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Bayesian and Classical Inference for Generalized Stress-Strength Parameter Under Generalized Logistic Distribution

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Abstract In this paper, we study generalized stress-strength model for generalized logistic distribution. The maximum likelihood estimator of this quantity is obtained and then a confidence interval is presented for it. Bayesian and bootstrap methods are also applied for the recommended model. A Markov Chain Monte Carlo (MCMC) simulation study for assessing the estimation methods is performed via the Metropolis-Hastings algorithm in each step of Gibbs algorithm. An application to real data set is addressed.

Keywords Stress-Strength Model, Generalized Stress-Strength Model, Generalized Logistic Distribution, MCMC, Bootstrap, Gibbs Sampling.

AMS 2010 subject classifications 94A08, 68U10

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1. Introduction

The stress-strength models have wide application in some fields especially in reliability. This model is related to a system with two components which that works truly if $Y < X$ So, the probability of working system is $R = P(Y < X)$ that is named stress-strength model (for more details about this model see *Kotz et al* [8]). *Saber et al* [11] introduced a generalization of R as following.

$$R^G = Pr(Y < \ X < Z). \tag{1}$$

The model in (1) is related to a system with three components where the quantity R is a special case from (1) if we set variable Z to $+\infty$. The major motivation for introducing this model can be justified by an example. Suppose that we have a of machine which have three distinct parts containing engine, steering and skid. Let lifetimes of these systems are random variables (rvs) $Y,\ X$ and Z respectively. Then, this machine works truly if $\{Y < X < Z\}$.

We use the following formula for computing R^G where

$$R^G = \int_{-\infty}^{+\infty} \int_{y}^{+\infty} \left[F_X\left(z\right) - F_X\left(y\right)\right] f_Y\left(y\right) f_Z\left(z\right) dz\, dy. \tag{2}$$

Many distributions have been studied by authors for estimating R which can also be applied to R^G. *Balakrishnan and Leung* [4] defined the Generalized Logistic (GL) distribution by generalizing the standard logistic distribution. For practical usage, the GL distribution has been noticed in estimating its parameters. For instance, *Asgharzadeh*

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et al [2] and *Rasekhi et al* [9] have surveyed on stress-strength model and multi-components stress-strength model when distribution of components is GL. A rv X have a $GL(\alpha, \lambda)$ distribution if its pdf is given by:

$$f(x) = \alpha\lambda\, exp(-\lambda x)\ [1 + exp(-\lambda x)]^{-\alpha-1}, x \in R. \tag{3}$$

Its cumulative distribution function is

$$F(x) = [1 + exp(-\lambda x)]^{-\alpha}. \tag{4}$$

Here $\alpha > 0$ and $\lambda > 0$ are the shape and scale parameters, respectively. The first parameter controls skewness of distribution in such way that has negative and positive skew for $\alpha > 1$ and $0 < \alpha < 1$, respectively. It becomes the standard logistic distribution when $\alpha = 1$. Also, it is unimodal and log-concave. A list of extensive works on GL distribution is found in *Asgharzadeh* [3], *Alkasasbeh and Raqab* [1] and *Gupta and Kundu* [7], among others. In this study, we shall concentrate on the statistical inference of R^G for the GL distribution.

The rest of paper is organized as follows: The R^G investigation when distribution of components is GL with common scale parameter is derived in Section 2. We estimate R^G via three approaches called the maximum likelihood estimations (MLE), Bayesian estimation and bootstrap method. Section 3 is devoted to similar work with previous section but in case of common shape parameter. A simulation study has been performed in Section 4 for surveying accuracy and performance of recommended methods in Section 2 and 3. We provide an application to a read life data set in Section 5. Finally, a discussion and concluding remark has been brought in Section 6.

2. Model for case of common scale parameter

In this section, generalized stress-strength model R^G is analyzed for GL distribution in the case of X, Y and Z are $GL(\alpha_1, \lambda)$, $GL(\alpha_2, \lambda)$ and $GL(\alpha_3, \lambda)$, respectively. By substitution (3) and (4) in (2) we arrive at

$$R^{\,G} = \frac{1}{(\alpha_1 + \alpha_2)(\alpha_1 + \alpha_2 + \alpha_3)}\alpha_1\alpha_3. \tag{5}$$

Let $X_1, \ldots, X_n \sim GL(\alpha_1, \lambda)$, $Y_1, \ldots, Y_m \sim GL(\alpha_2, \lambda)$ and $Z_1, \ldots, Z_L \sim GL(\alpha_3, \lambda)$, then the log likelihood function is:

$$\mathcal{L}_{i,j,k}^{n,m,L}(\alpha_1, \alpha_2, \alpha_3|\lambda) = n\ln(\alpha_1) + m\ln(\alpha_2) + L\ln(\alpha_3) + (n + m + L)\ln(\lambda)$$
$$-\lambda\left(\sum_{i=1}^{n} x_i + \sum_{j=1}^{m} y_j + \sum_{k=1}^{L} z_k\right) - (\alpha_1 + 1)\, S_1(\boldsymbol{x}, \lambda) - (\alpha_2 + 1)\, S_1(\boldsymbol{y}, \lambda) - (\alpha_3 + 1)\, S_1(\boldsymbol{z}, \lambda), \tag{6}$$

where $S_1(\boldsymbol{w}, a) = \sum_{i=1}^{q} \ln[1 + exp(-a w_i)]$ and $\boldsymbol{w} = (w_1, w_2, \ldots, w_q)$.

By derivation with respect to parameters we have

$$\frac{\partial}{\partial\alpha_1}\mathcal{L}_{i,j,k}^{n,m,L}(\alpha_1, \alpha_2, \alpha_3|\lambda) = \frac{n}{\alpha_1} - S_1(\boldsymbol{x}, \lambda), \frac{\partial}{\partial\alpha_2}\mathcal{L}_{i,j,k}^{n,m,L}(\alpha_1, \alpha_2, \alpha_3|\lambda) = \frac{m}{\alpha_2} - S_1(\boldsymbol{y}, \lambda),$$
$$\frac{\partial}{\partial\alpha_3}\mathcal{L}_{i,j,k}^{n,m,L}(\alpha_1, \alpha_2, \alpha_3|\lambda) = \frac{L}{\alpha_3} - S_1(\boldsymbol{z}, \lambda), \tag{7}$$

and

$$\frac{\partial}{\partial\lambda}\mathcal{L}_{i,j,k}^{n,m,L}(\alpha_1, \alpha_2, \alpha_3|\lambda) = \frac{n + m + L}{\lambda} - \left(\sum_{i=1}^{n} x_i + \sum_{j=1}^{m} y_j + \sum_{k=1}^{L} z_k\right)$$
$$+ (\alpha_1 + 1)\, S_2(\boldsymbol{x}, \lambda) + (\alpha_2 + 1)\, S_2(\boldsymbol{y}, \lambda) + (\alpha_3 + 1)\, S_2(\boldsymbol{z}, \lambda), \tag{8}$$

where $S_2(\boldsymbol{w},a)=\sum_{i=1}^{q}\frac{w_i e^{-a\ w_i}}{1+e^{-a\ w_i}}$ and $\boldsymbol{w}=(w_1,w_2,\ldots,w_q)$. Substituting (7) in (2)have

$$\frac{n+m+L}{\lambda}-\left(\sum_{i=1}^{n}x_i+\sum_{j=1}^{m}y_j+\sum_{k=1}^{L}z_k\right)+\left(\frac{n}{S_1(\boldsymbol{x},\lambda)}+1\right)S_2(\boldsymbol{x},\lambda)+$$
$$\left(\frac{m}{S_1(\boldsymbol{y},\lambda)}+1\right)S_2(\boldsymbol{y},\lambda)+\left(\frac{L}{S_1(\boldsymbol{z},\lambda)}+1\right)S_2(\boldsymbol{z},\lambda)=0. \tag{9}$$

