	0.00645	0.00610	0.01390	0.00464	0.00409	0.00562
20,10	1	0.00230	0.00269	0.00241	0.00400	0.00026	0.00547	0.00419	0.00717	0.00735	0.00340	0.00295	0.00380
	2	0.00095	0.00178	0.00191	0.00247	0.00146	0.00492	0.00339	0.00369	0.00668	0.00282	0.00258	0.00311
	3	0.00363	0.00339	0.00542	0.00121	0.00292	0.00324	0.00380	0.00378	0.00716	0.00310	0.00277	0.00344
40,12	1	0.00146	0.00263	0.00482	0.00285	0.00018	0.00427	0.00318	0.00494	0.00777	0.00280	0.00255	0.00302
	2	0.00192	0.00212	0.00284	0.00046	0.00277	0.00139	0.00224	0.00317	0.00497	0.00202	0.00193	0.00212
	3	0.00065	0.00033	0.00762	0.00168	0.00115	0.00365	0.00254	0.00275	0.01051	0.00224	0.00210	0.00240
40,20	1	0.00098	0.00036	0.00015	0.00475	0.00253	0.00609	0.00218	0.00321	0.00378	0.00199	0.00186	0.00210
	2	0.00092	0.00068	0.00203	0.00107	0.00085	0.00256	0.00178	0.00225	0.00458	0.00163	0.00156	0.00170
	3	0.00122	0.00107	0.00311	0.00158	0.00052	0.00305	0.00196	0.00210	0.00419	0.00178	0.00169	0.00187
60,18	1	0.00172	0.00212	0.00387	0.00169	0.00039	0.00292	0.00220	0.00391	0.00516	0.00200	0.00189	0.00210
	2	0.00037	0.00029	0.00106	0.00135	0.00021	0.00269	0.00138	0.00246	0.00358	0.00129	0.00125	0.00134
	3	0.00087	0.00152	0.00479	0.00088	0.00101	0.00235	0.00176	0.00217	0.00719	0.00161	0.00155	0.00169
60,30	1	0.00137	0.00179	0.00128	0.00152	0.00004	0.00257	0.00146	0.00197	0.00267	0.00136	0.00131	0.00141
	2	0.00027	0.00029	0.00032	0.00164	0.00036	0.00272	0.00109	0.00174	0.00235	0.00103	0.00100	0.00106
	3	0.00156	0.00197	0.00414	0.00045	0.00095	0.00155	0.00136	0.00161	0.00371	0.00128	0.00124	0.00132
100,20	1	0.00058	0.00066	0.00131	0.00210	0.00037	0.00323	0.00197	0.00375	0.00503	0.00183	0.00174	0.00190
	2	0.00067	0.00059	0.00058	0.00014	0.00119	0.00082	0.00097	0.00223	0.00127	0.00092	0.00091	0.00094
	3	0.00010	0.00102	0.00090	0.00089	0.00051	0.00208	0.00127	0.00193	0.00199	0.00120	0.00116	0.00124
100,50	1	0.00106	0.00135	0.00149	0.00083	0.00008	0.00158	0.00092	0.00114	0.00130	0.00088	0.00086	0.00090
	2	0.00011	0.00055	0.00104	0.00076	0.00001	0.00144	0.00064	0.00117	0.00089	0.00062	0.00061	0.00063
	3	0.00002	0.00007	0.00105	0.00124	0.00038	0.00198	0.00073	0.00102	0.00100	0.00070	0.00069	0.00072

Table 4. Bias and MSE of Θ with $T1 = X_{\frac{m}{4}}$ for different censoring schemes when $\Theta = 0.9$

		Bias						MSE					
		Classical			Bayes			Classical			Bayes		
n,m	C.s	MLE	AMLE	LS	SEL	LINEX		MLE	AMLE	LS	SEL	LINEX	
						$\lambda=1$	$\lambda=-1$					$\lambda=1$	$\lambda=-1$
20,4	1	0.00448	0.01124	0.05113	0.00316	0.00735	0.00607	0.00122	0.00349	0.01652	0.00074	0.00075	0.00089
	2	0.00400	0.00435	0.06685	0.00136	0.00916	0.00417	0.00092	0.00103	0.02142	0.00065	0.00078	0.00070
	3	0.00445	0.00468	0.02726	0.00050	0.00982	0.00501	0.00101	0.00119	0.00900	0.00066	0.00080	0.00074
20,10	1	0.00275	0.00351	0.00548	0.00565	0.00052	0.00776	0.00068	0.00079	0.00133	0.00050	0.00045	0.00056
	2	0.00169	0.00181	0.00456	0.00300	0.00290	0.00645	0.00053	0.00056	0.00130	0.00041	0.00042	0.00045
	3	0.00270	0.00229	0.00620	0.00365	0.00263	0.00667	0.00062	0.00059	0.00136	0.00046	0.00045	0.00051
40,12	1	0.00174	0.00307	0.00539	0.00405	0.00047	0.00608	0.00047	0.00072	0.00134	0.00038	0.00036	0.00042
	2	0.00147	0.00217	0.00330	0.00053	0.00298	0.00321	0.00032	0.00047	0.00077	0.00028	0.00029	0.00029
	3	0.00222	0.00197	0.01433	0.00109	0.00314	0.00386	0.00039	0.00040	0.00241	0.00032	0.00033	0.00034
40,20	1	0.00141	0.00134	0.00229	0.00384	0.00065	0.00563	0.00033	0.00033	0.00061	0.00028	0.00028	0.00030
	2	0.00120	0.00167	0.00373	0.00150	0.00139	0.00363	0.00025	0.00034	0.00074	0.00022	0.00022	0.00023
	3	0.00151	0.00149	0.00320	0.00225	0.00088	0.00433	0.00029	0.00031	0.00063	0.00025	0.00024	0.00026
60,18	1	0.00153	0.00172	0.00399	0.00295	0.00013	0.00470	0.00033	0.00041	0.00083	0.00028	0.00028	0.00030
	2	0.00112	0.00112	0.00430	0.00029	0.00203	0.00220	0.00022	0.00035	0.00067	0.00019	0.00020	0.00020
	3	0.00115	0.00158	0.01344	0.00124	0.00160	0.00333	0.00025	0.00034	0.00188	0.00022	0.00022	0.00023
60,30	1	0.00083	0.00064	0.00154	0.00296	0.00078	0.00444	0.00022	0.00021	0.00040	0.00020	0.00019	0.00021
	2	0.00076	0.00070	0.00243	0.00113	0.00077	0.00266	0.00016	0.00024	0.00038	0.00015	0.00015	0.00015
	3	0.00101	0.00098	0.00470	0.00165	0.00046	0.00324	0.00019	0.00021	0.00065	0.00017	0.00017	0.00018
100,20	1	0.00046	0.00144	0.00548	0.00301	0.00048	0.00458	0.00027	0.00039	0.00098	0.00025	0.00024	0.00024
	2	0.00060	0.00125	0.00101	0.00014	0.00144	0.00154	0.00015	0.00033	0.00020	0.00014	0.00014	0.00014
	3	0.00079	0.00140	0.00259	0.00059	0.00150	0.00231	0.00018	0.00032	0.00029	0.00017	0.00017	0.00017
100,50	1	0.00039	0.00043	0.00059	0.00211	0.00075	0.00317	0.00014	0.00012	0.00020	0.00012	0.00012	0.00013
	2	0.00078	0.00088	0.00098	0.00040	0.00072	0.00139	0.00010	0.00019	0.00015	0.00009	0.00009	0.00009
	3	0.00079	0.00078	0.00109	0.00092	0.00035	0.00198	0.00011	0.00015	0.00016	0.00010	0.00010	0.00010

Table 5. Bias and MSE of Θ with $T2 = X_{\frac{4m}{5}}$ for different censoring schemes when $\Theta = 0.9$

		Bias						MSE					
		Classical			Bayes			Classical			Bayes		
n,m	C.s	MLE	AMLE	LS	SEL	LINEX		MLE	AMLE	LS	SEL	LINEX	
						$\lambda = 1$	$\lambda = -1$					$\lambda = 1$	$\lambda = -1$
20,4	1	0.00362	0.01575	0.05071	0.00360	0.00714	0.00672	0.00115	0.00600	0.01714	0.00071	0.00072	0.00086
	2	0.00369	0.00398	0.06136	0.00108	0.00891	0.00444	0.00090	0.00102	0.01965	0.00064	0.00076	0.00069
	3	0.00434	0.00417	0.03015	0.00052	0.00968	0.00511	0.00108	0.00100	0.00988	0.00069	0.00084	0.00079
20,10	1	0.00361	0.00361	0.00544	0.00492	0.00127	0.00701	0.00069	0.00096	0.00131	0.00050	0.00046	0.00056
	2	0.00234	0.00268	0.00636	0.00245	0.00344	0.00587	0.00055	0.00059	0.00148	0.00042	0.00043	0.00046
	3	0.00305	0.00329	0.00508	0.00338	0.00290	0.00638	0.00062	0.00060	0.00135	0.00045	0.00045	0.00045
40,12	1	0.00184	0.00265	0.00518	0.00400	0.00050	0.00602	0.00049	0.00077	0.00133	0.00039	0.00037	0.00043
	2	0.00150	0.00204	0.00343	0.00050	0.00301	0.00318	0.00032	0.00046	0.00080	0.00027	0.00028	0.00028
	3	0.00176	0.00192	0.01310	0.00150	0.00273	0.00428	0.00041	0.00046	0.00245	0.00034	0.00034	0.00036
40,20	1	0.00158	0.00186	0.00258	0.00361	0.00042	0.00541	0.00033	0.00041	0.00059	0.00028	0.00027	0.00030
	2	0.00104	0.00138	0.00281	0.00166	0.00122	0.00378	0.00024	0.00033	0.00069	0.00021	0.00021	0.00023
	3	0.00191	0.00178	0.00319	0.00186	0.00129	0.00395	0.00028	0.00031	0.00065	0.00028	0.00024	0.00026
60,18	1	0.00108	0.00153	0.00350	0.00336	0.00030	0.00511	0.00034	0.00050	0.00086	0.00029	0.00028	0.00031
	2	0.00074	0.00144	0.00416	0.00066	0.00166	0.00256	0.00020	0.00036	0.00036	0.00018	0.00019	0.00019
	3	0.00131	0.00141	0.01360	0.00108	0.00175	0.00318	0.00025	0.00033	0.00180	0.00022	0.00022	0.00023
60,30	1	0.00096	0.00107	0.00190	0.00285	0.00067	0.00432	0.00021	0.00026	0.00041	0.00020	0.00019	0.00021
	2	0.00069	0.00113	0.00239	0.00119	0.00071	0.00273	0.00016	0.00025	0.00038	0.00015	0.00015	0.00015
	3	0.00096	0.00120	0.00494	0.00170	0.00041	0.00328	0.00019	0.00023	0.00067	0.00017	0.00017	0.00016
100,20	1	0.00144	0.00205	0.00679	0.00212	0.00041	0.00368	0.00028	0.00043	0.00100	0.00025	0.00024	0.00026
	2	0.00061	0.00129	0.00092	0.00012	0.00147	0.00153	0.00015	0.00033	0.00019	0.00014	0.00014	0.00014
	3	0.00074	0.00131	0.00254	0.00065	0.00144	0.00236	0.00018	0.00029	0.00028	0.00022	0.00017	0.00020
100,50	1	0.00093	0.00089	0.00109	0.00159	0.00023	0.00266	0.00014	0.00015	0.00018	0.00013	0.00012	0.00015
	2	0.00032	0.00069	0.00032	0.00085	0.00027	0.00184	0.00010	0.00018	0.00015	0.00011	0.00012	0.00010
	3	0.00063	0.00074	0.00091	0.00107	0.00020	0.00213	0.00011	0.00016	0.00016	0.00011	0.00010	0.00011