By solving (2) via numerically methods we get the $\widehat{\lambda}$. Also, we have

$$\widehat{\alpha}_1=\frac{n}{S_1(\boldsymbol{x},\widehat{\lambda})},\widehat{\alpha}_2=\frac{m}{S_1(\boldsymbol{y},\widehat{\lambda})},\widehat{\alpha}_3=\frac{L}{S_1(\boldsymbol{z},\widehat{\lambda})}. \tag{10}$$

By substitution $\widehat{\boldsymbol{\eta}}=(\widehat{\alpha}_1,\widehat{\alpha}_2,\widehat{\alpha}_3,\widehat{\lambda})$ in (5) we have:

$$\hat{R}^{\,G}=\frac{\widehat{\alpha}_1\widehat{\alpha}_3}{(\widehat{\alpha}_1+\widehat{\alpha}_2)(\widehat{\alpha}_1+\widehat{\alpha}_2+\widehat{\alpha}_3)}. \tag{11}$$

Also, the Fisher's information matrix is $\boldsymbol{J}(\boldsymbol{\eta})=[J_{ij}]_{i,j=1}^{4}$ which is obtained by $\boldsymbol{J}(\boldsymbol{\eta})=\mathbf{E}(\boldsymbol{I}(\boldsymbol{\eta}))$ and $\boldsymbol{I}(\boldsymbol{\eta})=-\frac{\partial^2\iota}{\partial\boldsymbol{\eta}^2}$ where $\boldsymbol{I}(\boldsymbol{\eta})=[I_{ij}]_{i,j=1}^{4}$ and $\boldsymbol{\eta}=(\alpha_1,\alpha_2,\alpha_3,\lambda)$, where

$$I_{11}=\frac{n}{\alpha_1^2},I_{22}=\frac{m}{\alpha_2^2},I_{33}=\frac{L}{\alpha_3^2},I_{12}=I_{21}=I_{13}=I_{31}=I_{23}=I_{32}=0,$$

$$I_{14}=I_{41}=S_2(\boldsymbol{x},\lambda),I_{24}=I_{42}=S_2(\boldsymbol{y},\lambda),I_{34}=I_{43}=S_2(\boldsymbol{z},\lambda),$$

and

$$I_{44}=\frac{n+m+L}{\lambda^2}+(\alpha_1+1)S_3(\boldsymbol{x},\lambda)+(\alpha_2+1)S_3(\boldsymbol{y},\lambda)+(\alpha_3+1)S_3(\boldsymbol{z},\lambda),$$

where $S_3(\boldsymbol{w},a)=\sum_{i=1}^{q}\frac{w_i^2e^{-a\ w_i}}{\left(1+e^{-a\ w_i}\right)^2}$. Therefore, by using (3)and integration we have:

$$J_{11}=\frac{n}{\alpha_1^2},J_{22}=\frac{m}{\alpha_2^2},J_{33}=\frac{L}{\alpha_3^2},J_{12}=J_{21}=J_{13}=J_{31}=J_{23}=J_{32}=0,J_{14}=J_{41}=\frac{n\alpha_1}{\lambda}K(1,\alpha_1+2),$$

$$J_{24}=J_{42}=\frac{m\alpha_2}{\lambda}K(1,\alpha_2+2),J_{34}=J_{43}=\frac{L\alpha_3}{\lambda}K(1,\alpha_3+2),$$

and

$$J_{44}=\frac{n+m+L}{\lambda^2}+\frac{n\alpha_1(\alpha_1+1)}{\lambda^2}K(2,\alpha_1+3)+\frac{n\alpha_2(\alpha_2+1)}{\lambda^2}K(2,\alpha_2+3)+\frac{n\alpha_3(\alpha_3+1)}{\lambda^2}K(2,\alpha_3+3),$$

where $K(a,b)=\int_0^{+\infty}t\,[\ln(t)\,]^a(1+t)^{-b}dt$.

According to *Gradshteyn and Ryzhik* [6], we have

$$K(2,\alpha\ +3)=\frac{\psi^{(1)}(\alpha\ )+\psi^2(\alpha)+\psi^2(1\ )+\frac{1}{6}\pi^2}{(\alpha+1)(\alpha+2)}+2\frac{[1-\alpha(\psi(1\ )+1)](\psi(\alpha\ )-1)}{\alpha(\alpha+1)(\alpha+2)},$$

and

$$K(1,\alpha\ +2)=-\frac{\psi(\alpha\ )-\psi(1\ )-1}{\alpha(\alpha+1)},$$

where $\psi^{(k)}(y) = \frac{\partial^{k+1}}{\partial y^{k+1}} Ln(\Gamma(y))$.

Therefore

$$J_{14} = J_{41} = -n\frac{\psi(\alpha_1) - \psi(1) - 1}{\lambda(\alpha_1 + 1)}, J_{24} = J_{42} = -m\frac{\psi(\alpha_2) - \psi(1) - 1}{\lambda(\alpha_2 + 1)},$$

$$J_{34} = J_{43} = -L\frac{\psi(\alpha_3) - \psi(1) - 1}{\lambda(\alpha_3 + 1)},$$

and

$$J_{44} = \frac{n+m+L}{\lambda^2} + \frac{n}{\lambda^2(\alpha_1+2)}\left(\alpha_1\begin{bmatrix}\psi^{(1)}(\alpha_1)\\ +\psi^2(\alpha_1)\\ +\psi^2(1)\\ +\frac{\pi^2}{6}\end{bmatrix} + 2\begin{pmatrix}\{1-\alpha_1[\psi(1)+1]\}\\ \times[\psi(\alpha_1)-1]\end{pmatrix}\right)$$

$$+\frac{m}{\lambda^2(\alpha_2+2)}\left(\alpha_2\begin{bmatrix}\psi^{(1)}(\alpha_2)\\ +\psi^2(\alpha_2)\\ +\psi^2(1)\\ +\frac{\pi^2}{6}\end{bmatrix} + 2\{1-\alpha_2[\psi(1)+1]\}[\psi(\alpha_2)-1]\right)$$

$$+\frac{L}{\lambda^2(\alpha_3+2)}\left\{\alpha_3\begin{bmatrix}\psi^{(1)}(\alpha_3)\\ +\psi^2(\alpha_3)\\ +\psi^2(1)\\ +\frac{\pi^2}{6}\end{bmatrix} + 2\{1-\alpha_3[\psi(1)+1]\}[\psi(\alpha_3)-1]\right\}.$$

Since $\widehat{\boldsymbol{\eta}} \to \boldsymbol{N}_4(\boldsymbol{\eta}, \boldsymbol{J}^{-1}(\boldsymbol{\eta}))$, by multivariate Delta method one can see that $\hat{R}^G \to N(R^G, \sigma^2(\boldsymbol{\eta}))$, where

$\sigma^2 = \frac{\partial R^G}{\partial \boldsymbol{\eta}} \boldsymbol{J}^{-1}(\boldsymbol{\eta}) \left(\frac{\partial R^G}{\partial \boldsymbol{\eta}}\right)^T$, $\frac{\partial}{\partial \boldsymbol{\eta}} R^G = \left(\frac{\partial R^G}{\partial \alpha_1}, \frac{\partial R^G}{\partial \alpha_2}, \frac{\partial R^G}{\partial \alpha_3}, 0\right)$, $\frac{\partial}{\partial \alpha_1} R^G = \frac{\alpha_3(\alpha_2^2+\alpha_2\alpha_3-\alpha_1^2)}{(\alpha_1+\alpha_2)^2(\alpha_1+\alpha_2+\alpha_3)^2}$,

$\frac{\partial}{\partial \alpha_2} R^G = \frac{-\alpha_1\alpha_3(2\alpha_1+2\alpha_2+\alpha_3)}{(\alpha_1+\alpha_2)^2(\alpha_1+\alpha_2+\alpha_3)^2}$, $\frac{\partial}{\partial \alpha_3} R^G = \frac{\alpha_1}{(\alpha_1+\alpha_2+\alpha_3)^2}$.

Now, a confidence interval for R^G can been obtained by following equation

$$R^G \in \left(\hat{R}^G - z_{1-\frac{\alpha}{2}}\,\sigma(\widehat{\boldsymbol{\eta}}),\ \hat{R}^G + z_{1-\frac{\alpha}{2}}\,\sigma(\widehat{\boldsymbol{\eta}})\right). \tag{12}$$

2.1. Bayesian estimation

In continuation of this section, a MCMC technique is applied for Bayesian estimation. To obtain the posterior distribution for quantities of interest in generalized stress-strength for a GL distribution, we use the Gamma priors $Gamma(a_1, b_1)$, $Gamma(a_2, b_2)$, $Gamma(a_3, b_3)$ and $Gamma(a_4, b_4)$ for parameters α_1, α_2, α_3 and λ, respectively. According to the range of changes in distribution parameters, gamma distribution has been used. On the other hand, because the gamma distribution is a general distribution, other distributions such as exponential and chi-square can be deduced from it.
All the hyper-parameters are assumed to be known. The full conditional distributions are as following.

$$\alpha_1|\alpha_2, \alpha_3, \lambda, \boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z} \sim Gamma(n + a_1, b_1 + S_1(\boldsymbol{X}, \lambda))$$

$$\alpha_2|\alpha_1, \alpha_3, \lambda, \boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z} \sim Gamma(m + a_2, b_2 + S_1(\boldsymbol{Y}, \lambda))$$

$$\alpha_3|\alpha_1, \alpha_2, \lambda, \boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z} \sim Gamma(L + a_3, b_3 + S_1(\boldsymbol{Z}, \lambda)),$$

then

$$f_{\lambda|\alpha_1,\alpha_2,\alpha_3,\boldsymbol{X},\boldsymbol{Y},\boldsymbol{Z}} \propto \lambda^{n+m+L+a_4-1}$$

$$exp\left[\begin{array}{c} -\lambda\left(b_4+\sum_{i=1}^{n}X_i+\sum_{j=1}^{m}Y_j+\sum_{k=1}^{L}Z_k\right) \\ -\left(\alpha_1+1\right)S_1\left(\boldsymbol{X},\lambda\right)-\left(\alpha_2+1\right)S_1\left(\boldsymbol{Y},\lambda\right) \\ -(\alpha_3+1)S_1(\boldsymbol{Z},\lambda) \end{array}\right],$$

where $\boldsymbol{X}=(X_1,\ldots,X_n)$, $\boldsymbol{Y}=(Y_1,\ldots,Y_m)$, $\mathbf{Z}=(Z_1,\ldots,Z_L)$. Since the posterior density $f_{\lambda|\alpha_1,\alpha_2,\alpha_3,\boldsymbol{X},\boldsymbol{Y},\boldsymbol{Z}}$ has not a known and closed form, Metropolis-Hastings (M-H) algorithm with normal proposal distribution is used for generation a sample of conditional distribution $\lambda|\alpha_1,\alpha_2,\alpha_3,\boldsymbol{X},\boldsymbol{Y},\boldsymbol{Z}$. Finally, a four staged Gibbs sampling is applied which its fourth stage is based on M-H algorithm.

2.2. *Bootstrap confidence intervals*

The confidence intervals (CIs) based on mentioned methods do not perform very well for small sample size. So, CI using the percentile bootstrap method (see *Efron* [5]) is proposed. Bootstrapping Algorithm for estimating the CIs of R ^{G}.
The algorithm for estimating the CIs of $R^{\,G}$ using this method is illustrated below:
Step 1:
From the sample $\{x_1, \ldots, x_n\}$, $\{y_1, \ldots, y_m\}$ and $\{z_1,\ldots,z_L\}$, compute $\widehat{\boldsymbol{\eta}}=(\widehat{\alpha}_1,\widehat{\alpha}_2,\widehat{\alpha}_3,\widehat{\lambda})$.
Step 2:
Use $\widehat{\alpha}_1$ and $\widehat{\lambda}$ to generate a bootstrap sample $\{x_1^*, \ldots, x_n^*\}$, and similarly use $\widehat{\alpha}_2$ and $\widehat{\lambda}$ to generate a sample $\{y_1^*, \ldots, y_m^*\}$, and also use $\widehat{\alpha}_3$ and $\widehat{\lambda}$ to generate a sample $\{z_1^*, \ldots, z_L^*\}$. Based on these samples compute $\hat{R}^{\,G}$ by (11)
Step 3:
Repeat step 2, where *N* boot times for generating $\hat{R}_1^G,\ldots,\hat{R}_N^G$.
Now, $\hat{R}_{boot}^G=\frac{1}{N}\sum_{i=1}^{N}\hat{R}_i^G$ and the approximate 100 $(1-\alpha)$% CI of $R^{\,G}$ is given by

$$\left(\hat{R}^G_{\left(\frac{\alpha}{2}\right)},\hat{R}^G_{\left(1-\frac{\alpha}{2}\right)}\right)$$

where $\hat{R}^G_{(\gamma)}$ shows quantile of order γ for $\hat{R}_1^G,\ldots,\hat{R}_N^G$.