Table 6. Bias and MSE of Θ with $T3 = X_m + 2$ for different censoring schemes when $\Theta = 0.9$

		Bias						MSE					
		Classical			Bayes			Classical			Bayes		
n,m	C.s	MLE	AMLE	LS	SEL	LINEX		MLE	AMLE	LS	SEL	LINEX	
						$\lambda = 1$	$\lambda = -1$					$\lambda = 1$	$\lambda = -1$
20,4	1	0.00555	0.02152	0.02152	0.00221	0.00846	0.00512	0.00118	0.00701	0.01701	0.00071	0.00074	0.00085
	2	0.00276	0.00361	0.06570	0.00040	0.00822	0.00522	0.00087	0.00100	0.02205	0.00062	0.00074	0.00068
	3	0.00321	0.00278	0.02641	0.00057	0.00868	0.00609	0.00098	0.00091	0.00939	0.00064	0.00077	0.00074
20,10	1	0.00275	0.00512	0.00487	0.00566	0.00052	0.00776	0.00066	0.00122	0.00130	0.00049	0.00044	0.00054
	2	0.00282	0.00303	0.00658	0.00201	0.00389	0.00545	0.00057	0.00062	0.00145	0.00043	0.00045	0.00047
	3	0.00250	0.00239	0.00537	0.00382	0.00249	0.00684	0.00060	0.00059	0.00131	0.00045	0.00044	0.00050
40,12	1	0.00229	0.00447	0.00619	0.00358	0.00091	0.00559	0.00049	0.00109	0.00137	0.00039	0.00037	0.00042
	2	0.00156	0.00156	0.00336	0.00154	0.00309	0.00309	0.00032	0.00047	0.00078	0.00028	0.00029	0.00029
	3	0.00169	0.00186	0.01141	0.00157	0.00266	0.00435	0.00038	0.00044	0.00211	0.00031	0.00032	0.00033
40,20	1	0.00142	0.00195	0.00279	0.00374	0.00056	0.00554	0.00032	0.00048	0.00059	0.00027	0.00026	0.00029
	2	0.00060	0.00092	0.00260	0.00206	0.00083	0.00419	0.00025	0.00034	0.00072	0.00023	0.00022	0.00024
	3	0.00130	0.00170	0.00256	0.00242	0.00073	0.00452	0.00028	0.00031	0.00063	0.00024	0.00023	0.00026
60,18	1	0.00127	0.00239	0.00330	0.00321	0.00015	0.00495	0.00033	0.00061	0.00080	0.00029	0.00027	0.00031
	2	0.00073	0.00104	0.00483	0.00067	0.00165	0.00258	0.00021	0.00036	0.00066	0.00019	0.00019	0.00019
	3	0.00133	0.00165	0.01376	0.00107	0.00176	0.00316	0.00025	0.00032	0.00179	0.00022	0.00022	0.00023
60,30	1	0.00088	0.00125	0.00162	0.00291	0.00073	0.00439	0.00022	0.00030	0.00040	0.00020	0.00020	0.00021
	2	0.00057	0.00086	0.00181	0.00057	0.00136	0.00222	0.00016	0.00025	0.00036	0.00015	0.00015	0.00015
	3	0.90309	0.00068	0.00496	0.00200	0.00010	0.00359	0.00018	0.00022	0.00022	0.00017	0.00017	0.00017
100,20	1	0.00080	0.00195	0.00659	0.00158	0.00109	0.00341	0.00028	0.00058	0.00103	0.00025	0.00025	0.00026
	2	0.00067	0.00167	0.00081	0.00006	0.00153	0.00153	0.00014	0.00035	0.00019	0.00013	0.00013	0.00013
	3	0.00067	0.00097	0.00243	0.00071	0.00139	0.00242	0.00018	0.00028	0.00029	0.00016	0.00016	0.00017
100,50	1	0.00092	0.00099	0.00115	0.00089	0.00052	0.00205	0.00014	0.00017	0.00019	0.00013	0.00012	0.00013
	2	0.00014	0.00034	0.00045	0.00058	0.00055	0.00161	0.00009	0.00019	0.00016	0.00009	0.00009	0.00009
	3	0.00077	0.00093	0.00095	0.00038	0.00091	0.00151	0.00011	0.00016	0.00016	0.00010	0.00010	0.00011

5. Real-life Examples

In this section, we examine two real-life data sets, to illustrate our proposed methods and further verify how our estimates work in practice. First, we checked the validity of the IW model using Kolmogorov-Smirnov $(K - S)$, Anderson-Darling $(A - D)$, and chi-square tests. Next, we created three different artificial adaptive Type-II hybrid progressive censored samples from each data using the same censoring schemes as those in Section 4.

Example 1: This application is provided by Nelson (2003). It represents the breakdown times (in minutes) of an insulating fluid between two electrodes recorded at different voltages; 34 kilo-volts (data I) and 36 Kilo-volt (data II), as presented in table 7. We have fitted the IWD (0.70151, 1.8886) for data set I and IWD (1.0823,1.3309) for data set II. The results are summarized in table 8 with a significance level of 0.05.