3. Model for case of common shape parameter

In this section, generalized stress-strength model R^G is analyzed for GL distribution in the case of X, Y and Z are GL(α,λ_1), GL(α,λ_2) and GL(α,λ_3), respectively. After cumbersome computation, we have

$$R^{\,G}=\alpha\int_0^{+\infty}(1+y)^{-\alpha-1}\left\{\begin{array}{c}\left(1+y^{\frac{\lambda_1}{\lambda_3}}\right)^{-\alpha}\left(1+y^{\frac{\lambda_2}{\lambda_3}}\right)^{-\alpha} \\ -\left(1+y^{\frac{\lambda_1}{\lambda_2}}\right)^{-\alpha}\left[1-\left(1+y^{\frac{\lambda_3}{\lambda_2}}\right)^{-\alpha}\right]\end{array}\right\}dy. \quad (13)$$

Equation (13) has been derived by substitution (3) and (4) in(2). In continuation of this section we shall find MLEs of parameters.
Let $X_1,\ldots,X_n\sim GL(\alpha,\lambda_1)$, $Y_1,\ldots,Y_m\sim GL(\alpha,\lambda_2)$ and $Z_1,\ldots,Z_L\sim GL(\alpha,\lambda_3)$, then the likelihood function is

$$\mathcal{L}_{i,j,k}^{n,m,L}\left(\lambda_1,\lambda_2,\lambda_3|\alpha\right)=nln\left(\lambda_1\right)\;+\;mln\left(\lambda_2\right)\;+Lln\left(\lambda_3\right)\;+\left(n+m+L\right)ln\left(\alpha\right)$$

$$-\lambda_1\sum_{i=1}^{n}x_i-\lambda_2\sum_{j=1}^{m}y_j-\lambda_3\sum_{k=1}^{L}z_k-\left(\alpha+1\right)\left(S_1\left(\boldsymbol{x},\lambda_1\right)+S_1\left(\boldsymbol{y},\lambda_2\right)+S_1\left(\boldsymbol{z},\lambda_3\right)\right). \quad (14)$$

By derivation with respect to parameters we have

$$\frac{\partial}{\partial\alpha}\mathcal{L}_{i,j,k}^{n,m,L}\left(\lambda_1,\lambda_2,\lambda_3|\alpha\right)=\frac{n+m+L}{\alpha}-S_1\left(\boldsymbol{x},\lambda_1\right)+S_1\left(\boldsymbol{y},\lambda_2\right)+S_1\left(\boldsymbol{z},\lambda_3\right)). \tag{15}$$

$$\frac{\partial}{\partial\lambda_1}\mathcal{L}_{i,j,k}^{n,m,L}\left(\lambda_1,\lambda_2,\lambda_3|\alpha\right)=\frac{n}{\lambda_1}-\sum_{i=1}^{n}x_i+(\alpha+1)\,S_2\left(\boldsymbol{x},\lambda_1\right), \tag{16}$$

$$\frac{\partial}{\partial\lambda_2}\mathcal{L}_{i,j,k}^{n,m,L}\left(\lambda_1,\lambda_2,\lambda_3|\alpha\right)=\frac{m}{\lambda_2}-\sum_{i=1}^{n}y_i+(\alpha+1)\,S_2\left(\boldsymbol{y},\lambda_2\right), \tag{17}$$

and

$$\frac{\partial}{\partial\lambda_3}\mathcal{L}_{i,j,k}^{n,m,L}\left(\lambda_1,\lambda_2,\lambda_3|\alpha\right)=\frac{L}{\lambda_3}-\sum_{i=1}^{n}z_i+(\alpha+1)\,S_2\left(\boldsymbol{z},\lambda_3\right). \tag{18}$$

Using (15) and replacing in (16) -(18), we encounter with a system of nonlinear equations. This system is solved by numerically methods. Then, resulted MLEs of $\widehat{\lambda}_1$, $\widehat{\lambda}_2$ and $\widehat{\lambda}_3$ are used in (15) which leads to

$$\widehat{\alpha}=\frac{1}{S_1\left(\boldsymbol{x},\widehat{\lambda}_1\right)+S_1\left(\boldsymbol{y},\widehat{\lambda}_2\right)+S_1\left(\boldsymbol{z},\widehat{\lambda}_3\right)}(n+m+L).$$

By substitution $\widehat{\boldsymbol{\eta}}=(\widehat{\alpha}\ ,\widehat{\lambda}_1,\widehat{\lambda}_2,\widehat{\lambda}_3)$ in (13) we have

$$\hat{R}^{\,G}=\widehat{\alpha}\int_0^{+\infty}(1+y)^{-\widehat{\alpha}\,-1}\left\{\begin{array}{c}\left(1+y^{\frac{\widehat{\lambda}_1}{\widehat{\lambda}_3}}\right)^{-\widehat{\alpha}}\left(1+y^{\frac{\widehat{\lambda}_2}{\widehat{\lambda}_3}}\right)^{-\widehat{\alpha}}\\ -\left(1+y^{\frac{\widehat{\lambda}_1}{\widehat{\lambda}_2}}\right)^{-\widehat{\alpha}}\left[1-\left(1+y^{\frac{\widehat{\lambda}_3}{\widehat{\lambda}_2}}\right)^{-\widehat{\alpha}}\right]\end{array}\right\}dy. \tag{19}$$

Also, the Fisher's information matrix is $\boldsymbol{J}\left(\boldsymbol{\eta}\right)=\left[J_{ij}\right]_{i,j=1}^{4}$ which is obtained by $\boldsymbol{J}\left(\boldsymbol{\eta}\right)=E(\boldsymbol{I}\left(\boldsymbol{\eta}\right))$ and $\boldsymbol{I}\left(\boldsymbol{\eta}\right)=-\frac{\partial^2\iota}{\partial\boldsymbol{\eta}^2}$, where $\boldsymbol{I}\left(\boldsymbol{\eta}\right)=\left[I_{ij}\right]_{i,j=1}^{4}$and $\boldsymbol{\eta}=(\alpha,\lambda_1,\lambda_2,\lambda_3)$. Then, we have

$$I_{11}=\frac{n+m+L}{\alpha^2},\ I_{12}=I_{21}=S_2(\boldsymbol{x},\lambda_1),\ I_{13}=I_{31}=S_2(\boldsymbol{y},\lambda_2),\ I_{14}=I_{41}=S_2(\boldsymbol{z},\lambda_3)$$

$$I_{22}=\frac{n}{\lambda_1^2}+(\alpha+1)S_3(\boldsymbol{x},\lambda_1),I_{23}=I_{32}=I_{24}=I_{42}=I_{34}=I_{43}=0.$$

$$I_{33}=\frac{m}{\lambda_2^2}+(\alpha+1)S_3(\boldsymbol{y},\lambda_2),I_{44}=\frac{L}{\lambda_3^2}+(\alpha+1)S_3(\boldsymbol{z},\lambda_3).$$

Now, using same methods and formulas as Section 2 one can reach to following results.

$$J_{11}=\frac{n+m+L}{\alpha^2},\ J_{12}=J_{21}=-\frac{nQ_1}{\lambda_1},\ J_{13}=J_{31}=-\frac{mQ_1}{\lambda_2},\ J_{14}=J_{41}=-\frac{LQ_1}{\lambda_3},$$

$$J_{22}=\frac{n}{\lambda_1^2}Q_2,\ J_{33}=\frac{m}{\lambda_2^2}Q_2,\ J_{23}=J_{32}=J_{24}=J_{42}=J_{34}=J_{43}=0$$

and

$$J_{44}=\frac{L}{\lambda_3^2}Q_2$$

where

$$Q_1 = \frac{1}{\alpha+1}\left[\psi\left(\alpha\right) - \psi\left(1\right) - 1\right],$$

$$Q_2{=}1 + \frac{\alpha\left[\psi^{(1)}\left(\alpha\right) + \psi^2\left(\alpha\right) + \psi^2\left(1\right) + \frac{\pi^2}{6}\right] + 2[1-\alpha(\psi\left(1\right)+1)](\psi\left(\alpha\right)-1)}{\alpha+2}.$$

Since $\widehat{\boldsymbol{\eta}} \to \boldsymbol{N}_4(\boldsymbol{\eta}, \boldsymbol{J}^{-1}(\boldsymbol{\eta}))$, by multivariate Delta method we have $\hat{R}^{\,G} \to N(R^{\,G}, \sigma^2(\boldsymbol{\eta}))$ where $\sigma^2 = \frac{\partial R^{\,G}}{\partial \boldsymbol{\eta}} \boldsymbol{J}^{-1}(\boldsymbol{\eta}) \left(\frac{\partial R^{\,G}}{\partial \boldsymbol{\eta}}\right)^T$ and $\frac{\partial}{\partial \boldsymbol{\eta}} R^{\,G} = \left(\frac{\partial R^{\,G}}{\partial \alpha}, \frac{\partial R^{\,G}}{\partial \lambda_1}, \frac{\partial R^{\,G}}{\partial \lambda_2}, \frac{\partial R^{\,G}}{\partial \lambda_3}\right)$.

$$\frac{\partial}{\partial \alpha} R^{\,G} = \int_0^{+\infty} \left(\begin{array}{c} \left\{ \frac{\left(1+y^{\frac{\lambda_1}{\lambda_3}}\right)^{-\alpha}\left(1+y^{\frac{\lambda_2}{\lambda_3}}\right)^{-\alpha} - \left(1+y^{\frac{\lambda_1}{\lambda_2}}\right)^{-\alpha}\left[1-\left(1+y^{\frac{\lambda_3}{\lambda_2}}\right)^{-\alpha}\right]}{\left[(1+y)^{-\alpha-1} - \alpha(1+y)^{-\alpha-1}\ln(1+y)\right]^{-1}} \right\} \\ - \left\{ \begin{array}{c} \left[\begin{array}{c} \ln\left(1+y^{\frac{\lambda_1}{\lambda_3}}\right) \\ +\ln\left(1+y^{\frac{\lambda_2}{\lambda_3}}\right) \end{array} \right] \left(1+y^{\frac{\lambda_1}{\lambda_3}}\right)^{-\alpha}\left(1+y^{\frac{\lambda_2}{\lambda_3}}\right)^{-\alpha} \\ +\ln\left[\left(1+y^{\frac{\lambda_1}{\lambda_2}}\right)\right]\left(1+y^{\frac{\lambda_1}{\lambda_2}}\right)^{-\alpha} \\ -\left[\begin{array}{c} \ln\left(1+y^{\frac{\lambda_1}{\lambda_2}}\right) \\ +\ln\left(1+y^{\frac{\lambda_3}{\lambda_2}}\right) \end{array} \right] \left(1+y^{\frac{\lambda_1}{\lambda_2}}\right)^{-\alpha}\left(1+y^{\frac{\lambda_3}{\lambda_2}}\right)^{-\alpha} \end{array} \right\} \alpha(1+y)^{-\alpha-1} \end{array} \right) dy, \tag{20}$$

$$\frac{\partial}{\partial \lambda_1} R^{\,G} = \alpha \int_0^{+\infty} (1+y)^{-\alpha-1} \left\{ \begin{array}{c} -\alpha\left(1+y^{\frac{\lambda_1}{\lambda_3}}\right)^{-\alpha-1} \frac{\ln(y)}{\lambda_3} y^{\frac{\lambda_1}{\lambda_3}} \left(1+y^{\frac{\lambda_2}{\lambda_3}}\right)^{-\alpha} \\ +\alpha\left(1+y^{\frac{\lambda_1}{\lambda_2}}\right)^{-\alpha-1} \frac{\ln(y)}{\lambda_2} y^{\frac{\lambda_1}{\lambda_2}} \left[1-\left(1+y^{\frac{\lambda_3}{\lambda_2}}\right)^{-\alpha}\right] \end{array} \right\} dy, \tag{21}$$

$$\frac{\partial}{\partial \lambda_2} R^{\,G} = \alpha \int_0^{+\infty} (1+y)^{-\alpha-1} \left\{ \begin{array}{c} -\alpha\left(1+y^{\frac{\lambda_1}{\lambda_3}}\right)^{-\alpha} \frac{\ln(y)}{\lambda_3} y^{\frac{\lambda_2}{\lambda_3}} \left(1+y^{\frac{\lambda_2}{\lambda_3}}\right)^{-\alpha-1} \\ -\alpha\left(1+y^{\frac{\lambda_1}{\lambda_2}}\right)^{-\alpha-1} \frac{\ln(y)\lambda_1}{\lambda_2^2} y^{\frac{\lambda_1}{\lambda_2}} \left[1-\left(1+y^{\frac{\lambda_3}{\lambda_2}}\right)^{-\alpha}\right] \\ +\alpha\left(1+y^{\frac{\lambda_1}{\lambda_2}}\right)^{-\alpha} \frac{\ln(y)\lambda_3}{\lambda_2^2} y^{\frac{\lambda_3}{\lambda_2}} \left(1+y^{\frac{\lambda_3}{\lambda_2}}\right)^{-\alpha-1} \end{array} \right\} dy, \tag{22}$$