Table 7. Breakdown times (in minutes) for data set I and II

Data I	0.19	0.78	0.96	1.31	2.78	3.16	4.15	4.67	4.85	6.5
	7.35	8.01	8.27	12.06	31.75	32.52	33.91	36.71	72.89	
Data II	0.35	0.59	0.96	0.99	1.69	1.97	2.07	2.58	2.71	2.9
	3.67	3.99	5.35	13.77	25.50					

Table 8. Test statistic and p-value associated with each test for example 1

Data	K-S (p-value)	A-D (p-value)	Chi-Squared (p-value)
I	0.1873 (0.4625)	0.7723 (0.4986)	0.8865 (0.6420)
II	0.2037 (0.4991)	0.4929 (0.7509)	1.4421 (0.2298)

Based on the above results, it appears that the IWD model fits the data well. Further, Figures 6 - 9 show that Nelson's data are well-suited to the IWD.

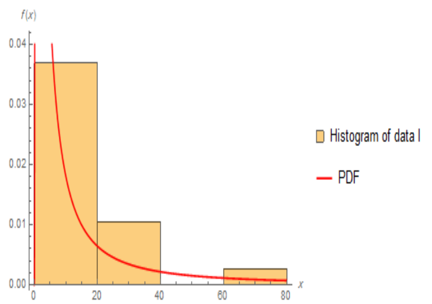

Figure 6. Estimated PDF of data I

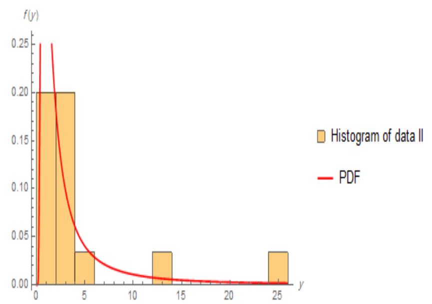

Figure 7. Estimated PDF of data II

Table 9 presents the generated adaptive Type-II hybrid progressive censored samples and the associated stopping time for each scheme. The estimates of θ are calculated based on $m_1 = 11$, $m_2 = 9$. Bayes estimates are computed based on a non-informative prior.

Table 10 shows that estimates of θ based on adaptive Type-II hybrid progressive samples are comparable to those derived from complete data. Furthermore, the classical and Bayesian estimates of θ are pretty close to each other.

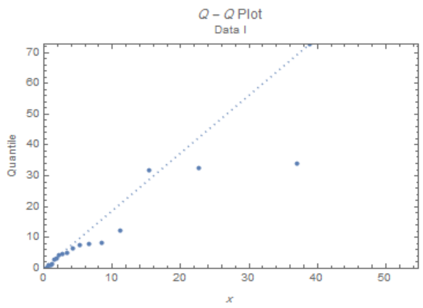

Figure 8. Q-Q plot for data I

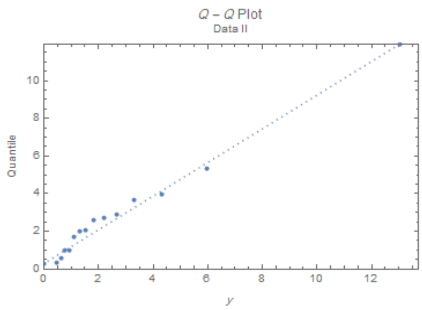

Figure 9. Q-Q plot for data II

Table 9. Adaptive Type-II censored samples form data I and data II

Scheme	T_1	Censored sample from data I										
1	2	0.19	0.78	1.31	2.78	3.16	4.67	7.35	8.01	12.06	31.75	32.52
2	8	0.19	0.78	0.96	1.31	2.78	3.16	4.15	4.67	4.85	6.5	7.35
3	7.5	0.19	0.78	1.31	3.16	4.15	4.67	4.85	7.35	8.01	12.06	31.75
Scheme	T_2	**Censored sample from data II**										
1	1	0.35	0.59	0.99	1.97	2.07	2.58	2.9	3.67	3.99		
2	3	0.35	0.59	0.96	0.99	1.69	1.97	2.07	2.58	2.71		
3	2	0.35	0.59	0.99	1.69	1.97	2.58	2.71	3.67	5.35		

Table 10. Estimates of θ for example 1

C.s	MLE	AMLE	LSE	SEL	LINEX $\lambda=1$	LINEX $\lambda=-1$
Complete	0.5690	0.5771	0.5914	0.5690	0.5687	0.5693
1	0.5626	0.5845	0.6087	0.562	0.5601	0.5636
2	0.5695	0.5089	0.5859	0.5694	0.5691	0.5698
3	0.5714	0.5304	0.5811	0.5707	0.5695	0.5719

In addition, the parametric bootstrap percentile method is used to compute 95% bootstrap confidence intervals, standard error, and average values of θ. All results are shown in Table 11 & 12. As can be seen from Tables 11 & 12, the standard error is the least for most of the estimates of θ under the second censoring scheme. Moreover, Bayes estimates under LINEX loss function when $\lambda = 1$ has the lowest error under the second scheme. Average values of the estimates of θ are close to those in Table 10.

Table 11. Bootstrap confidence intervals for the considered estimates of θ for example 1

C.s	1	2	3
MLE	(0.3019, 0.807)	(0.383, 0.7388)	(0.3279, 0.7865)
AMLE	(0.3486, 0.7793)	(0.2912, 0.7414)	(0.2767, 0.7833)
LSE	(0.3455, 0.801)	(0.3799, 0.7629)	(0.3975, 0.7746)
SEL	(0.3021, 0.8077)	(0.3829, 0.7389)	(0.3275, 0.7863)
LINEX ($\lambda = 1$)	(0.3138, 0.7957)	(0.3905, 0.7304)	(0.3391, 0.7754)
LINEX ($\lambda = -1$)	(0.2935, 0.8168)	(0.3755, 0.7458)	(0.3189, 0.7953)

Table 12. Standard error and average value for each estimate after bootstrapping for example 1

C.s	MLE	AMLE	LSE	SEL	LINEX $\lambda = 1$	LINEX $\lambda = -1$
Standard Error						
1	0.1302	0.1132	0.1102	0.13	0.1239	0.1349
2	0.0908	0.1198	0.0929	0.0909	0.0866	0.0947
3	0.12	0.1269	0.0971	0.1201	0.1138	0.1251
Average Value						
1	0.5668	0.5782	0.6083	0.5665	0.5654	0.5674
2	0.5713	0.5272	0.5841	0.5713	0.5709	0.5716
3	0.5688	0.5405	0.5917	0.5684	0.5678	0.569

Example 2: Adaptive hybrid progressive Type-II is used here to analyze data used by Efron (1988). Head and neck cancer (HNC) patients are divided into two groups. Radiotherapy (RT) was used to treat patients in the first group, and their survival times were recorded in days (Data 1). While a combination of chemotherapy and radiotherapy (CT+RT) was administered to patients in the second group, and their survival times were also recorded in days (Data 2). Failure times for the two data sets are reported in Table (13). The survival times in bold indicate that a patient left the treatment center and never reported back. Efron (1988) compared the two therapies based on estimated survival functions under each model and found that (CT+RT) provides better HNC patient survival time than (RT).

Table 13. Survival times (in days) for data 1 and data 2

Data 1	7	34	42	63	64	**74**	83	84	91	108
	112	119	133	133	139	140	140	146	149	154
	157	160	160	165	173	176	**185**	218	225	241
	248	273	277	**279**	279	**319**	405	417	420	440
	523	**523**	583	594	1101	**1116**	1146	**1226**	**1349**	**1412**
	1417									
Data 2	37	84	92	94	110	112	119	127	130	133
	140	146	155	159	169	173	179	194	195	209
	249	281	319	339	432	469	519	**528**	**547**	**613**
	633	725	**759**	817	**1092**	**1245**	**1331**	1557	**1642**	**1771**
	1776	**1897**	**2023**	**2146**	**2297**					

Makkar et al. (2014) converted the survival times into months by dividing them by 30.438 before generating the unknown censored data using the truncated lognormal distribution. Retrieved survival times are reported in Table (14).

We checked the validity of the IW model based on the parameters $\alpha_1 = 1.0657$ and $\beta_1 = 4.8044$ for Data I and $\alpha_2 = 1.0021$ and $\beta_2 = 7.117$ for Data II, respectively. Results are summarized in Table (15) with a significance level of 0.05.

Table 14. Survival times (in months) for data 1 and data 2

Data 1	0.23	1.12	1.38	2.07	2.10	2.73	2.76	2.99	3.55
	3.68	4.24	4.37	4.37	4.57	4.60	4.60	4.80	4.90
	5.06	5.16	5.26	5.26	5.42	5.68	5.78	6.53	7.16
	7.39	7.02	8.15	8.97	9.10	9.76	10.42	13.31	13.70
	13.80	14.46	14.48	16.10	17.18	19.15	19.52	22.70	36.17
	37.65	41.55	45.28	46.55	49.40	53.62			
Data 2	1.22	2.76	3.02	3.09	3.61	3.68	3.91	4.17	4.27
	4.37	4.60	4.80	5.09	5.22	5.68	5.88	6.37	6.41
	6.87	8.18	9.23	10.48	11.14	12.20	14.91	15.41	17.05
	20.80	23.56	23.74	23.82	25.87	26.84	31.98	41.35	47.38
	51.15	55.46	58.38	58.36	63.47	68.46	74.47	78.26	81.43

Table 15. Test statistic and p-value associated with each test for example 2

Data	K-S (p-value)	A-D (p-value)	Chi-Squared (p-value)
1	0.1606 (0.1290)	1.383 (0.2065)	5.6975 (0.3368)
2	0.1175 (0.5248)	0.7716 (0.5003)	3.7435 (0.4418)

Table (15) indicates that the IW model is a good fit for both data sets. In addition, the fitted PDFs and Q-Q plots are plotted for both data sets and reported in Figures 10 - 13 also confirm that the IW model is a good fit for Efron's data sets.