and

$$\frac{\partial}{\partial \lambda_3} R^{\,G} = \alpha \int_0^{+\infty} (1+y)^{-\alpha-1} \left[\begin{array}{c} \alpha\left(1+y^{\frac{\lambda_1}{\lambda_3}}\right)^{-\alpha-1} \frac{\ln(y)\lambda_1}{\lambda_3^2} y^{\frac{\lambda_1}{\lambda_3}} \left(1+y^{\frac{\lambda_2}{\lambda_3}}\right)^{-\alpha} \\ +\alpha\left(1+y^{\frac{\lambda_2}{\lambda_3}}\right)^{-\alpha-1} \frac{\ln(y)\lambda_2}{\lambda_3^2} y^{\frac{\lambda_2}{\lambda_3}} \left(1+y^{\frac{\lambda_1}{\lambda_3}}\right)^{-\alpha} \\ -\alpha\left(1+y^{\frac{\lambda_1}{\lambda_2}}\right)^{-\alpha} \frac{\ln(y)}{\lambda_2} y^{\frac{\lambda_3}{\lambda_2}} \left(1+y^{\frac{\lambda_3}{\lambda_2}}\right)^{-\alpha-1} \end{array} \right] dy. \tag{23}$$

Now, a CI for $R^{\,G}$ can been obtained by

$$R^{\,G} \in \left(\hat{R}^{\,G} - z_{1-\frac{\alpha}{2}}\,\sigma(\widehat{\boldsymbol{\eta}})\,,\, \hat{R}^{\,G} + z_{1-\frac{\alpha}{2}}\,\sigma(\widehat{\boldsymbol{\eta}})\right). \tag{24}$$

3.1. Bayesian estimation

Again, for having more statistical findings, we use of MCMC methods for a Bayesian inference on $R^{\,G}$. We apply the Metropolis-Hastings algorithm in a Gibbs sampler to provide the Bayesian estimation of $R^{\,G}$. We use the independent gamma prior densities $Gamma(a_1, b_1)$, $Gamma(a_2, b_2)$, $Gamma(a_3, b_3)$ and $Gamma(a_4, b_4)$ for parameters $\alpha,\ \lambda_1,\ \lambda_2$ and λ_3, respectively. Now, the full conditional distributions are as follows

$$\alpha|\lambda_1,\lambda_2,\lambda_3,\boldsymbol{X},\boldsymbol{Y},\boldsymbol{Z}\sim Gamma\begin{pmatrix} n+m+L \\ +a_1, b_1 \\ +S_1\left(\boldsymbol{X},\lambda_1\right) \\ +S_1\left(\boldsymbol{Y},\lambda_2\right) \\ +S_1(\boldsymbol{Z},\lambda_3) \end{pmatrix},$$

$$f_{\lambda_1|\alpha,\lambda_2,\lambda_3,\boldsymbol{X},\boldsymbol{Y},\boldsymbol{Z}}\propto{\lambda_1}^{n+a_2-1}exp\left[-\lambda_1\left(b_2+\sum_{i=1}^{n}X_i\right)-(\alpha+1)\,S_1\left(\boldsymbol{X},\lambda_1\right)\right],$$

$$f_{\lambda_2|\alpha,\lambda_1,\lambda_3,\boldsymbol{X},\boldsymbol{Y},\boldsymbol{Z}}\propto{\lambda_2}^{m+a_3-1}exp\left[-\lambda_2\left(b_3+\sum_{j=1}^{m}Y_j\right)-(\alpha+1)\,S_1\left(\boldsymbol{Y},\lambda_2\right)\right],$$

and

$$f_{\lambda_3|\alpha,\lambda_1,\lambda_2,\boldsymbol{X},\boldsymbol{Y},\boldsymbol{Z}}\propto{\lambda_3}^{L+a_4-1}exp\left[-\lambda_3\left(b_4+\sum_{k=1}^{L}Z_k\right)-(\alpha+1)\,S_1\left(\boldsymbol{z},\lambda_3\right)\right].$$

This is obvious that the else of the first posterior density, all three other densities do not have an explicit and closed form. So, M-H algorithm with normal proposal distribution is applied for generation a sample of them. Therefore, we use of Gibbs sampling with four stages whose three last stages are three separate M-H algorithms.

3.2. Bootstrap CIs

Analogously to Section 2.2, we study confidence interval based on the percentile bootstrap method for case of common shape parameter**.** Bootstrapping algorithm can be addressed as follows.
Step 1: From the sample $\{x_1,\ \dots\ ,\ x_n\}$, $\{y_1,\ \dots\ ,\ y_m\}$ and $\{z_1,\dots,z_L\}$, compute $\widehat{\boldsymbol{\eta}}=(\widehat{\alpha}\ ,\widehat{\lambda}_1,\widehat{\lambda}_2,\widehat{\lambda}_3)$.
Step 2: Use $\widehat{\alpha}$ and $\widehat{\lambda}_1$ to generate a bootstrap sample $\{x_1^*,\ \dots\ ,\ x_n^*\}$, and similarly use $\widehat{\alpha}$ and $\widehat{\lambda}_2$ to generate a sample $\{y_1^*,\ \dots\ ,\ y_m^*\}$, and also use $\widehat{\alpha}$ and $\widehat{\lambda}_3$ to generate a sample $\{z_1^*,\ \dots\ ,\ z_L^*\}$. Based on these samples compute $\hat{R}^{\,G}$ by (19).
Step 3: Repeat step 2, *N* boot times for generating $\hat{R}_1^G,\dots,\hat{R}_N^G$.
Now, $\hat{R}_{boot}^G=\frac{1}{N}\sum_{i=1}^{N}\hat{R}_i^G$ and the approximate 100 $(1-\alpha)\%$ confidence interval of $R^{\,G}$ is given by $(\hat{R}_{(\frac{\alpha}{2})}^G,\hat{R}_{(1-\frac{\alpha}{2})}^G)$ where $\hat{R}_{(\gamma)}^G$ shows quantile of order γ for $\hat{R}_1^G,\dots,\hat{R}_N^G$.

4. Numerical results

In this section, a simulation study is done for both classic and Bayesian approaches. The results of Sections 2 and 3 have been presented separately. Throughout this section, coefficient of confidence for all confidence intervals has been considered to 0.95. We compute four well known and important criteria in order to investigate content of accuracy and performance of estimation. They are Bias, Mean Square Error (MSE), Length of confidence interval (L) and Cover of Probability (CP) for confidence interval. In the first step, the results have been represented in Tables 1-3 for three fixed cases of parameters λ, α_1, α_2 and α_3 and some variable values of sample sizes $n,\ m,\ L$. As these tables demonstrate both methods MLE and Bayesian have been worked well. This finding is resulted in small values of Bias, MSE and L along with near to 0.95 values for CP. In both methods proficiency of estimation is an increasing function of sample sizes $n,\ m,\ L$. For instance, all four criteria for case $n=\ m=\ L=70$ show better estimation than case $n=\ m=\ L=25$. This is completely logic by attention to using of asymptotic distribution. A comparison between MLE and Bayesian methods denote that approximately Bayesian method has more reliable results and better performance than MLE w.r.t Bias and MSE criteria. These results are approximately reversed for both criteria L and CP. However, in total this seems Bayesian method has more suitable results.