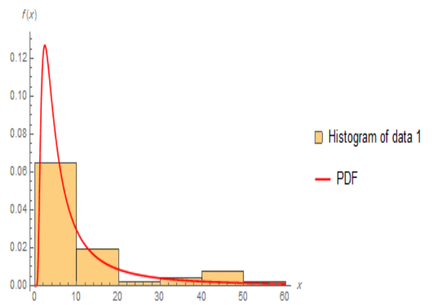

Figure 10. Estimated PDF of data 1

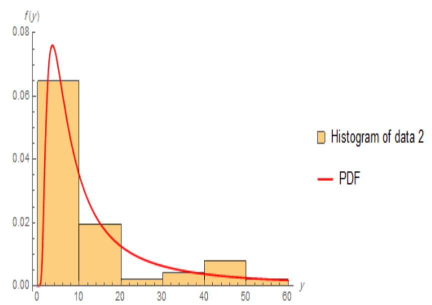

Figure 11. Estimated PDF of data 2

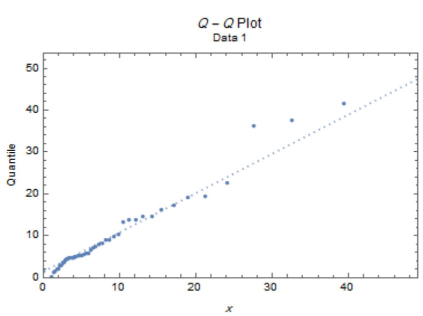

Figure 12. Q-Q plot for data 1

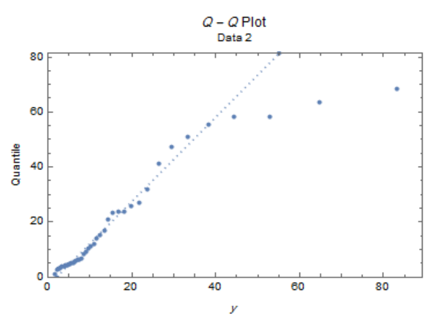

Figure 13. Q-Q plot for data 2

The three adaptive Type-II censoring schemes for the simulation study are used to generate adaptive Type-II hybrid progressive censored samples, the associated stopping time for each scheme and the generated censored samples are given in Table (16).

Table 16. Adaptive Type-II censored samples form Efron's data sets

Scheme	T_1	Censored sample from data 1					
1	2.1	0.23	1.12	1.38	2.07	2.73	2.99
		3.55	3.68	4.24	4.37	4.60	4.60
		5.06	5.16	5.42	5.68	6.53	7.16
		7.92	9.76	10.42	13.31	13.80	14.46
		17.18	19.52				
2	7	0.23	1.12	1.38	2.07	2.10	2.73
		2.76	2.99	3.55	3.68	4.24	4.37
		4.37	4.57	4.60	4.60	4.80	4.90
		5.06	5.16	5.26	5.26	5.42	5.68
		5.78	6.53				
3	5	0.23	1.12	2.10	2.76	2.99	3.55
		3.68	4.24	4.37	4.60	5.06	5.26
		5.26	5.78	7.16	7.39	8.15	9.76
		10.42	13.31	13.80	14.46	17.18	19.15
		19.52	22.70				
Scheme	T_2	Censored sample from data 2					
1	3.5	1.22	2.76	3.02	3.61	4.17	4.37
		4.60	5.22	5.68	6.37	8.18	9.23
		10.48	12.20	14.91	17.05	20.80	23.82
		26.84	31.98	41.35	47.38		
2	11	1.22	2.76	3.02	3.09	3.61	3.68
		3.91	4.17	4.27	4.37	4.60	4.80
		5.09	5.22	5.68	5.88	6.37	6.41
		6.87	8.18	9.23	10.48		
3	9	1.22	2.76	3.09	3.68	4.17	4.27
		4.60	4.80	5.09	5.88	6.41	9.23
		10.48	11.14	14.91	15.41	20.80	23.56
		26.84	31.98	41.35	47.38		

The estimates of θ are calculated based on $m_1 = 26$, $m_2 = 22$. Bayes estimates are computed based on a non-informative prior. Table 17 shows that estimates of θ based on adaptive Type-II hybrid progressive samples are comparable to those derived from complete data. Furthermore, the classical and Bayesian estimates of θ are pretty close to each other. It is worth mentioning that when calculating the MPSE, repeated data points are handled differently, as suggested by Cheng and Amin (1983).

Table 17. Estimates of θ for example 2

C.s	MLE	AMLE	LSE	SEL	LINEX $\lambda = 1$	LINEX $\lambda = -1$
Complete	0.3444	0.3937	0.4115	0.3444	0.3512	0.3383
1	0.2925	0.3641	0.4075	0.2907	0.3009	0.2833
2	0.3482	0.4388	0.4443	0.3482	0.3551	0.3421
3	0.3131	0.3628	0.3262	0.3129	0.322	0.3052

In addition, the parametric bootstrap percentile method is used to compute 95% bootstrap confidence intervals, standard error, and average values of $\hat{\theta}$. All results are shown in Table 18 & 19. As can be seen from Tables 18 & 19, the standard error is the least for all estimates of θ under the second censoring scheme. Moreover, average values of the estimates of θ are close to those in Table 17.

Table 18. Bootstrap confidence intervals for each estimate of θ for example 2

C.s		1	2	3
MLE		(0.1287, 0.548)	(0.2243, 0.5102)	(0.1607, 0.5563)
AMLE		(0.2616, 0.4969)	(0.3916, 0.5136)	(0.2815, 0.4728)
LSE		(0.1419, 0.5555)	(0.2696, 0.5729)	(0.07535, 0.5864)
SEL		(0.1287, 0.5477)	(0.2242, 0.5102)	(0.1606, 0.5562)
LINEX	$\lambda = 1$	(0.136, 0.549)	(0.2326, 0.5115)	(0.1698, 0.5564)
	$\lambda = -1$	(0.1227, 0.5481)	(0.2165, 0.5091)	(0.1533, 0.5553)

Table 19. Standard error and average value for each estimate after bootstrapping for example 2

Standard Error

C.s	MLE	AMLE	LSE	SEL	LINEX $\lambda = 1$	LINEX $\lambda = -1$
1	0.1175	0.06017	0.09677	0.1176	0.1158	0.1191
2	0.07964	0.02808	0.07796	0.07968	0.07755	0.08156
3	0.1106	0.04649	0.1262	0.1107	0.1079	0.113

Average Value

C.s	MLE	AMLE	LSE	SEL	LINEX $\lambda = 1$	LINEX $\lambda = -1$
1	0.3149	0.3712	0.3857	0.3141	0.3218	0.3083
2	0.3674	0.4369	0.4245	0.3673	0.3734	0.3619
3	0.3391	0.3678	0.3627	0.3389	0.3458	0.3332

6. Conclusions and Recommendations

Progressive censoring is widely used in reliability and life testing studies to address various concerns that experimenters may have, such as reducing total test time, conserving experimental units, and developing efficient estimation methods. But there is always a trade-off between these three concerns to reduce the cost and the total test time of the experiment. Different types of progressive censoring have been developed to help mitigate these concerns. For example, the adaptive Type-II hybrid progressive censoring allows more flexibility during the experiment. It provides more control over the experiment, resulting in a shorter test duration and more observed failures.

In this article, we study the statistical inference of the reliability model under adaptive Type-II hybrid progressive censoring when the random stress and strength components are independent IWDs that share the same shape parameter. Due to the inability to obtain the MLE in closed forms, the AMLE of θ is derived. LSE of θ is also calculated. Bayesian estimators are obtained based on the SEL and LINEX loss function using Lindleys approximation method due to the lack of explicit forms.

Our extensive simulations conclude that Bayes estimators derived by the LINEX loss function have the smallest Bias and MSE for all sample sizes. Hence, we recommended using the Bayes estimates for estimating the reliability under adaptive Type-II progressive hybrid censoring of the IWD distribution under the second censoring scheme, where the random variables are independent and have common shape parameters.

A. Appendix

The entries for Lindley's approximation are given by the following equations

$$\hat{\sigma} = \begin{bmatrix} \hat{\sigma}_{\beta_1\beta_1} & \hat{\sigma}_{\beta_1\beta_2} \\ \hat{\sigma}_{\beta_2\beta_1} & \hat{\sigma}_{\beta_2\beta_2} \end{bmatrix} = -\begin{bmatrix} (\frac{\partial^2 l}{\partial\beta_1^2}|_{(\beta_1=\hat{\beta}_{1_{MLE}},\beta_1=\hat{\beta}_{1_{MLE}})}) & (\frac{\partial^2 l}{\partial\beta_1\partial\beta_2}|_{(\beta_1=\hat{\beta}_{1_{MLE}},\beta_1=\hat{\beta}_{1_{MLE}})}) \\ (\frac{\partial^2 l}{\partial\beta_2\partial\beta_1}|_{(\beta_1=\hat{\beta}_{1_{MLE}},\beta_1=\hat{\beta}_{1_{MLE}})}) & (\frac{\partial^2 l}{\partial\beta_1^2}|_{(\beta_1=\hat{\beta}_{1_{MLE}},\beta_1=\hat{\beta}_{1_{MLE}})}) \end{bmatrix}^{-1}$$
$$= \begin{bmatrix} -(\frac{\partial^2 l}{\partial\beta_1^2}|_{(\beta_1=\hat{\beta}_{1_{MLE}},\beta_1=\hat{\beta}_{1_{MLE}})})^{-1} & 0 \\ 0 & -(\frac{\partial^2 l}{\partial\beta_1^2}|_{(\beta_1=\hat{\beta}_{1_{MLE}},\beta_1=\hat{\beta}_{1_{MLE}})})^{-1} \end{bmatrix}$$

$$\hat{l}_{\beta_1} = \frac{\partial l}{\partial\beta_1}|_{\beta_1=\hat{\beta}_{1_{MLE}}} = \frac{m_1}{\hat{\beta}_{1_{MLE}}} + \frac{e^{-\hat{\beta}_{1_{MLE}}x_{m_1}^{-\alpha}}R_{m_1}x_{m_1}^{-\alpha}}{1-e^{-\hat{\beta}_{1_{MLE}}x_{m_1}^{-\alpha}}} - \sum_{i=1}^{m_1}x_i^{-\alpha} + \sum_{i=1}^{J_1}\frac{(e^{-\hat{\beta}1x_i^{-\alpha}})R_ix_i^{-\alpha}}{1-e^{-\hat{\beta}_{1_{MLE}}x_i^{-\alpha}}}$$

$$\hat{l}_{\beta_1\beta_1} = \frac{\partial^2 l}{\partial\beta_1^2}|_{\beta_1=\hat{\beta}_{1_{MLE}}} = \frac{-m_1}{\hat{\beta}_{1_{MLE}}^2} - R_{m_1}(\frac{e^{-2\hat{\beta}_{1_{MLE}}x_{m_1}^{-\alpha}x_{m_1}^{-2\alpha}}}{(1-e^{-\hat{\beta}_{1_{MLE}}x_{m_1}^{-\alpha}})^2} + \frac{e^{-\hat{\beta}_{1_{MLE}}x_{m_1}^{-2\alpha}}}{1-e^{-\hat{\beta}_{1_{MLE}}x_{m_1}^{-\alpha}}})$$
$$-\sum_{i=1}^{J_1}R_i(\frac{e^{-2\hat{\beta}_{1_{MLE}}x_i^{-\alpha}x_i^{-2\alpha}}}{(1-e^{-\hat{\beta}_{1_{MLE}}x_i^{-\alpha}})^2} + \frac{e^{-\hat{\beta}_{1_{MLE}}x_i^{-2\alpha}}}{1-e^{-\hat{\beta}_{1_{MLE}}x_i^{-\alpha}}})$$