Table 1. Results for α_1= 2,α_2= 1.5,α_3= 4, λ= 1.25, R^G=0.3048.

n	25	45	70	25	50	65
m	25	45	70	45	35	80
L	25	45	70	30	70	20
$\widehat{R}^G$	0.3080	0.3059	0.3057	0.3068	0.3039	0.3036
Bias($\widehat{R}^G$)	0.0032	0.0012	0.0009	0.0020	-0.0009	-0.0012
MSE($\widehat{R}^G$)	0.0027	0.0015	0.0009	0.0017	0.0015	0.0013
Length	0.1941	0.1449	0.1163	0.1599	0.1471	0.1539
CP	0.9314	0.9386	0.9464	0.9394	0.9412	0.9804
$\widehat{R}^G_{Bayes}$	0.3102	0.3098	0.3092	0.3063	0.3047	0.3035
Bias($\widehat{R}^G_{Bayes}$)	-0.1006	-0.0649	-0.0136	-0.0284	-0.0101	-0.0413
MSE($\widehat{R}^G_{Bayes}$)	0.0123	0.0060	0.0017	0.0032	0.0025	0.0038
Length	0.1491	0.1246	0.1114	0.1482	0.1395	0.1431
CP	0.7580	0.7900	0.9600	0.9300	0.9550	0.8250

Table 2. Results for α_1= 0.25,α_2= 5,α_3= 1.2, λ= 7, R^G=0.0089.

n	25	40	70	30	50	60
m	25	40	70	25	35	100
L	25	40	70	15	70	85
$\widehat{R}^G$	0.0092	0.0090	0.0089	0.0092	0.0090	0.0090
Bias($\widehat{R}^G$)	0.0003	0.0002	0.0000	0.0004	0.0001	0.0001
MSE($\widehat{R}^G$)	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
Length	0.0177	0.0139	0.0105	0.0186	0.0134	0.0094
CP	0.8840	0.8982	0.9176	0.8682	0.9002	0.9264
$\widehat{R}^G_{Bayes}$	0.0097	0.0084	0.0097	0.0104	0.0091	0.0098
Bias($\widehat{R}^G_{Bayes}$)	0.0009	0.0005	0.0002	0.0006	0.0003	0.0001
MSE($\widehat{R}^G_{Bayes}$)	0.0004	0.0002	0.0001	0.0004	0.0002	0.0000
Length	0.0415	0.0246	0.0193	0.0408	0.0277	0.0122
CP	0.9800	0.9162	0.9600	0.8900	0.9500	0.9854

Regarding a statistical point of view approach this has not been caused randomly. Since we use from proper prior distributions there are more information rather than MLE which this information leads to exacter estimation.

The same results have been acquired for different values of sample sizes $n,\ m,\ L$ and parameters $\alpha, \lambda_1, \lambda_2$ and λ_3 in Tables 4 and 5. Although performance of both estimation methods is suitable as previous case, comparison between them has a bit difference with results of Tables 1-3. Here, Bayesian method has approximately better results than MLE w.r.t all four criteria. A justification for this finding is this fact that in MLE method we need to compute 4 integrals (20)-(23) by numerically methods which may lead to some bit errors in estimation. However, for Bayesian method there is no necessity for computing such integrals.

In order to check the ability of model for small sample sizes, results of bootstrapping estimation have been represented in Tables 6-8. The results show that the bootstrap method is absolutely appropriate for both point estimation and interval estimation of R^G. Its performance is appropriate even for small sample sizes. As tables demonstrate $\hat{R}^G_{boot}$ is really near to R^G. Furthermore, R^G has been lied in bootstrap confidence interval by considering values of its lower bound and upper bound.

In order to investigate about effect of sample size and have a more reliable result, we have worked with more sample sizes in Table 8. This table along with two Tables 6 and 7 denote that although bootstrap method has suitable results for small sample sizes its proficiency is improved with increasing sample sizes.

Table 3. $\alpha_1 = 5, \alpha_2 = 2, \alpha_3 = 3$, $\lambda = 1.5$, R^G=0.2143.

n	15	40	50	70	50	60
m	15	40	50	70	35	65
L	15	40	50	70	70	85
$\widehat{R}^G$	0.2272	0.2141	0.2136	0.2140	0.2126	0.2132
Bias($\widehat{R}^G$)	0.0129	-0.0001	-0.0007	-0.0002	-0.0017	-0.0011
MSE($\widehat{R}^G$)	0.0048	0.0009	0.0007	0.0005	0.0007	0.0005
Length	0.1950	0.1169	0.1044	0.0883	0.1017	0.0847
CP	0.8000	0.9465	0.9392	0.9484	0.9394	0.9452
$\widehat{R}^G_{Bayes}$	0.2078	0.2133	0.2139	0.2131	0.2215	0.2184
Bias($\widehat{R}^G_{Bayes}$)	-0.0065	-0.0010	-0.0003	-0.0012	0.0072	0.0041
MSE($\widehat{R}^G_{Bayes}$)	0.0032	0.0015	0.0013	0.0009	0.0013	0.0010
Length	0.1802	0.1147	0.1031	0.0872	0.1038	0.0852
CP	0.9938	0.9738	0.9650	0.9650	0.9700	0.9600

Table 4. $\alpha = 1.8, \lambda_1 = 2.5, \lambda_2 = 2$, $\lambda_3 = 2$, R^G=0.183090.

n	15	30	45	100	50	120
m	15	30	45	100	60	75
L	15	30	45	100	75	80
$\widehat{R}^G$	0.1735	0.1798	0.1897	0.1857	0.1802	0.1875
Bias($\widehat{R}^G$)	0.0891	0.0679	0.0698	0.0288	0.0433	0.0405
MSE($\widehat{R}^G$)	0.1201	0.0925	0.0839	0.0311	0.0459	0.0362
Length	0.0177	0.0139	0.0105	0.0186	0.0134	0.0094
CP	0.8840	0.8982	0.9076	0.9482	0.9102	0.9264
$\widehat{R}^G_{Bayes}$	0.1782	0.1728	0.1865	0.1820	0.1878	0.1828
Bias($\widehat{R}^G_{Bayes}$)	0.0153	0.0106	0.0069	0.0034	0.0056	0.0036
MSE($\widehat{R}^G_{Bayes}$)	0.0011	0.0006	0.0004	0.0002	0.0003	0.0002
Length	0.0864	0.0652	0.0551	0.0382	0.0468	0.0403
CP	0.9430	0.9240	0.9460	0.9620	0.9340	0.9460