$$\hat{l}_{\beta_1\beta_1\beta_1} = \frac{\partial^3 l}{\partial\beta_1^3}|_{\beta_1=\hat{\beta}_{1_{MLE}}} = \frac{2m_1}{\hat{\beta}_{1_{MLE}}^3} + R_{m_1}(\frac{2e^{-3\hat{\beta}_{1_{MLE}}x_{m_1}^{-\alpha}x_{m_1}^{-3\alpha}}}{(1-e^{-\hat{\beta}_{1_{MLE}}x_{m_1}^{-\alpha}})^3} + \frac{3e^{-2\hat{\beta}_{1_{MLE}}x_{m_1}^{-\alpha}x_{m_1}^{-3\alpha}}}{(1-e^{-\hat{\beta}_{1_{MLE}}x_{m_1}^{-\alpha}})^2} + \frac{e^{-2\hat{\beta}_{1_{MLE}}x_{m_1}^{-\alpha}x_{m_1}^{-3\alpha}}}{1-e^{-\hat{\beta}_{1_{MLE}}x_{m_1}^{-\alpha}}})$$
$$+\sum_{i=1}^{J_1}(\frac{2e^{-3\hat{\beta}_{1_{MLE}}x_i^{-\alpha}x_i^{-3\alpha}}}{(1-e^{-\hat{\beta}_{1_{MLE}}x_i^{-\alpha}})^3} + \frac{3e^{-2\hat{\beta}_{1_{MLE}}x_i^{-\alpha}x_i^{-3\alpha}}}{(1-e^{-\hat{\beta}_{1_{MLE}}x_i^{-\alpha}})^2} + \frac{e^{-2\hat{\beta}_{1_{MLE}}x_i^{-\alpha}x_i^{-3\alpha}}}{1-e^{-\hat{\beta}_{1_{MLE}}x_i^{-\alpha}}})$$

$$\hat{l}_{\beta_2} = \frac{\partial l}{\partial\beta_2}|_{\beta_2=\hat{\beta}_{2_{MLE}}} = \frac{m_2}{\hat{\beta}_{2_{MLE}}} + \frac{e^{-\hat{\beta}_{2_{MLE}}y_{m_2}^{-\alpha}}S_{m_2}y_{m_2}^{-\alpha}}{1-e^{-\hat{\beta}_{2_{MLE}}y_{m_2}^{-\alpha}}} - \sum_{i=1}^{m_2}y_i^{-\alpha} + \sum_{i=1}^{J_2}\frac{(e^{-\hat{\beta}2y_i^{-\alpha}})S_iy_i^{-\alpha}}{1-e^{-\hat{\beta}_{2_{MLE}}y_i^{-\alpha}}}$$

$$\hat{l}_{\beta_2\beta_2} = \frac{\partial^2 l}{\partial \beta_2^2}|_{\beta_2=\hat{\beta}_{2_{MLE}}} = \frac{-m_2}{\hat{\beta}_{2_{MLE}}^2} - S_{m_2}(\frac{e^{-2\hat{\beta}_{2_{MLE}} y_{m_2}^{-\alpha}} y_{m_2}^{-2\alpha}}{(1-e^{-\hat{\beta}_{2_{MLE}} y_{m_2}^{-\alpha}})^2} + \frac{e^{-\hat{\beta}_{2_{MLE}} y_{m_2}^{-2\alpha}}}{1-e^{-\hat{\beta}_{2_{MLE}} y_{m_2}^{-\alpha}}})$$
$$- \sum_{i=1}^{J_2} S_i(\frac{e^{-2\hat{\beta}_{2_{MLE}} y_i^{-\alpha}} y_i^{-2\alpha}}{(1-e^{-\hat{\beta}_{2_{MLE}} y_i^{-\alpha}})^2} + \frac{e^{-\hat{\beta}_{2_{MLE}} y_i^{-2\alpha}}}{1-e^{-\hat{\beta}_{2_{MLE}} y_i^{-\alpha}}})$$

$$\hat{l}_{\beta_2\beta_2\beta_2} = \frac{\partial^3 l}{\partial \beta_2^3}|_{\beta_2=\hat{\beta}_{2_{MLE}}} = \frac{2m_2}{\hat{\beta}_{2_{MLE}}^3} + S_{m_2}(\frac{2e^{-3\hat{\beta}_{2_{MLE}} y_{m_2}^{-\alpha}} y_{m_2}^{-3\alpha}}{(1-e^{-\hat{\beta}_{2_{MLE}} y_{m_2}^{-\alpha}})^3} + \frac{3e^{-2\hat{\beta}_{2_{MLE}} y_{m_2}^{-\alpha}} y_{m_2}^{-3\alpha}}{(1-e^{-\hat{\beta}_{2_{MLE}} y_{m_2}^{-\alpha}})^2} + \frac{e^{-2\hat{\beta}_{2_{MLE}} y_{m_2}^{-\alpha}} y_{m_2}^{-3\alpha}}{1-e^{-\hat{\beta}_{2_{MLE}} y_{m_2}^{-\alpha}}})$$
$$+ \sum_{i=1}^{J_2}(\frac{2e^{-3\hat{\beta}_{2_{MLE}} y_i^{-\alpha}} y_i^{-3\alpha}}{(1-e^{-\hat{\beta}_{2_{MLE}} y_i^{-\alpha}})^3} + \frac{3e^{-2\hat{\beta}_{2_{MLE}} y_i^{-\alpha}} y_i^{-3\alpha}}{(1-e^{-\hat{\beta}_{2_{MLE}} x_i^{-\alpha}})^2} + \frac{e^{-2\hat{\beta}_{2_{MLE}} y_i^{-\alpha}} y_i^{-3\alpha}}{1-e^{-\hat{\beta}_{2_{MLE}} y_i^{-\alpha}}})$$

$$\hat{l}_{\beta_2\beta_1} = \hat{l}_{\beta_1\beta_2} = \hat{l}_{\beta_2\beta_2\beta_1} = \hat{l}_{\beta_2\beta_1\beta_2} = \hat{l}_{\beta_1\beta_2\beta_2} = \hat{l}_{\beta_1\beta_1\beta_2} = \hat{l}_{\beta_1\beta_2\beta_1} = \hat{l}_{\beta_2\beta_1\beta_1} = 0$$

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