Table 5. $\alpha = 0.5, \lambda_1 = 1.52, \lambda_2 = 4$, $\lambda_3 = 2.5$, $R^G = 0.026$.

n	15	40	70	90	45	75
m	15	40	70	90	35	45
L	15	40	70	90	60	80
$\widehat{R}^G$	0.0293	0.0262	0.0270	0.0260	0.0260	0.0270
Bias($\widehat{R}^G$)	0.0122	0.0054	0.0140	0.0043	0.0060	0.0033
MSE($\widehat{R}^G$)	0.0009	0.0002	0.0004	0.0003	0.0002	0.0001
Length	0.0555	0.0324	0.0290	0.0244	0.0318	0.0243
CP	0.8750	0.9200	0.9390	0.9468	0.8870	0.9370
$\widehat{R}^G_{Bayes}$	0.0232	0.0275	0.0266	0.0264	0.0273	0.0252
Bias($\widehat{R}^G_{Bayes}$)	0.0112	0.0065	0.0052	0.0044	0.0073	0.0045
MSE($\widehat{R}^G_{Bayes}$)	0.0006	0.0003	0.0003	0.0002	0.0003	0.0002
Length	0.0662	0.0547	0.0530	0.0517	0.0539	0.0522
CP	0.8750	0.9960	0.9811	1.0000	0.9950	0.9990

Table 6. $\alpha = 1, \lambda_1 = 0.25, \lambda_2 = 4$, $\lambda_3 = 7$, $R^{G} = 0.056065$

n	5	15	30	50	10	35
m	5	15	30	50	15	25
L	5	15	30	50	5	40
$\widehat{R}^{G}_{boot}$	0.0403	0.0412	0.0474	0.0580	0.0358	0.0466
Lower bound	0.0112	0.0206	0.0270	0.0343	0.0169	0.0292
Upper bound	0.1464	0.0932	0.0779	0.0908	0.1105	0.0726

Table 7. $\alpha = 1.5$,$\lambda_1 = 2, \lambda_2 = 1$, $\lambda_3 = 0.1$, $R^{G} = 0.2769$

n	10	20	35	50	20	30
m	10	20	35	50	15	15
L	10	20	35	50	10	45
$\widehat{R}^{G}_{boot}$	0.2750	0.2761	0.2765	0.2768	0.2755	0.2775
Lower bound	0.2389	0.2533	0.2591	0.2619	0.2485	0.2569
Upper bound	0.3324	0.3154	0.3047	0.2951	0.3198	0.3136

Table 8. $\alpha_1 = 2, \alpha_2 = 1.5, \alpha_3 = 4$, $\lambda = 1.25$, R^{G}=0.3047619

n	5	10	15	25	50	70	100	200	500	1000
m	5	10	15	25	50	70	100	200	500	1000
L	5	10	15	25	50	70	100	200	500	1000
$\widehat{R}^{G}_{boot}$	0.3092	0.3102	0.3061	0.3091	0.3056	0.3065	0.3043	0.3046	0.3048	0.3050
Lower bound	0.0947	0.1531	0.1889	0.2054	0.2368	0.2474	0.2545	0.2700	0.2839	0.2894
Upper bound	0.6148	0.5049	0.4490	0.4166	0.3765	0.3633	0.3510	0.3409	0.3276	0.3202

Table 9. The results of fitted GL distribution to three data sets.

Voltage	α	λ	K-S	P-value
28	597.2997941	1.316873717	0.30447	0.6464
30	68.75303647	1.07050723	0.36317	0.08321
36	2.354178797	1.324146519	0.11882	0.9671

5. Real data study

In this section, a real data by *Nelson* for illustrative and comparative purposes is analyzed. Data are Ln times to breakdown of an insulating fluid in an accelerated test. They have been measured at different voltages of 26, 28, 30, 32, 34, 36 and 38 kV. We have fitted the generalized logistic distributions to the all data sets separately and we understood that we may consider voltages 28, 30 and 36 for case of common scale parameter. The results of these fitness have been represented in Table 9. We performed the Kolmogorov-Smirnov (K-S) test for these three data whose null hypothesis is that distribution of data is GL. As this table demonstrates all P-values are bigger than 0.05 so we cannot reject null hypothesis in no of these tests. We conclude that these three data sets have GL distribution. Now, we investigate this problem that whether we can assume case of common scale parameter. Similar with Table 9, the estimates of the parameters and K-S test for case of common scale parameter are demonstrated in Table 10. It is clear that, we cannot reject the hypothesis that the three scale parameters are equal. Therefore, we let scale parameters are equal for these data.
Therefore, based on the results given in Section 2, we can obtain different estimates of R^G. The MLE and Bayes estimates and intervals of R^G are presented in Table 11.

Table 10. The results of fitted GL distribution to three data sets assuming that the three scale parameters are equal.

Voltage	α	λ	K-S	P-value
28	372.896	1.21385	0.30906	0.6282
30	51.96263		0.24182	0.4689
36	2.323052		0.11882	0.9671

Table 11. The results of fitted GL distribution to all 7 data sets.

Method	R^G	CIs or HPD
MLE	0.8355663	(0.7397633, 0.9313694)
Bayesian	0.8327827	(0.7767816, 0.8766773)

The value of hyper parameters in priors are $a_1 = 13, b_1 = 1,\ a_2 = 5.5, b_2 = 3, a_3 = 25, b_3 = 0.08,\ a_4 = 5, b_4 = 0.1$. Also, the standard deviation σ for normal proposal distribution is 0.25. For this stage, we have used from M-H algorithm in a Gibbs sampler with iteration 10000. The MLEs have been considered for starting points in Gibbs sampler, too. As we see HPD confidence interval has less length than classical confidence interval based on Equation (12).

6. Discussion and conclusion

In this paper, a generalized stress-strength model has been studied for GL distribution. We considered two cases for which common scale parameters and common shape parameters, respectively. For both cases three prominent and well-known statistical estimation methods containing MLE, Bayesian and Bootstrap have been applied. Our results illustrate priority of Bootstrap method in comparison with two other methods while we deal with small sample sizes. A reverse finding exists for large sample sizes. The other result whose conclusion has accordance with previous statistical papers and textbooks is an approximate similarity of MLE and Bayesian estimation. Although, Bayesian method has more reliable results rather than MLE.

As a future work, we shall estimate R^G for many distributions such as the Kumaraswamy distribution, the generalized Pareto distribution, the generalized failure rate distribution and the power Lindley distribution which can be considered for estimating the R^G. For a comprehensive list of such distributions see *Rezaei et al* [10]. Estimating the R^G under power Lindley is considered as a current project.

In many situations, we may know system's components had been alive for a known time until we are going to have some inferences about R. Therefore, $R^{|a,b}{=}P(X{>}Y|X{>}a, Y{>}b)$ was defined by *Saber and Khorshidian* [12]. The study of $R^{|a,b}$ for GL distribution is another ongoing work.

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